# **Teacher's Resource Binder**

to accompany

# **ROGAWSKI'S CALCULUS** for AP\*

## Early Transcendentals

## **Second Edition**

Jon Rogawski Ray Cannon

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## Preface

Written to support *Calculus for AP*\* *Early Transcendentals, Second Edition,* by John Rogawski and Ray Cannon, this *Teacher's Resource Binder* offers both supplementary and complementary material. We understand a teacher's time is precious; as a result, we try to keep our notes succinct for ease of reference. Our overarching goal is to be concise, practical, and easy to use. Hopefully it will save some time in planning classes!

Regardless of a teacher's level of experience, we hope that our readers share a similar enjoyment of teaching and exploring calculus. We developed a number of features to help the veteran instructor or the first time teacher. Many teachers find different techniques effective in conveying the key topics in calculus. In this TRB, we made an effort to address the different approaches that may be employed. We make suggestions and provide guidance to those who prefer lectures and also supply material for instructors who prefer other approaches. We hope this resource will help you to try new methods in approaching a difficult topic.

On the following pages we note the features in this guide and provide brief descriptions about how each feature should best be utilized.

We would like to acknowledge the editorial staff at W. H. Freeman and Company.

## Features

#### Each Section includes the following information:

1) Class Time - covering all of the material for the AP exams is difficult.

- We make suggestions about how much time should be spent on each section for AB and BC calculus.
- We also note how critical each section is to a student's understanding of calculus.

2) Key Points - It may be difficult to discern some of the main ideas in the text. We provide

- a streamlined list of all the important topics from each chapter.
- concise points for quick reference.
- a bulleted list that identifies the main ideas in each section.

**3)** Lecture Material - In this feature, we take a more theoretical approach to the section's material to foster a conceptual understanding for the student. The material

- is concise in its presentation.
- is based on our own teaching experiences.
- intertwines key examples and exercises from the text.
- guides the teacher in a lecture or lecture and discussion setting.

4) **Discussion Topics/Class Activities** - With any class, some of the more interesting topics require some deviation from the main concepts, and there are some common issues that repeatedly give students problems. These topics

- are engaging examples from our own experience and also examples and problems from the text.
- will force a student to think and reflect on the material, allowing the student to formulate a distinct understanding of the material at hand.
- provide an opportunity to get outside the typical lecture.

**5)** Suggested Problems - There a large number and variety of problems at the end of each section. To help the teacher identify good problems, especially those that are in the AP style, we provide the following:

- a quick reference guide for homework problems.
- suggestions about some core problems that cover a variety of topics and problem types.
- problems that cover graphical, numerical, abstract, and algebraic genres. Often, we also note the difficulty, and if the problem relates to a specific topic in that particular section.

6) Worksheet and Worksheet Solutions - We provide material that can be distributed in class. The best way to learn calculus is by doing calculus. It is very helpful, especially to struggling students, when the first attempt at a type of problem takes place in class, with the teacher available to help. Thus, we provide problems that relate directly to the material.

- Exercises that provide additional practice to students who are having trouble with specific topics.
- Exercises that provide feedback for both students and the instructor.

#### Each Chapter includes the following:

**AP Style Questions** - After talking to many AP teachers, we understand that there can never be enough practice and preparation for the exam. As a result, we developed multiple choice and free response questions that correspond with sections of Rogawski's *Calculus*. These can be used as practice quizzes or for testing material.

## **Teacher's Resource Manual Author Team**

Lin McMullin has taken the lead role in developing the TRB for both the first and second editions of Jon Rogawski's *Calculus*. He is an author and consultant working in mathematics education. He taught high school mathematics, including AB and BC Calculus for 34 years. He has led many workshops and institutes for AP Calculus teachers in the United States and Europe. He served as an AP Calculus exam reader and table leader for 14 years. He is the author of *Teaching AP Calculus*, which is a resource book for teachers based on the material he teaches in his one-week summer institutes.

**Ray Cannon** (Baylor University) wrote the chapter overviews that begin each chapter. Ray has long been interested in the articulation between high school and college mathematics and has served the AP Calculus program in a variety of ways: as a Reader of the exams, as a Table Leader, as Exam Leader (both AB and BCO, and, finally, through four years as Chief Reader. He has also served on the College Board's Test Development Committee for AP Calculus. Ray is a frequent consultant for the College Board, presenting at workshops and leading week-long summer institutes. Additionally, Ray served on Mathematical Association of America (MAA) committees concerned with the issue of proper placement of students in precalculus and calculus courses. Ray has won numerous awards for his teaching and service, including university-wide teaching awards from the University of North Carolina and Baylor University. He was named a Piper Professor in the state of Texas in 1997 and has twice been given awards by the Southwestern region of the College Board for outstanding contributions to the Advanced Placement Program.

#### **AP** Question Writers for the Teacher's Resource Binder

**John Jensen** is currently the Faculty Chair in Mathematics at Rio Salado College in Tempe, Arizona. Before arriving at Rio Salado, he taught high school mathematics for 30 years in the Paradise Valley School District in Phoenix, Arizona. For 25 years, he taught Advanced Placement Calculus.

John has been an AP Calculus reader and table leader for 17 years and has conducted over 150 workshops and institutes in the United States, Canada, Europe, and Asia. During the course of his career, he has received the following honors: Presidential Award for Excellence in Teaching Mathematics in 1987; the first Siemens Advanced Placement Award in 1998; the Distinguished Service Award (1998) and the Exemplar Award (2001) by the College Board; and the Tandy Technology (Radio Shack) Award in 1997.

John is also a former fellow of the Woodrow Wilson Mathematics Institute at Princeton University and holds a National Board Adolescence and Young Adulthood Certificate in Mathematics.

**Haika Karr** teaches AP Calculus AB and BC at Liberty Hill High School in Liberty Hill, Texas. Her 12 years of teaching experience include 7 years of teaching Calculus. She is currently the Mathematics Department Chairperson, coaches UIL Mathematics and Number Sense, and sponsors her school's chapter of the National Honor's Society.

**Bret Norvilitis** has taught high school and middle school math for the past 16 years, the past 10 at Orchard Park High School in Orchard Park, NY, where he is currently the department chair. He became a Calculus teacher five years ago. He has attended many weekend AP seminars and two week-long AP conferences in both AB and BC. He has taught the AB curriculum seven times and the BC curriculum twice.

Correlation to The College Board's AP* Topic Outline
(AB and BC)

AB	BC	College Board Topic Outline	Rogawski ET	Rogawski LT
AB	BC	I. Function, Graphs, and Limits		
AB	BC	Analysis of graphs	1.7	1.5
AB	BC	Limits of Functions (including one-sided limits)	Ch. 2	Ch. 2
AB	BC	• Intuitive understanding of the limiting process	2.1-2.2	2.1-2.2
AB	BC	Calculating limits using algebra	2.3, 2.5-2.6	2.3, 2.5-2.6
AB	BC	• Estimating limits from graphs or tables of data	2.2, 2.6	2.2, 2.6
AB	BC	Asymptotic and unbounded behavior	4.6	4.5
		• Understanding asymptotes in terms of graphical		
AB	BC	behavior	4.6	4.5
		• Describing asymptotic behavior in terms of		
AB	BC	limits involving infinity	4.6	4.5
		• Comparing relative magnitudes of functions and		
AB	BC	their rates of change	4.5	7.7
AB	BC	Continuity as a property of functions	2.4	2.4
AB	BC	• Intuitive understanding of continuity	2.4	2.4
AB	BC	• Understanding continuity in terms of limits	2.4	2.4
		• Geometric understanding of graphs of		
AB	BC	continuous functions	2.8	2.8
	BC	Parametric, polar and vector functions	11.5	12.5
AB	BC	II. Derivatives	Chs 3-4	Chs 3-4
AB	BC	Concept of the derivative	3.1	3.1
		• Derivative presented graphically, numerically,		
AB	BC	and analytically	3.1, 3.3	3.1, 3.3
4.5	DC	• Derivative interpreted as an instantaneous rate	2.4	2.4
AB	BC	of change	3.4	3.4
	DC	• Derivative defined as the limit of the difference	2.1	2.1
AB	BC		3.1	3.1
٨D	DC	• Relationship between differentiability and	2.0	2.0
AB	BC	Continuity Desirative at a point	3.2	3.2
	DC PC	• Slope of a surve at a point	3.1	3.1
AD	DC	• Slope of a curve at a point	5.1	5.1
٨D	PC	• Tangent line to a curve at a point and local linear approximation	4.1	4.1
AD	DC	• Instantaneous rate of shange as the limit of	4.1	4.1
ΔB	BC	• Instantaneous fate of change as the finit of	3.1	3.1
AD	DC	• Approximate rate of change from graphs and	5.1	5.1
AB	BC	tables of values	3.1	3.1
AB	BC	Derivative as a function	3.2	3.2
		• Corresponding characteristics of graphs of f and	5.2	
AB	BC	f	3.2. 4.2. 4.4. 4.6	3.2. 4.2. 4.4-4 5
		• Relationship between the increasing and	,,, 1.0	,, 1.5
AB	BC	decreasing behavior of f and the sign of f	4.2, 4.4, 4.6	4.2, 4.4-4.5

		• The Mean Value Theorem and its geometric		
AB	BC	interpretation	3.9, 4.2-4.4, 4.6	4.2-4.5, 7.1, 7.3
AB	BC	• Equations involving derivatives	4.4, 4.6	4.4-4.5
AB	BC	Second derivatives	4.4, 4.6	4.4-4.5
		• Corresponding characteristics of the graphs of f,		
AB	BC	f',and f"	4.4, 4.6	4.4-4.5
AB	BC	• Relationship between the concavity of f and the sign of f"	4.4, 4.6	4.4-4.5
٨D	PC	• Points of inflection as places where concavity	1116	4 4 4 5
	BC	Applications of derivatives	4.4, 4.0	4.4-4.3
	BC	• Analysis of curves	131116	1315
AD	DC	Analysis of curves	4.3-4.4, 4.0	4.5-4.5
	BC	• Analysis of planar curves given in parametric form, polar form, and vector form	4.6, 11.1, 11.3, 11.5	4.5, 12.1, 12.3, 12.5
		• Optimization, both absolute (global) and relative		
AB	BC	(local) extrema	4.7	4.6
		• Modeling rates of change, including related		
AB	BC	rates	3.11	3.9
		• Use of implicit differentiation to find the		
AB	BC	derivative of an inverse function	3.8, 3.10	3.8, 7.2
		• Interpretation of the derivative as a rate of		
AB	BC	change in varied applied contexts	3.4	3.4
		Geometric interpretation of differential		
		equations via slope fields and the relationship		
		between slope fields and solution curves for		
AB	BC	differential equations	9.3	10.2
		• Numerical solution of differential equations		
	BC	using Euler's method	9.2	7.6
	BC	• L'Hospital's Rule	4.5, 7.6, 10.3	7.7, 8.6, 11.3
AB	BC	Computation of derivatives	3.3, 3.5	3.3, 3.5
AB	BC	• Knowledge of derivatives of basic functions	3.2-3.3, 3.5-3.6	3.2-3.3, 3.5-3.6
		• Derivative rules for sums, products, and		
AB	BC	quotients of functions	3.2-3.3, 3.6	3.2-3.3, 3.6
AB	BC	Chain rule and implicit differentiation	3.7, 3.10	3.7-3.8
		• Derivatives of parametric, polar, and vector		
	BC	functions	11.1, 11.3, 11.7	12.1, 12.3, 12.7
AB	BC	III. Integrals		
		Interpretations and properties of definite		
AB	BC	integrals	5.1-5.2	5.1-5.2
AB	BC	• Definite integral as a limit of Riemann sums	5.2	5.2
		• Definite integral of the rate of change of a		
		quantity over an interval interpreted as the		
AB	BC	change of the quantity over the interval	5.5	5.5
AB	BC	Basic properties of definite integrals	5.2	5.2
	-		6.1-6.3, 8.1.	6.1-6.3, 9.1.
AB	BC	Applications of integrals	11.2, 11.4	12.2, 12.4
AB	BC	Fundamental Theorem of Calculus	5.7	7.1, 7.3, 7.8

		• Use of the Fundamental Theorem to evaluate		
AB	BC	definite integrals	5.3-5.4	5.3-5.4
		• Use of the Fundamental Theorem to represent a		
AB	BC	particular antiderivative	5.3-5.4	5.3-5.4
AB	BC	Techniques of antidifferentiation	3.5, 5.7	3.5, 7.1, 7.3, 7.8
		<ul> <li>Antiderivatives following directly from</li> </ul>		
AB	BC	derivatives of basic functions	4.9, 5.3-5.4, 5.6	4.8, 5.3-5.4, 5.6
AB	BC	• Antiderivatives by substitution of variables	5.6	5.6
		• Antiderivatives by substitution of variables,		
	BC	parts, and simple partial fractions	5.6, 7.1, 7.5	5.6, 8.1, 8.5
	BC	• Improper integrals as limits of definite integrals	7.6	8.6
AB	BC	Applications of antidifferentiation	4.8	4.7
		• Finding specific antiderivatives using initial		
AB	BC	conditions	4.9, 9.1, 9.3	4.8, 10.1-10.2
		• Solving separable differential equations and		
AB	BC	using them in modeling	5.8, 9.1, 9.3	7.4, 10.1-10.2
		• Solving logistic differential equations and using		
	BC	them in modeling	9.4	10.3
AB	BC	Numerical approximations to definite integrals	7.8	8.8
		• Use of Riemann sums and trapezoidal sums to		
AB	BC	approximate definite integrals of functions	8.1	9.1
	BC	IV. Polynomial approximations to		
	BC	Concent of series	10.1-10.2	11 1-11 2
	BC	Series of constants	10.1-10.2	11.1-11.2
	DC	Motivating examples including decimal	10.2-10.5	11.2-11.3
	BC	expansion	10 2-10 3	11 2-11 3
	BC	Geometric series with applications	10.2-10.3	11.2-11.3
	BC	The harmonic series	10.2-10.3	11.2-11.3
	BC	• Alternating series with error bound	10.4	11.4
	DC	<ul> <li>Terms of series as areas of rectangles and their</li> </ul>	10.1	11.1
	BC	relationship to improper integrals	10.3	11.3
	BC	• The ratio test for convergence and divergence	10.5	11.5
		Comparing series to test for convergence or	10.0	11.0
	BC	divergence	103 105	113 115
	BC	Taylor series	10.6-10.7	11.6-11.7
	20	• Taylor polynomial approximation with	1010 1017	1110 1117
	BC	graphical demonstration of convergence	8.4, 10.6-10.7	9.4, 11.6-11.7
		Maclaurin series and the general Taylor series		
		0 5		0 4 11 6 11 7
	BC	centered at $x = a$	8.4, 10.6-10.7	9.4, 11.6-11./
	BC	<ul> <li>centered at x = a</li> <li>Maclaurin series for the functions ex, sin x, cos</li> </ul>	8.4, 10.6-10.7	9.4, 11.0-11.7
	BC BC	<ul> <li>centered at x = a</li> <li>Maclaurin series for the functions ex, sin x, cos x and 1/1-x</li> </ul>	8.4, 10.6-10.7 8.4, 10.6-10.7	9.4, 11.6-11.7
	BC BC	<ul> <li>centered at x = a</li> <li>Maclaurin series for the functions ex, sin x, cos x and 1/1-x</li> <li>Formal manipulation of Taylor series and</li> </ul>	8.4, 10.6-10.7 8.4, 10.6-10.7	9.4, 11.6-11.7
	BC BC BC	<ul> <li>centered at x = a</li> <li>Maclaurin series for the functions ex, sin x, cos x and 1/1-x</li> <li>Formal manipulation of Taylor series and shortcuts to computing Taylor series</li> </ul>	8.4, 10.6-10.7 8.4, 10.6-10.7 10.6-10.7	9.4, 11.6-11.7 9.4, 11.6-11.7 11.6-11.7

	• Radius and interval of convergence of power		
BC	series	10.6-10.7	11.6-11.7
BC	• Lagrange error bound for Taylor polynomials	8.4	9.4

## **Class Pacing Guide Recommended time allocation per section.**

#### **Chapter 1: Precalculus Review**

The material in this chapter is a quick review of precalculus material. This chapter can be greatly shortened or omitted in the AP Calculus course. These time suggestions are for those who need to cover the material for the first time in the calculus course.

Section	AB time in 40-minute periods	BC time in 40-minute periods	AP course description
1.1 Real Numbers, Functions, and Graphs	0-1	0-1	See prerequisites
1.2 Linear and Quadratic Functions	0-1	0-1	See prerequisites
1.3 The Basic Classes of Functions	0-1	0-1	See prerequisites
1.4 Trigonometric Functions	0-2	0-1	See prerequisites
1.5 Inverse Functions	0-2	0-1	See prerequisites
1.6 Exponential and Logarithmic Functions	0-2	0-1	See prerequisites
1.7 Technology: Calculators and Computers	0-2	0-1	1-1
Catch-up, review, and testing	0-2	0-1	
Total	0-13	0-8	

## Chapter 2: Limits

The material in this chapter is often taught in precalculus courses. This saves time in the calculus course. If this is your situation, then the material in this chapter can be greatly shortened or omitted in the AP Calculus course. These time suggestions are for those who need to cover the material for the first time in the calculus course.

<b>a</b>			4 D. C
Section	AB time in 40-minute	BC time in 40-minute	AP Course
	periods	periods	Description
2.1 Limits, Rates of Change, Tangent Lines	1	1	I-2-a
2.2 Limits: A Numerical and Graphical Approach	2	2	1-2-a, c
2.3 Basic Limit Laws	2	1	I-2-b
2.4 Limits and Continuity	2	1-2	I-4-a,b
2.5 Evaluating Limits Algebraically	2	1	I-2-b
2.6 Trigonometric Limits	1	1	I-2-b, c
2.7 Limits at Infinity	1	1	I-2-a, b
2.8 Intermediate Value Theorem	1	1-2	I-2-b, c
Catch-up, review, and testing	3	2.5	
Total	15	10-13	

The bisection method is not tested on either the AB or BC exams.

Chapter 3: Differentiation					
Section	AB time in 40-minute	BC time in 40-minute	AP Course		
	periods	periods	Description		
3.1 Definition of	2	1-2	II-l-a,c, II-2a,c,d		
the Derivative					
3.2 The Derivative	2	1-2	II-l-a,d;n-2a,b,d;U-3-		
as a Function			aII-6 a, b		
3.3 The Product	2	1	II-1-a; n-6-a,b		
and Quotient					
3.4 Rates of Change	1	1	n-1-b;II-5-f		
3.5 Higher Derivatives	1	1	II-6-a; ni-4		
3.6 Trigonometric Functions	1	1	II-6-a, b		
3.7 The Chain Rule	1	1	II-6-c		
3.8 Derivatives of Inverse Functions	2	1-2	II-6		
3.9 Derivatives of Exponential and Logarithmic Functions	1	1	II-6		
3.10 Implicit Differentiation	2	1-2	Н-5-е; ІІ-6-с		
3.11 Related Rates	2	2	II-5-d		
Catch-up, review, and testing	3	3			
Total	20	16-20			

Chapter 4: Applications of the Derivative					
Section	AB time in 40-minute	BC time in 40-minute	AP Course		
	periods	periods	Description		
4.1 Linear	2	1	Н-2-b		
Approximations and					
Applications					
4.2 Extreme Values	2	2	II-3-a,b,c;		
			II-4-a,b,c		
4.3 The Mean Value	2	2	Н-3-с; II-5-а		
Theorem and			,		
Monotonicity					
4.4 The Shape of a	2	2	II-3-a,b,c;		
Graph			II-4-a,b,c;		
			II-5-a		
4.5 L'Hôpital's	0	1	II-5-i		
Rule					
4.6 Graph	3	2-3	I-3-a,b;		
Sketching and			II-3-a,b,c;		
Asymptotes*			II-4-a,b,c; II-5-a,c		
4.7 Applied	2	2	II-5-c		
Optimization					
4.8 Newton's Method	Omit	Omit	Not testedon either the		
			AB or BC exams		
4.9 Antiderivatives**	3	2	III-5-a; III-4-a		
Catch-up, review, and	3	3			
testing					
Total	19	15-18			

\* 4.5 Asymptotes are often covered thoroughly in precalculus courses. If your students have studied this already, then this material may be quickly reviewed. Otherwise a third full period will be needed to teach about asymptotes.

\*\* 4.8 Antiderivatives and initial value problems may be considered here, or with the material in Chapter 5. If placed after 5.4 (FTC), student will have a reason to need to know about antiderivatives.

Chapter 5: The Integral					
Section	AB time in 40-minute	BC time in 40-minute	AP Course		
	periods	periods	Description		
5.1 Limits: Approximating and Computing Area	2	2	III-i-a		
5.2 The Definite Integral	2	1	III-l-a,b,d		
5.3 Fundamental Theorem of Calculus, Part I	2	1	III-3-a, b; ni-4-a		
5.4 Fundamental Theorem of Calculus, Part II	2	2	III-3-a, b; III-4-a		
5.5 Net or Total Change as the Integral of a Rate	2	1	III-l-c		
5.6 Substitution Method	2	2	III-4-a, b		
5.7 Further Transcendental Functions	1	1	III-3, III-4		
5.8 Exponential Growth and Decay	2	2	III-5-b		
Catch-up, review, and testing	3	3			
Total	19	15			

Chapter 6: Applications of the Integral			
Section	AB time in 40-minute	BC time in 40-minute	AP Course
	periods	periods	Description
6.1 Area Between	2	1	III-2
Two Curves			
6.2 Setting Up	3	2	III-2
Integrals:			
Volume,			
Density,			
Average Value			
6.3 Volumes of	3	2-3	ra-2
Revolution			
6.4 The Method of	2: Optional	2: Optional	Not tested on either
Cylindrical Shells			the AB or BC exams
6.5 Work and Energy	0	0	Not tested on either
			the AB or BC exams
Catch-up, review, and	3	3	
testing			
Total	12	8-9	

Chapter 7: Techniques of Integration			
Section	AB time in 40-	BC time in 40-	AP Course
	minute periods	minute periods	Description
7.1 Integration by	0	1	III-4-b
Parts			BC only
7.2	0	0	Not tested on either
Trigonometric			the AB or BC
7 3 Trigonomotrio	0	0	Not tosted on either
Substitution	0	0	the AB or BC
			exams
7.4 Integrals of	0	0	Not tested on either
Hyperbolic			the AB or BC
Functions and			exams
Hyperbolic			
Functions			
7.5 The Method of	0	1-2	III-4-b
Partial Fractions			BC only
7.6 Improper Integrals	0	1-2	II-5-I;
			III-4-c
			BC only
7.8 Numerical	2	1	III-6
Integration			
Catch-up, review,	1	3	
and testing			
Total	3	7-9	

Chapter 8: Further Applications of Integration and Taylor Polynomials			
Section	AB time in 40 minute periods	BC time in 40 minute Periods	AP Course Description
8.1 Arc Length and Surface Area	0	1	III-2 BC only
8.2 Fluid Pressure and Force	0	0	Not tested on either AB or BC exams
8.3 Center of Mass	0	0	Not tested on either AB or BC exams
8.4 Taylor Polynomials	0	2-3	IV-a,b,c,g BC Only
Catch up, Review and testing	0	2	
Total	0	5-6	

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Chapter 9: Introduction to Differential Equations			
Section	AB time in 40-	BC time in 40-	AP Course Description
9.1 Solving Differential Equations	3	2	III-5-a,b
9.2 Models involving y' = k(y-b)	0	0	Not tested on either the AB or BC exams
9.3 Graphical and Numerical Methods	3	2	II-5-g III-5-a,b
9.4 The Logistic Equation	0	1-2	BC only III-5-c
9.5 First-Order Linear Equations	0	0	Not tested on either the AB or BC exams
Catch-up, review, and testing	2	2	
Total	8	7-8	

## Chapter 10: Infinite Series BC only (Not tested on the AB Exam)

Section	AB time in 40-	BC time in 40-	AP Course
	minute periods	minute periods	Description
10.1 Sequences	0	2	BC only
			IV-i
10.2 Summing	0	2	PC only
an Infinite Series	0	2	IV_i
an minine Series			IV-2-a,b,c
10.3 Convergence of	0	2	BC only;
a Series with Positive			II-5-i
Terms			IV-2-a,b,c,e,g
10.4 Absolute and	0	1	BC only
Conditional			IV-d
Convergence			
10.5 The Ratio	0	1	BC only
and Root Tests			IV-2-f,g
10.6 Power Series	0	2	BC only IV-3-
			a,b,c,d,e,f
10.7 Taylor Series	0	2	BC only IV-3
			a,b,c,d,e,f
Catch-up, review,	0	3-4	
and testing			
Total	0	15-16	

Chapter 11: BC only (Not tested on the AB Exam)			
Section	AB time in 40-	BC time in 40-	AP Course
	minute periods	minute periods	Description
11.1 Parametric	0	1*	BC only
Equations			II-5-b
			II-6-d
11.2 Arc Length	0	1	BC only
and Speed			III-2
11.0.D.1	0	1. Oskak	
11.3 Polar	0	1-2**	BC only
Coordinates			11-5-b 11-
			<u>6-d</u>
11.4 Area and	0	1	BC only
Arc Length in			111-2
Polar Coordinates			
11.5 Vectors In the	0	2	Not tested on either
Plane			the AB or BC
			exams
11.6 Dot Product and	0	0	Not tested on either
the Angle Between			the AB or BC
Two Vectors			exams
11-7 Calculus of	0	2	
Vector-Valued			
Functions			
Catch-up, review,	0	2	
and testing			
Total	0	10-11	

Chapter	AB time in 40-minute periods	BC time in 40-minute periods
1	Optional 0-13	Optional 0-8
2	14	10-11
3	20	16-20
4	19	15 - 18
5	19	15
6	11 (13)	8 (11)
7	3	7-9
8	0	15-16
9	8	7-8
10	0	15-16
11	0	10-11
APExam Review	20	15
Total	130	118 - 142

#### 1.1. Real Numbers, Functions, and Graphs.

#### **Class Time** AB and BC, 0–1 period. Essential.

The material in this section is a review of precalculus material. This section can be greatly shortened or omitted in the AP Calculus course. The items listed below are important for calculus and your students should understand them.

#### **Key Points**

- The absolute value is defined by  $|a| = \begin{cases} -a & \text{if } a < 0 \\ a & \text{if } a \ge 0 \end{cases}$ .
- Triangle inequality:  $|a + b| \leq |a| + |b|$  with equality if and only if a and b have the same sign.
- There are four types of intervals with endpoints a and b:

• Open and closed intervals can be expressed with inequalities:

$$(a,b) = \{x : a < x < b\}, \qquad [a,b] = \{x : a \le x \le b\}$$

or

$$(a,b) = \{x : |x-c| < r\}, \qquad [a,b] = \{x : |x-c| \le r\}$$

where c = (a + b)/2 is the midpoint and r = (a - b)/2 is the radius.

- The distance d between  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $d = \sqrt[r]{(x_2 x_1)^2 + (y_2 y_1)^2}$ . An equation of the circle of radius r with center (a, b) is  $(x a)^2 + (y b)^2 = r^2$ .
- A zero or root of a function f(x) is a value c such that f(c) = 0.
- The definitions of increasing and decreasing on an interval distinguish between increasing / decreasing and *strictly* increasing / decreasing.
- Vertical Line Test: A curve in the plane is the graph of a function if and only if each vertical line x = a intersects the curve in at most one point.
- Even function: A function f is even if f(-x) = f(x), in which case the graph is symmetric about the y-axis.
- Odd function: A function f is odd if f(-x) = -f(x), in which case the graph is symmetric about the origin.
- There are four common ways to transform the graph of f(x) to obtain the graph of a related function:
  - (1) f(x) + c; shifts the graph of f(x) vertically c units.
  - (2) f(x-c); shifts the graph of f(x) horizontally c units to the right, c > 0.
  - (3) kf(x); scales the graph of f(x) vertically by a factor of k.

(4) f(kx); scales the graph of f(x) horizontally by a factor |k| (this is a compression if |k| > 1), if k < 0 the function is reflected in the y-axis.

#### Lecture Material

As this material should all be review, it should be covered as quickly as possible. Start with the usual terminology used to describe real numbers: real  $\mathbb{R}$ , rational  $\mathbb{Q}$ , whole, and irrational numbers. Define the absolute value function, and emphasize that the definition will be used in this course. State the basic properties of absolute value |a| = |-a| and  $|ab| = |a| \cdot |b|$ , as well as the triangle inequality  $|a+b| \leq |a|+|b|$ . Define the three types of intervals (open, closed, and half-open), and write down the equivalent inequalities. Work Exercise 16. Now state the basic terminology of the Cartesian plane (x, y-coordinates, origin, axis) and state the distance formula. Work Exercise 36(a). Derive the standard form of a circle using the distance formula. Work Exercise 37(b). Define a function, as well as stating the usual terminology associated with functions (value, domain, range, independent variable, dependent variable, graph, zero or root, odd, even, increasing, decreasing). Work Exercises 47, 52, and 54. State the Vertical Line Test. Discuss translation (shifting) and scaling of graphs. Work Exercise 72.

Be sure that students understand the concepts of increasing and decreasing on an interval. They should know this graphically, numerically (how increasing / decreasing looks in a table), analytically, and verbally (the definitions).

#### **Discussion Topics/Class activities**

Work Exercise 79 for the class, and then show the students that a polynomial function is even if and only if every exponent is even and is odd if and only if every exponent is odd. Constants are considered to have exponent 0. Then have the class determine whenever the product and quotient of even functions is even and whenever the product and quotient of odd functions is odd.

#### Suggested Problems

Exercises 3, 5, 11, 13, 19, 23, 28 (numerical), 41–55 odd (numerical and graphical), 59, 63 (graphical), 74, 76, 78 (graphical)

## Worksheet 1.1. Real Numbers, Functions, and Graphs

- 1. Express the set of numbers x satisfying the condition |x + 7| < 2 as an interval.
- **2.** Plot the points (1, 4) and (3, 2) and calculate the distance between them.



**3.** Determine the equation of the circle with center (2, 4) passing through (1, -1).

4. Find the domain and range of  $f(x) = \frac{1}{x^2}$ .

- 5. Find the interval on which the function  $f(x) = \frac{1}{x^2 + 1}$  is increasing.
- 6. Find the zeros of the function  $f(x) = 2x^2 4$  and sketch its graph by plotting points. Use symmetry and increasing/decreasing information if appropriate.



- 7. Let  $f(x) = x^2$ . Sketch the graphs of the following functions over [-2, 2].
  - a. f(x+1)
  - b. f(x) + 1
  - c. f(5x)
  - d. 5f(x)



### Solutions to Worksheet 1.1

- 1. Express the set of numbers x satisfying the condition |x + 7| < 2 as an interval. The expression |x + 7| < 2 is equivalent to -2 < x + 7 < 2. Therefore, -9 < x < -5 which represents the interval (-9, -5).
- **2.** Plot the points (1, 4) and (3, 2) and calculate the distance between them.



- **3.** Determine the equation of the circle with center (2, 4) passing through (1, -1). The equation of the indicated circle is  $(x - 2)^2 + (y - 4)^2 = 3^2 = 9$ .
- 4. Find the domain and range of  $f(x) = \frac{1}{x^2}$ .  $D: \{x: x \neq 0\}$  $R: \{y: y > 0\}$
- 5. Find the interval on which the function  $f(x) = \frac{1}{x^2 + 1}$  is increasing. A graph of the function  $y = \frac{1}{x^2 + 1}$  follows.



From the graph, we see that the function is increasing on the interval  $(-\infty, 0]$ .

6. Find the zeros of the function  $f(x) = 2x^2 - 4$  and sketch its graph by plotting points. Use symmetry and increasing/decreasing information if appropriate.



- 7. Let  $f(x) = x^2$ . Sketch the graphs of the following functions over [-2, 2].
  - a. f(x+1)
  - b. f(x) + 1
  - c. f(5x)
  - d. 5f(x)



#### 1.2. Linear and Quadratic Functions.

#### Class Time AB and BC, 0–1 period. Essential.

The material in this section is a review of precalculus material. The section can be greatly shortened or omitted in the AP course. The items listed below are important for calculus and your students should understand them.

#### Key Points

- A function of the form f(x) = mx + b is a linear function.
- The general equation of a line is ax + by = c. The line y = c is horizontal and the line x = k is vertical.
- There are three convenient forms for writing the equation of a nonvertical line:
  - (1) Point-slope form: y b = m(x a), where m is the slope and the line passes through the point (a, b).
  - (2) Point-point form: The line through  $\mathbf{P} = (a_1, b_1)$  and  $\mathbf{Q} = (a_2, b_2)$  has slope  $m = \frac{b_2 - b_1}{a_2 - a_1} \text{ and equation } y - b_1 = m(x - a_1).$ (3) Slope-intercept form: y = mx + b, where m is the slope and b is the y-
  - intercept.
- Two lines with slopes  $m_1$  and  $m_2$ , respectively, are parallel if and only if  $m_1 = m_2$ and they are perpendicular if and only if  $m_1 = -1/m_2$  (provided that  $m_2 \neq 0$ ).
- The roots of a quadratic polynomial  $f(x) = ax^2 + bx + c$  are given by the quadratic formula  $x = \frac{-b \pm \sqrt{D}}{2a}$ , where  $D = b^2 - 4ac$  is the discriminant. The roots are real if  $D \ge 0$  and complex with nonzero imaginary part if D < 0.
- Completing the square consists of writing a quadratic function as a constant multiple of a square plus a constant.

#### Lecture Material

Begin by reminding the students that a linear function has the form f(x) = mx + b. where m is the slope of the line, and b is the y-intercept. Setting y = f(x), we have y = mx + b, the slope-intercept form of the line. Setting  $\Delta x = x_2 - x_1$  and  $\Delta y = x_2 - x_1$  $y_2 - y_1 = f(x_2) - f(x_1)$ , we have

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}}$$

(illustrated graphically in Figure 1). Point out that slope measures steepness; a negative slope indicates that the line points down from left to right (and so is strictly increasing if m > 0 and strictly decreasing if m < 0; a horizontal line has slope 0 (and equation y = b; and a vertical line has "infinite" (or undefined) slope (and equation x = c for

some constant c). Discuss parallel and perpendicular lines: Two lines of slope  $m_1$  and  $m_2$  are parallel if and only if  $m_1 = m_2$  and perpendicular if and only if  $m_1 = -1/m_2$ , or  $m_1m_2 = -1$ . Now turn to the standard forms of equations of lines. First, the general linear equation is ax + by = c, where a and b are not both 0. Other useful forms are the point-slope form y - b = m(x - a), where (a, b) is a point on the line, and the point-point form

$$y - b_1 = m(x - a_1)$$
, where  $m = \frac{b_2 - b_1}{a_2 - a_1}$ 

Now work Exercises 2, 10, 14, and 18.

A quadratic function is a function of the form  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ . The graph of f(x) is a parabola, and the parabola opens upward if a > 0 and downward if a < 0. The discriminant of f(x) is  $D = b^2 - 4ac$ , and the roots of f(x) are given by the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

The sign of D determines the number of real roots of f(x). If D > 0, then f(x) has two real roots, if D = 0, then f(x) has one real root, and if D < 0, then f(x) has no real roots. Now show how to complete the square, and write f(x) in the form  $f(x) = a(x-h)^2 + k$ . The point (h, k) is the vertex of the parabola, and k is either the maximum or minimum value of f(x), depending upon whether the parabola opens upward or downward. Point out that the quadratic formula is obtained by completing the square with the equation  $ax^2 + bx + c = 0$ . Now work Exercises 38 and 42.

#### **Discussion Topics/Class Activities**

Have students work Exercises 53 and 58.

#### Suggested Problems

Exercises 1–19 odd (numerical), 33–45 odd (numerical)

## Worksheet 1.2. Linear and Quadratic Functions

1. Find the slope, y-intercept, and x-intercept of the line with equation y = 4 - x.

In Exercises 2, 3, and 4, find the equation of the line with the given description. 2. Slope -2, y-intercept 3.

**3.** Passes through (-1, 4) and (2, 7).

**4.** Vertical, passes through (-4, 9).

5. Complete the square and find the minimum or maximum value of the quadratic function  $y = 2x^2 - 4x - 7$ .

6. Sketch the graph of  $y = x^2 - 6x + 8$  by plotting the roots and the minimum point.



## Solutions to Worksheet 1.2

1. Find the slope, y-intercept, and x-intercept of the line with equation y = 4 - x.

Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x, and the y-intercept is the constant term; that is, m = -1 and the y-intercept is 4. To determine the x-intercept, substitute y = 0 and then solve for x: 0 = 4 - x or x = 4.

In Exercises 2, 3, and 4, find the equation of the line with the given description.

- 2. Slope -2, y-intercept 3. The equation is y = -2x + 3.
- **3.** Passes through (-1, 4) and (2, 7).

The slope of the line that passes through (-1, 4) and (2, 7) is

$$m = \frac{7-4}{2-(-1)} = 1$$

Using the point-slope form for the equation of a line, y - 7 = 1(x - 2) or y = x + 5.

**4.** Vertical, passes through (-4, 9).

A vertical line has the equation x = c for some constant c. Because the line needs to pass through the point (-4, 9), we must have c = -4. The equation of the desired line is then x = -4.

5. Complete the square and find the minimum or maximum value of the quadratic function  $y = 2x^2 - 4x - 7$ .

$$y = 2(x^{2} - 2x + 1 - 1) - 7 = 2(x^{2} - 2x + 1) - 7 - 2 = 2(x - 1)^{2} - 9$$

Therefore, the minimum value of the quadratic polynomial is -9, which occurs at x = 1.
6. Sketch the graph of  $y = x^2 - 6x + 8$  by plotting the roots and the minimum point.



#### 1.3. The Basic Classes of Functions.

#### Class Time AB and BC, 0–1 period. Essential.

The material in this section is a review of precalculus material. The section can be greatly shortened or omitted in the AP course. The items listed below are important for calculus and your students should understand them.

#### **Key Points**

• The function  $x^m$  is called the power function with exponent m. A polynomial P(x) is a sum of multiples of power functions  $x^m$ , where m is a whole number:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

P(x) has degree n (provided  $a_n \neq 0$ ) and  $a_n$  is the leading coefficient.

- A rational function is a quotient P(x)/Q(x) of two polynomials.
- An algebraic function is produced by taking sums, products, and *n*th roots of polynomials and rational functions.
- An exponential function has the form  $f(x) = b^x$ , where b > 0 is the base.
- The composite function  $f \circ g$  is defined by  $(f \circ g)(x) = f(g(x))$ . The domain of  $f \circ g$  is the set of x such that g(x) belongs to the domain of f.

#### Lecture Material

Begin by defining a polynomial function  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$ , and note that the numbers  $a_0, \ldots, a_n$  are coefficients, the degree of P(x) is n (provided that  $a_n \neq 0$ ),  $a_n$  is the leading coefficient, and the domain of any polynomial is  $\mathbb{R}$ . A rational function is a quotient of polynomials: f(x) = P(x)/Q(x), where P(x) and Q(x) are polynomials. Note that the domain of a rational function is all real numbers except where Q(x) = 0. The next class of functions is the algebraic functions, produced by taking sums, multiples, and quotients of roots of polynomials and rational functions. The domains of algebraic functions are more subtle and best handled by example. Basically, though, one needs to exclude any numbers that will give a negative number under an even radical or which will produce division by 0. Work Exercises 8 and 12. Exponential functions have the form  $f(x) = b^x$ , where b > 0, have domain  $\mathbb{R}$  and range  $(0, \infty)$ , are increasing if b > 1, and are decreasing if 0 < b < 1. Their inverses are the logarithmic functions  $\log_b x$ . Finally, trigonometric functions are built from  $\sin x$  and  $\cos x$  and will be studied in Section 1.4.

There are several methods for constructing new functions from old. The most familiar are addition, subtraction, multiplication, and division of functions. Perhaps the most important way to combine functions is composition, defined by  $f \circ g(x) = f(g(x))$ , for values of x such that g(x) lies in the domain of f. Work Exercises 28 and 32.

# **Discussion Topics/Class Activities** Have students work Exercises 40 and 41.

## Suggested Problems

Exercises 1–11 odd (numerical), 13–25 odd (descriptive), 27–33 odd (numerical)

# Worksheet 1.3. The Basic Classes of Functions

In Exercises 1 and 2, determine the domain of the function.

1. 
$$f(x) = \frac{\sqrt{x}}{x^2 - 9}$$

**2.** 
$$f(x) = \frac{x + x^{-1}}{(x - 3)(x + 4)}$$

In Exercises 3 and 4, calculate the composite functions  $f \circ g$  and  $g \circ f$  and determine their domains.

**3.** 
$$f(x) = \frac{1}{x}, g(x) = x^{-4}$$

4. 
$$f(x) = \frac{1}{x^2 + 1}, g(x) = x^{-2}$$

# Solutions to Worksheet 1.3

In Exercises 1 and 2, determine the domain of the function.

1. 
$$f(x) = \frac{\sqrt{x}}{x^2 - 9}$$
  
 $x \ge 0, x \ne \pm 3$   
2.  $f(x) = \frac{x + x^{-1}}{(x - 3)(x + 4)}$   
 $x \ne 0, 3, -4$ 

In Exercises 3 and 4, calculate the composite functions  $f \circ g$  and  $g \circ f$  and determine their domains.

3. 
$$f(x) = \frac{1}{x}, g(x) = x^{-4}.$$
  
 $f(g(x)) = x^4; D: \mathbb{R} \setminus \{0\}$   
 $g(f(x)) = x^4: D: \mathbb{R} \setminus \{0\}$   
4.  $f(x) = \frac{1}{x^2 + 1}, g(x) = x^{-2}.$   
 $f(g(x)) = \frac{1}{(x^{-2})^2 + 1} = \frac{1}{x^{-4} + 1}; D: x \neq 0$ 

$$g(f(x)) = \left(\frac{1}{x^2 + 1}\right)^{-2} = (x^2 + 1)^2; D: \mathbb{R}$$

#### 1.4. Trigonometric Functions.

### Class Time AB 0-2 periods; BC 0-1 period. Very important.

The material in this section is a review of precalculus material. The section can be greatly shortened or omitted in the AP course. The items listed below are important for calculus and your students should understand them.

#### **Key Points**

- An angle of  $\theta$  radians subtends an arc of length  $\theta r$  on a circle of radius r.
- To convert from radians to degrees, multiply by  $180/\pi$ .
- To convert from degrees to radians, multiply by  $\pi/180$ .
- Unless otherwise stated, all angles in this text are in radians.
- The functions  $\cos \theta$  and  $\sin \theta$  are defined in terms of right triangles for acute angles and as coordinates of a point on the unit circle for general angles:



$$\sin \theta = \frac{b}{c} = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}$$



- Some basic properties of sine and cosine:
  - (1) Periodicity:  $\sin(\theta + 2\pi k) = \sin \theta$ ,  $\cos(\theta + 2\pi k) = \cos \theta$ , k = any integer.
  - (2) Parity:  $\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$
- (3) Basic identity:  $\sin^2 \theta + \cos^2 \theta = 1$
- The four additional trigonometric functions:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}$$

- The graphs of all 6 trigonometric functions.
- The values of the trigonometric functions at 0,  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ ,  $\frac{\pi}{3}$ , and  $\frac{\pi}{2}$ .

#### Lecture Material

Begin with the two common methods of measuring angles, degrees and radians, and show that to convert from degrees to radians one multiplies by  $\frac{\pi}{180}$ , while to convert from radians to degrees one multiplies by  $\frac{180}{\pi}$ . Give an example of each. Give the usual right triangle definitions of  $\sin \theta$  and  $\cos \theta$  (so that  $0 \le \theta \le \frac{\pi}{2}$ ), and then show how these definitions can be extended to all angles using the unit circle. Use the unit circle to show that  $\sin \theta$  is odd while  $\cos \theta$  is even. Discuss calculating  $\sin \theta$  and  $\cos \theta$  for the special angles 0,  $\pi/6$ ,  $\pi/4$ ,  $\pi/3$ ,  $\pi/2$  using appropriate special triangles or the unit circle (this information is tabulated in Table 2). Point out using the unit circle that both  $\sin \theta$  and  $\cos \theta$  are periodic of period  $2\pi$ . Also discuss the sign of  $\sin \theta$  and  $\cos \theta$  in each of the four quadrants. Now show the graph of  $\sin \theta$  and  $\cos \theta$  using Figure 6 for the graph of  $\sin \theta$ . (It may be useful to point out to the students that much of the information here is stored in the graphs of  $\sin \theta$  and  $\cos \theta$ .) Now define the other four trigonometric functions  $\tan x$ ,  $\cot x$ ,  $\sec x$ , and  $\csc x$ , as well as their graphs (found in Figure 10). Now work Exercises 8, 10, 20, 22, and 32.

Turning to identities, show that  $\sin^2 \theta + \cos^2 \theta = 1$  using the unit circle definitions of  $\sin \theta$  and  $\cos \theta$ , and show the equivalent versions  $\tan^2 \theta + 1 = \sec^2 \theta$  and  $1 + \cot^2 \theta = \csc^2 \theta$ . Point out the Basic Trigonometric Identities listed in the text, as well as the Law of Cosines (Theorem 1), which is a generalization of the Pythagorean Theorem. Work Exercises 24 and 46.

#### **Discussion Topics/Class Activities**

Have the students work Exercise 57 at their desks.

#### Suggested Problems

Exercises 1, 3, 4, 6, 7 (numerical), 9–13 odd, 16 (numerical), 19–25 odd (numerical), 35, 37, 39

# Worksheet 1.4. Trigonometric Functions

1. Find the values of the six standard trigonometric functions at  $\theta = 11\pi/6$ .

**2.** Find all angles between 0 and  $2\pi$  satisfying  $\tan \theta = 1$ .

**3.** Find  $\cos \theta$  and  $\tan \theta$  if  $\sin \theta = \frac{3}{5}$  and  $0 \le \theta < \pi/2$ .

**4.** Find  $\sin \theta$ ,  $\cos \theta$ , and  $\sec \theta$  if  $\cot \theta = 4$  and  $0 \le \theta < \pi/2$ .

- 5. Sketch the graph of  $y = \cos\left(2\left(\theta \frac{\pi}{2}\right)\right)$  over the interval  $[0, 2\pi]$ .
- **6.** Find  $\sin 2\theta$  and  $\cos 2\theta$  if  $\tan \theta = \sqrt{2}$  and  $0 \le \theta < \pi/2$ .

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7. Derive the identity  $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$  using the identities listed in this section.

### Solutions to Worksheet 1.4

1. Find the values of the six standard trigonometric functions at  $\theta = 11\pi/6$ . We see that  $\sin \frac{11\pi}{6} = -\frac{1}{2}$  and  $\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$ 

Then

$$\tan \frac{11\pi}{6} = \frac{\sin \frac{11\pi}{6}}{\cos \frac{11\pi}{6}} = -\frac{\sqrt{3}}{3}$$
$$\cot \frac{11\pi}{6} = \frac{\cos \frac{11\pi}{6}}{\sin \frac{11\pi}{6}} = -\sqrt{3}$$
$$\csc \frac{11\pi}{6} = \frac{1}{\sin \frac{11\pi}{6}} = -2$$
$$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{2\sqrt{3}}{3}$$

- 2. Find all angles between 0 and  $2\pi$  satisfying  $\tan \theta = 1$ .  $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$
- **3.** Find  $\cos \theta$  and  $\tan \theta$  if  $\sin \theta = \frac{3}{5}$  and  $0 \le \theta < \pi/2$ . Using the Pythagorean Theorem we see that

$$\cos \theta = \frac{4}{5}$$
 and  $\tan \theta = \frac{3}{4}$ 

4. Find  $\sin \theta$ ,  $\cos \theta$ , and  $\sec \theta$  if  $\cot \theta = 4$  and  $0 \le \theta < \pi/2$ . Using the Pythagorean Theorem, we see that

$$\sin \theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}, \quad \cos \theta = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17} \text{ and } \sec \theta = \frac{\sqrt{17}}{4}.$$

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5. Sketch the graph of  $y = \cos\left(2\left(\theta - \frac{\pi}{2}\right)\right)$  over the interval  $[0, 2\pi]$ .



6. Find  $\sin 2\theta$  and  $\cos 2\theta$  if  $\tan \theta = \sqrt{2}$  and  $0 \le \theta < \pi/2$ . By the double-angle formulas,  $\sin 2\theta = 2 \sin \theta \cos \theta$  and  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ . Using the Pythagorean Theorem,

$$\sin \theta = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$
 and  $\cos \theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ .

Finally,

$$\sin 2\theta = 2\frac{\sqrt{6}}{3} \cdot \frac{\sqrt{3}}{3} = \frac{2\sqrt{2}}{3}$$
$$\cos 2\theta = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

7. Derive the identity  $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$  using the identities listed in this section.

Substitute  $x = \theta/2$  into the double-angle formula for cosine,  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ , to obtain  $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$ .

#### 1.5. Inverse Functions.

Class Time AB and BC, 0–2 periods. Essential.

### **Key Points**

- Inverse of a function
  - (i) One-to-one functions
  - (ii) Calculating the inverse of a function
  - (iii) Relation between the graphs of f(x) and  $f^{-1}(x)$
- Inverse trigonometric functions

#### Lecture Material

Calculate inverses of functions as in Examples 1 and 2, pointing out the relations between the domains and ranges of f and  $f^{-1}$  and the fact that the graph of  $f^{-1}(x)$  is the reflection of the graph of f across the line y = x.

A function f is invertible if and only if f is one-to-one onto its range. A function f is one-to-one on its domain if and only if  $f(x_1) = f(x_2)$  implies that  $x_1 = x_2$  for all  $x_1, x_2$  in its domain. A function f is onto its range if and only if for every y in the range of f, there exists an x in the domain of f such that f(x) = y. A graphical test for one-to-oneness is that all horizontal lines intersect the graph at most once. Sketch one-to-one and not one-to-one functions. Explain that it is often possible to make a function one-to-one by restricting the domain. Illustrate this idea with  $y = x^2$  on  $[0, \infty)$ .

Show that the graph of  $f^{-1}$  is obtained by reflecting the graph of f(x) across the line y = x. Illustrate this concept with  $f(x) = x^3$  on [-2, 2] and  $f^{-1}(x = x^{1/3})$  on [-8, 8].

Next introduce inverse trigonometric functions by graphing  $y = \sin x$  on  $[-\pi, \pi]$  and reflecting it across y = x to get the inverse sine function denoted  $\sin^{-1}(x)$  on [-1, 1]. Explain that  $\cos x$  is restricted to  $[0, \pi]$  to obtain the inverse and  $\tan x$  is restricted to  $[-\pi, \pi]$  to obtain its inverse. If time permits, discuss the remaining three trigonometric functions.

#### **Discussion Topics/Class Activities**

Discuss Exercise 51 about the inverses of even and odd functions.

#### Suggested Problems

Exercises 2, 3 (basic), 4 (algebraic), 9, 13 (algebraic and graphical), 16 (graphical), 19, 23, 25, 27, 29, 31, 39, 43 (basic inverse trig problems)

# Worksheet 1.5. Inverse Functions

1. Find a domain on which f is one-to-one and a formula for the inverse of f restricted to this domain.

a. 
$$f(x) = \frac{1}{x+1}$$

b. 
$$f(s) = \frac{1}{s^2}$$

2. Evaluate without using a calculator. a.  $\sin^{-1}\frac{1}{2}$ 

b. 
$$\sec^{-1} \frac{2}{\sqrt{3}}$$
  
c.  $\sin^{-1} \left( \sin \frac{4\pi}{3} \right)$ 

d. 
$$\tan\left(\cos^{-1}\frac{2}{3}\right)$$

# Solutions to Worksheet 1.5

- 1. Find a domain on which f is one-to-one and a formula for the inverse of f restricted to this domain.
  - a.  $f(x) = \frac{1}{x+1}$ f is one-to-one on  $(\infty, -1) \cup (-1, \infty)$ .  $f^{-1}(x) = \frac{1-x}{x}$  for all  $x \neq 0$ b.  $f(s) = \frac{1}{s^2}$ . f is one-to-one on  $(0, \infty)$ .  $f^{-1}(x) = \frac{1}{\sqrt{x}}$  for all  $x \in (0, \infty)$ .
- 2. Evaluate without using a calculator.

a. 
$$\sin^{-1}\frac{1}{2}$$
$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$
  
b. 
$$\sec^{-1}\frac{2}{\sqrt{3}}$$
$$\sec^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{6}$$
  
c. 
$$\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$$
$$\sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \frac{4\pi}{3}$$
  
d. 
$$\tan\left(\cos^{-1}\frac{2}{3}\right)$$
$$\tan\left(\cos^{-1}\frac{2}{3}\right) = \frac{\sqrt{5}}{2}$$

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#### 1.6. Exponential and Logarithmic Functions.

Class Time AB and BC, 0–2 periods. Essential.

### **Key Points**

- Exponential function  $y = b^x$  for  $b > 0, b \neq 1$
- $b^x$  is increasing if b > 1 and decreasing if b < 1.
- The number  $e \approx 2.71828$  is the unique number such that the area of the region under the hyperbola  $y = \frac{1}{x}$  for  $1 \le x \le e$  is equal to 1. • For b > 0 with  $b \ne 1$ , the logarithmic function  $\log_b x$  is the inverse of  $b^x$ . That is
- $x = b^y \iff y = \log_b x.$
- The natural logarithm is the logarithm to the base e and is denoted  $\ln x$ .
- Important logarithmic properties:
  - (i)  $\log_b(xy) = \log_b x + \log_b y$

(ii) 
$$\log_b(\frac{x}{y}) = \log_b x - \log_b y$$

(iii) 
$$\log_b(x^n) = n \log_b x$$

- (iv)  $\log_b 1 = 0$  and  $\log_b b = 1$
- Hyperbolic trig functions are not tested on the AP exams and may be omitted.

#### Lecture Material

Define an exponential function as a function of the form  $f(x) = b^x$  for  $b > 0, b \neq 1$ . The number b is called the base. If b > 1, the  $b^x$  is increasing. If 0 < b < 1, then  $b^x$  is decreasing. Graph  $y = 2^x$  and  $y = (\frac{1}{2})^x$ . Exponential functions are very important in applications such as population growth and radioactive decay.

Discuss the Laws of Exponents given in Theorem 1 and work several problems such as Example 1 and Exercises 4, 6, and 26.

There is a unique number  $e \approx 2.71828$  such that the area of the region under the hyperbola  $y = \frac{1}{x}$  for  $1 \le x \le e$  is equal to 1. The function  $y = e^x$  is especially important in applications. We will come back to this later.

The inverse of the exponential function  $b^x$  is called the logarithmic function to the base b and is denoted  $\log_b x$ . When b = e, it is called the natural logarithm and is denoted  $\ln x$ . Graph  $e^x$  and  $\ln x$  together on the same coordinate axes. State the logarithmic properties given in the Key Points and explain that they follow easily from the exponential properties. Work Example 3 and Exercises 12, 16, 18, and 30.

An important formula is the change of base formula  $\log_b x = \frac{\log_a x}{\log_a b}$ . It is most often

used with a = e. Show that  $\log_2 10 = \frac{\ln 10}{\ln 2}$ .

# Discussion Topics/Class Activities

Discuss Exercise 49.

## Suggested Problems

Exercises 1 (basic), 3, 5, 7, 9 (algebraic), 11–25 odd (log properties)

# Worksheet 1.6. Exponential and Logarithmic Functions

- 1. Solve for the unknown variable. a.  $e^{t^2} = e^{4t-3}$ 
  - b.  $(\sqrt{5})^x = 125$
  - c.  $6e^{-4t} = 2$
  - d.  $\log_3 y + 3\log_3(y^2) = 14$
- 2. Calculate directly without using a calculator. a.  $\log_5 \frac{1}{25}$ 
  - b.  $\log_7(49)^2$
  - c.  $\log_{25} 30 + \log_{25} \frac{5}{6}$
- **3.** Compute sinh 1 using a calculator.
- 4. Compute  $\sinh(\ln 3)$  without using a calculator.

### Solutions to Worksheet 1.6

- 1. Solve for the unknown variable.
  - a.  $e^{t^2} = e^{4t-3}$

Equating exponents gives  $t^2 - 4t + 3 = 0$ . Thus (t - 3)(t - 1) = 0. So t = 3 and t = 1 are the answers.

- b.  $(\sqrt{5})^x = 125$  $(\sqrt{5})^x = 125 \implies 5^{\frac{x}{2}} = 5^3 \implies \frac{x}{2} = 3 \implies x = 6$
- c.  $6e^{-4t} = 2$ Taking the natural logarithm of both sides gives  $-4t = \ln \frac{1}{3} = -\ln 3 \implies t = \frac{\ln 3}{4}$
- d.  $\log_3 y + 3 \log_3(y^2) = 14.$  $\log_3 y + 3 \log_3(y^2) = 14 \implies \log_3 y^7 = 14 \implies y^7 = 3^{14} \implies y^7 = (3^2)^7 \implies y = 9$
- 2. Calculate directly without using a calculator.

a. 
$$\log_5 \frac{1}{25}$$
  
 $\log_5 \frac{1}{25} = \log_5(5^{-2}) = -2$   
b.  $\log_7(49)^2$   
 $\log_7(49)^2 = \log_7(7^4) = 4$   
c.  $\log_{25} 30 + \log_{25} \frac{5}{6}$   
 $\log_{25} 30 + \log_{25} \frac{5}{6} = \log_{25}(30)(\frac{5}{6}) = \log_{25} 25 = 1$ 

- 3. Compute  $\sinh 1$  using a calculator.  $\sinh 1 = \frac{e^1 - e^{-1}}{2} \approx 1.1752.$
- 4. Compute  $\sinh(\ln 3)$  without using a calculator.  $e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{5} = 8 = 4$

$$\sinh(\ln 3) = \frac{e^{\ln 3} - e^{-\ln 3}}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{8}{6} = \frac{4}{3}$$

#### 1.7. Technology: Calculators and Computers.

Class Time AB 0–2 periods; BC 0–1 period. Essential.

AP Calculus students are expected to use technology, specifically graphing calculators, in their work in calculus. The use of Computer Algebra Systems (CAS) and computer graphing software is encouraged. As with the other topics in this chapter, students should have experience with using technology to do mathematics before coming to calculus. This section touches on some of the principal uses of graphing calculators. It may be omitted if your students are familiar with them.

On the AP Calculus exams, students are expected to know how to do the following 4 things on their graphing calculator. These items are tested on the exams. Items (1) and (2) are discussed in this section; items (3) and (4) are discussed in later chapters.

- (1) Graph of a function in a given viewing rectangle or in a convenient viewing window of their choosing.
- (2) Solve an equation numerically. This may be done by finding where the graphs of the left and right sides intersect (using the built-in intersection operation), or by using any built-in equation solving operation.
- (3) Find the numerical value of a derivative at a point.
- (4) Find the numerical value of a definite integral.

Students are expected to show all the work leading to any other answer (e.g. finding a maximum value) even if the calculator has a built-in operation for doing it.

#### **Key Points**

- The appearance of a graphs depends upon the chosen viewing rectangle. One should experiment with different viewing rectangles to obtain one that displays the relevant information. Note that the scales along the x and y-axis may change as you vary the viewing rectangle.
- The following are some ways in which graphing calculators and computer algebra systems can be used in calculus:
  - (1) Visualizing the behavior of a function.
  - (2) Finding solutions graphically or numerically.
  - (3) Conducting graphical or numerical experiments.
  - (4) Illustrating theoretical ideas (for example, local linearity).

#### Lecture Material

Discuss the general uses of calculators in calculus (as listed in the Key Points). Then stress the importance of a correct viewing window (Figure 3 giving three viewing rectangles for  $f(x) = 12 - x - x^2$  may be useful for this). That is, the viewing window should be chosen, usually by trial and error, so that it contains all of the relevant information for the problem under consideration. You are looking for minima and maxima, places where the graph hits the x and y axis, and vertical and horizontal asymptotes. Work Exercises 6, 8, 10, and 16.

### Discussion Topics/Class Activities

Discuss Chebyshev polynomials as outlined in Exercise 24.

#### Suggested Problems

Exercises 1–17 odd (calculator)

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# Worksheet 1.7. Technology: Calculators and Computers



2. Plot the graph of  $f(x) = \frac{8x+1}{8x-4}$  in an appropriate viewing rectangle. What are the vertical and horizontal asymptotes?



**3.** Illustrate local linearity for  $f(x) = x^2$  by zooming in on the graph at x = 0.5.



4. Investigate the behavior of the function  $f(x) = \left(\frac{x+6}{x-4}\right)^x$  as x grows large by making a table of function values and plotting a graph. Describe the behavior in words.



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## Solutions to Worksheet 1.7

1. How many solutions does  $\cos x = x^2$  have? The equation  $\cos x = x^2$  has exactly two solutions. If  $|x| > \pi/2$ , then  $x^2 > (\pi/2)^2 > \pi/2$  $1 > \cos(x).$ 



2. Plot the graph of  $f(x) = \frac{8x+1}{8x-4}$  in an appropriate viewing rectangle. What are the vertical and horizontal asymptotes?

$$f(x) = \frac{8x+1}{8x-4}$$
 has vertical asymptote  $x = \frac{1}{2}$  and horizontal asymptote  $y = 1$ .

**3.** Illustrate local linearity for  $f(x) = x^2$  by zooming in on the graph at x = 0.5.



4. Investigate the behavior of the function  $f(x) = \left(\frac{x+6}{x-4}\right)^x$  as x grows large by making a table of function values and plotting a graph. Describe the behavior in words.



The function  $f(x) = \left(\frac{x+6}{x-4}\right)^x$  has a vertical asymptote at x = 4 (from the right) and decreases to approximately 16300 at  $x \approx 16$ . Then f(x) increases and appears to have a vertical asymptote at about y = 22026.5.

### Ray Cannon's Chapter 2 Overview

The notion of limit is central to calculus, and is what distinguishes calculus from algebra. The chapter starts in Section 2.1 with the key idea of defining *instantaneous* rate of change as the limit of average rates of change over intervals whose lengths go to zero. An understanding of this approach here provides the underpinning for the importance of the derivative developed in Chapter 3.

AP students are expected to be able to work with functions represented numerically and graphically, and this is part of the goal of Section 2.2, along with the development of one-sided limits and infinite limits. Note that infinite limits are not limits in the true sense of the word, but are used to describe the geometric property of vertical asymptotes of a graph. Section 2.3 lays the groundwork for the more precise computation of limits by showing the laws that govern algebraic manipulation of limits. Section 2.4 then introduces the important concept of continuity, and makes the point that since all our familiar functions are continuous on their domains, limits of these functions can be computed by simple functional evaluation.

Section 2.5 then shows how to manipulate the formula for f(x) when f is not continuous and simple evaluation does not work. Again, this section is very important for understanding how to compute the value of a derivative using the definition, which follows in Chapter 3. Section 2.6 develops some limits involving trigonometric functions, which are transcendental functions; transcendental means that algebraic tools are not enough for handling these functions. The last type of limit, limits at infinity, is discussed in Section 2.7, and the geometric application here is horizontal asymptotes. The geometric flavor continues in Section 2.8 as The Intermediate Value Theorem guarantees that the graph of a function continuous on an interval is a connected piece. Lastly, Section 2.9 deals with the formal definition of limit. This is not a required element of the AP Course Description, but many teachers like to give this as an added feature to their course to present a more rigorous treatment of limits.

### 2.1. Limits, Rates of Change, and Tangent Lines.

Class Time AB and BC, 1 period. Essential.

#### **Key Points**

- Average velocity  $= \frac{\text{change in position}}{\text{change in time}}$ .
- The average rate of change (ROC) of a function y = f(x) over an interval  $[x_0, x_1]$ is

Avg ROC = 
$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$
.

Graphically, this may be interpreted as the slope of the secant line, that is, the line passing through the points  $(x_0, f(x_0))$  and  $(x_1, f(x_1))$ .

• Estimating the instantaneous rate of change at a point; the slope of the tangent line.

#### Lecture Material

The notion of rate of change is fundamental and is used throughout the course. Here, the intention is to introduce the concepts of average rate of change of a function over an interval and the instantaneous rate of change at a point as the limit of average rates of change. Graphically, average rates of change correspond to slopes of secant lines, while the instantaneous rate of change in f at a point  $x = x_0$  is the slope of the line tangent to the graph of f at the point  $(x_0, f(x_0))$ .

An especially important case is rectilinear motion. In this case, the average rate of change in position with respect to time over the interval  $[t_0, t_1]$  is the average velocity of the particle and the instantaneous rate of change of position with respect to time at  $t = t_0$  is the instantaneous velocity at time  $t = t_0$ .

Use examples to illustrate the computation of instantaneous rates of change both numerically and graphically.

#### **Discussion Topics/Class Activities**

You could lead a discussion on what the tangent line means in a physical context, such as a train derailing or a car hitting ice. Then you could provide students with pictures of graphs of functions and have them draw what they think is the tangent line at different points. Dynamic graphing software, such as Winplot or Geometer's Sketchpad, can help illustrate this concept.

### Suggested Problems

Exercises 1, 3, 5, 6 (basic, numerical), 7, 9 (numerical), 19, 21, 23 (graphical), 25 (verbal and graphical), 29 (graphing calculator)

# Worksheet 2.1. Limits, Rates of Change, and Tangent Lines

1. A ball is dropped from a state of rest at time t = 0. The distance traveled after t seconds is  $s(t) = 16t^2$  ft. Compute the average velocity over the time intervals [3,3.01], [3,3.005], [3,3.001], and [3,3.0005]. Use this computation to estimate the ball's instantaneous velocity at t = 3. Compare this velocity to s'(3).

2. Draw the graph of  $f(x) = \sqrt{x}$  by plugging in x = 0, 1, 4, 9. Graphically find the slope of the secant line between (1, 1) and (4, 2). Compare it to the average rate of change over the interval [1, 4]. Estimate the instantaneous rate of change at x = 1 graphically. Compare it to f'(1).

**3.** Again consider  $f(x) = \sqrt{x}$ . Is the rate of change of f with respect to x greater at low or high x values?

4. Which graph has the property that for all x, the average rate of change over [0, x] is greater than the instantaneous rate of change at x?



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### Solutions to Worksheet 2.1

1. A ball is dropped from a state of rest at time t = 0. The distance traveled after t seconds is  $s(t) = 16t^2$  ft. Compute the average velocity over the time intervals [3,3.01], [3,3.005], [3,3.001], and [3,3.0005]. Use this computation to estimate the ball's instantaneous velocity at t = 3. Compare this velocity to s'(3).

$$\frac{h}{\frac{s(3+h)-s(3)}{h}} = 96.$$

$$\frac{h}{h \to 0} \frac{s(3+h)-s(3)}{h} = 96.$$

2. Draw the graph of  $f(x) = \sqrt{x}$  by plugging in x = 0, 1, 4, 9. Graphically find the slope of the secant line between (1, 1) and (4, 2). Compare it to the average rate of change over the interval [1, 4]. Estimate the instantaneous rate of change at x = 1 graphically. Compare it to f'(1).

Slope of the line through (1, 1) and (4, 2): m = 1/3Instantaneous rate of change at x = 1: f'(1) = 1/2



**3.** Again consider  $f(x) = \sqrt{x}$ . Are the rates of change of f with respect to x greater at low or high x values?

Slopes are decreasing, so the instantaneous rates of change in f at x are getting smaller as x increases.

4. Which graph has the property that for all x, the average rate of change over [0, x] is greater than the instantaneous rate of change at x?



Graph (B).

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Class Time AB and BC, 2 periods. Essential.

#### **Key Points**

- $\lim f(x)$ 
  - (i) Definition.
  - (ii) Estimates of limits by graphical and numerical methods.
- One-sided limits.
- Functions that approach infinity as a limit.
- Limits that do not exist.

#### Lecture Material

The concept of limit plays a fundamental role in all aspects of calculus. Emphasis in this section is on understanding this concept numerically and graphically. Use graphs and tables of values to investigate limits such as

$$\lim_{x \to 0} \frac{\cos x}{x^2}, \quad \lim_{x \to 1} x \frac{|x-1|}{x-1}, \text{ and } \lim_{x \to 0} \sin(1/x)$$

The zoom feature of graphing calculators is a very effective tool in this regard.

Point out that  $\lim_{x\to c} f(x)$  may exist even if f(c) is not defined, as for example  $\lim_{x\to 0} \frac{\sin x}{x}$ . Moreover, if f(x) approaches a limit as  $x \to c$ , then the limiting value L is unique. In particular, if  $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$ , then f has no limit at x = c.

Infinite limits are not "true" limits; they describe the behavior of the function near a point.

#### Discussion Topics/Class Activities

Use the Zoom feature of a graphing calculator to investigate limits as in Exercises 30, 38, 48 and 58.

Ask students to use the definition of limit to control errors: How close must x be to 1 for f(x) = 5 - 3x to be within  $10^{-5}$  of 2?

Suggested Problems (spread over 2 assignments)

Exercises 1, 2, 4 (numerical), 5, 6 (graphical), 12, 15 (definition), 17 to 37 odd (numerical), 49, 51, 53, 54 (graphical), 55, 57, 63 (graphing calculator)

# Worksheet 2.2. Limits: A Numerical and Graphical Approach

1. Use your graphing calculator to graph  $f(x) = \frac{\cos x}{x^2}$ . Make a guess as to the value of  $\lim_{x\to 0} \frac{\cos x}{x^2}$ . Construct a table of values for f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.0001). Estimate  $\lim_{x\to 0} \frac{\cos x}{x^2}$ .

2. Graph  $f(x) = x \frac{|x-1|}{x-1}$ . What is the  $\lim_{x \to 1^+} f(x)$  and  $\lim_{x \to 1^-} f(x)$ ? Construct a table of values for f(.9), f(.99), f(.999), f(1.001), f(1.)1), f(1.1). What is the  $\lim_{x \to 1^+} f(x)$  and  $\lim_{x \to 1^-} f(x)$ ?

**3.** Using a graphing calculator, graph  $f(x) = \sin \frac{1}{x}$ . Does if look as if  $\lim_{x \to 0} f(x)$  exists? Construct a table of values for f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.001). What do you conclude about  $\lim_{x \to 0} f(x)$ ?

4. Using a graphing calculator, graph  $f(x) = \frac{\sin x}{x}$ . Make a guess as to the  $\lim_{x \to 0} f(x)$ . Construct a table of values for f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.001). Estimate  $\lim_{x \to 0} f(x)$ .

### Solutions to Worksheet 2.2

1. Use your graphing calculator to graph  $f(x) = \frac{\cos x}{x^2}$  Make a guess as to the value of  $\lim_{x \to 0} \frac{\cos x}{x^2}$ . Construct a table of values for f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.0001). Estimate  $\lim_{x \to 0} \frac{\cos x}{x^2}$ .  $\frac{x \pm .1 \pm .01 \pm .001 \pm .0001}{f(x) - 0.49958347 - 0.49999583 - 0.49999996 - 0.50000000}$   $\lim_{x \to 0} \frac{\cos x}{x^2} = -\frac{1}{2}$ 

2. Graph  $f(x) = x \frac{|x-1|}{x-1}$ . What is the  $\lim_{x \to 1+} f(x)$  and  $\lim_{x \to 1-} f(x)$ ? Construct a table of values for f(.9), f(.99), f(.999), f(1.001), f(1.)1), f(1.1). What is the  $\lim_{x \to 1+} f(x)$  and  $\lim_{x \to 1-} f(x)$ ?



**3.** Using a graphing calculator, graph  $f(x) = \sin \frac{1}{x}$ . Does if look as if  $\lim_{x\to 0} f(x)$  exists? Construct a table of values for f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.001). What do you conclude about  $\lim_{x\to 0} f(x)$ ?



4. Using a graphing calculator, graph  $f(x) = \frac{\sin x}{x}$ . Make a guess as to the  $\lim_{x \to 0} f(x)$ . Construct a table of values for f(-.1), f(-.01), f(-.001), f(-.0001), f(.1), f(.01), f(.001), f(.001). Estimate  $\lim_{x \to 0} f(x)$ .



#### 2.3. Basic Limit Laws.

Class Time AB 2 periods; BC 1 period. Essential.

#### **Key Points**

The limit of a complicated function can be computed in terms of the limits of simpler constituents. If  $\lim_{x\to c} f(x) = L$  and  $\lim_{x\to c} g(x) = M$ , then

• 
$$\lim (f(x) \pm g(x)) = L \pm M$$

•  $\lim_{x \to c} f(x) g(x) = LM$ 

• If 
$$M \neq 0$$
, then  $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ .

#### Lecture Material

Since  $\lim_{x\to c} x = c$  and  $\lim_{x\to c} k = k$  for every c and for every constant k, repeated applications of the limit properties show that  $\lim_{x\to c} kx^n = kc^n$  for every natural number n. Use limit properties to evaluate limits of polynomial and rational functions at points in their domains.

It is important to point out that the limit properties hold in general only for finite limits. (Indeterminate forms are discussed in Sections 2.5 and 4.5.)

These limits, along with continuity and the limits of the Elementary functions, allow you to find most limits by substituting c into the function. Use several examples from Exercises 1 - 22 to prepare for this idea.

### **Discussion Topics/Class Activities**

Challenge students to show that if  $\lim_{x\to c} \frac{f(x)}{x-c}$  exists (finite), then  $\lim_{x\to c} f(x) = 0$ . (This is essentially Exercise 39.)

#### Suggested Problems

Exercises 5, 9, 11, 13, 15, 19, 21 (basic), 25, 28, 35 (abstract)
## Worksheet 2.3. Basic Limit Laws

1. 
$$\lim_{x \to -1} (3x^4 - 2x^3 + 4x) =$$

$$\lim_{x \to 2} (x+1)(3x^2 - 9) =$$

**3.** 
$$\lim_{t \to 4} \frac{3t - 14}{t + 1} =$$

**4.** Assuming that 
$$\lim_{x \to 6} f(x) = 4$$
, find  $\lim_{x \to 6} f(x)^2$ ,  $\lim_{x \to 6} \frac{1}{f(x)}$ , and  $\lim_{x \to 6} xf(x)$ .

5. Assuming that  $\lim_{x \to -1} f(x) = 3$  and  $\lim_{x \to -1} g(x) = 4$ , find  $\lim_{x \to -1} \frac{f(x)g(x) - 2}{3g(x) + 2}$ .

- 1.  $\lim_{x \to -1} (3x^4 2x^3 + 4x) = 1$
- 2.  $\lim_{x \to 2} (x+1)(3x^2 9) = 1$  $3t 14 \qquad 2$
- 3.  $\lim_{t \to 4} \frac{3t 14}{t + 1} = -\frac{2}{5}$
- 4. Assuming that  $\lim_{x \to 6} f(x) = 4$ , then  $\lim_{x \to 6} f(x)^2 = 16$ ,  $\lim_{x \to 6} \frac{1}{f(x)} = \frac{1}{4}$ , and  $\lim_{x \to 6} xf(x) = 24$ .
- 5. Assuming that  $\lim_{x \to -1} f(x) = 3$  and  $\lim_{x \to -1} g(x) = 4$ , then  $\lim_{x \to -1} \frac{f(x)g(x) 2}{3g(x) + 2} = \frac{5}{7}$ .

### 2.4. Limits and Continuity.

Class Time AB 2 periods; BC 1–2 periods. Essential.

### **Key Points**

- Continuous functions.
  - (i) Continuity at a point.
  - (ii) Examples and types of discontinuities.
  - (iii) One-sided continuity.
  - (iv) Continuity on an interval.
- Combinations of continuous functions; in particular, composition of continuous functions.
- Continuity of polynomials and rational functions.
- Continuity of trigonometric and exponential functions and *n*th-root functions.

### Lecture Material

Discontinuities of a function correspond to breaks in the function's graph. Three common types are removable, jump, and infinite discontinuities, illustrated respectively by the functions  $\frac{\sin x}{x}$ ,  $\frac{|x|}{x}$ , and  $\frac{x^2 + x + 1}{x}$  at 0. Another "worse" type of discontinuity is that of  $\sin(1/x)$  at 0.

Discuss examples of piecewise-defined functions and continuity on intervals. See Example 2 and Exercise 6 in particular. Use limit properties to establish continuity of polynomials and rational functions. Continuity of trigonometric and exponential functions and nth-root functions follows from an examination of their graphs.

Emphasize the fact that the composition of continuous functions is continuous. Limits of continuous functions can be evaluated by substitution. This idea is used in Section 2.5.

### Discussion Topics/Class Activities

Exercises 50 and 80 present good discussion topics.

### Suggested Problems (spread over 2 assignments)

Exercises 1, 2 (graphical), 9, 10, 12, 13 (using theorems), 17, 18, 21, 23, 25, 27, 29, 31, 33 (testing definitions), 39, 40, 42, 45 (abstract), 52, 55, 57, 65, 67, 70 (substitution method), 79, 82 (graphical), 84

## Worksheet 2.4. Limits and Continuity

1. Using a graphing calculator, graph  $f(x) = \frac{\sin x}{x}$ . Show f has a removable discontinuity at x = 0.

**2.** Graph 
$$f(x) = \frac{|x|}{x}$$
. Show f has a jump discontinuity at  $x = 0$ .



**3.** Graph  $f(x) = \frac{x^2 + x + 1}{x}$ . Show f has an infinite discontinuity at x = 0.



- 4. Determine where the following functions are discontinuous and classify the type of discontinuity.
  - a.  $f(x) = \llbracket \frac{1}{2}x \rrbracket$
  - b.  $f(t) = 3t^{-3/2} 9t^3$
  - c.  $f(x) = \frac{x^2 9}{x 3}$ d.  $f(x) = \frac{x - 9}{x - 3}$
  - e.  $f(t) = \tan 2t$

1. Using a graphing calculator, graph  $f(x) = \frac{\sin x}{x}$ . Show f has a removable discontinuity at x = 0.



**2.** Graph  $f(x) = \frac{|x|}{x}$ . Show f has a jump discontinuity at x = 0.



**3.** Graph 
$$f(x) = \frac{x^2 + x + 1}{x}$$
. Show f has an infinite discontinuity at  $x = 0$ .



- 4. Determine where the following functions are discontinuous and classify the type of discontinuity.
  - a.  $f(x) = [\![\frac{1}{2}x]\!]$ : f has a jump discontinuity at every even integer,  $n = 0, \pm 2, \pm 4, \ldots$

- b.  $f(t) = 3t^{-3/2} 9t^3$ :  $\lim_{t \to 0+} f(t) = \infty$ . f is continuous at every t > 0.
- c.  $f(x) = \frac{x^2 9}{x 3}$ : f has a removable discontinuity at x = 3.
- d.  $f(x) = \frac{x-9}{x-3}$ : f has an infinite discontinuity at x = 3.
- e.  $f(t) = \tan 2t$ : f has an infinite discontinuity at every odd multiple of  $\frac{\pi}{4}$ :  $t = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}, \pm \frac{5\pi}{4}, \dots$

#### 2.5. Evaluating Limits Algebraically.

Class Time AB 2 periods; BC 1 period. Essential.

### **Key Points**

- Indeterminate forms.
- Evaluating limits by transformation and substitution.

### Lecture Material

The limit of a function at points where it is continuous is found by substitution. In this section, we consider limits at points where functions are not defined yet still have limits.

Basic Limit Laws do not apply to limits of indeterminate forms such as  $\frac{0}{0}$ ,  $\frac{\infty}{\infty}$ , or  $\infty - \infty$ . But if f(x) is indeterminate at x = c, it is often possible to algebraically simplify f(x) for x close to  $c, x \neq c$ , so that the resulting expression g(x) is continuous at x = c. Consider  $f(x) = \frac{x^2 - x - 2}{x - 2}$ . Then f has form  $\frac{0}{0}$  at x = 2 since  $x^2 - x - 2$  and

x - 2 each approach 0 as  $x \to 2$ .

Step 1: Transform and cancel. If  $x \neq 2$ , then  $f(x) = \frac{x^2 - x - 2}{x - 2} = \frac{(x - 2)(x + 1)}{x - 2} = x + 1$ , and the function g(x) = x + 1 is continuous everywhere; in particular, g is continuous at x = 2.

Step 2: Substitute (evaluate using continuity). Since f(x) = x + 1 for all  $x \neq 2$ ,  $\lim_{x \to 2} f(x) = \lim_{x \to 2} x + 1 = 3$ . The textbook offers several examples worked in detail. Slopes of tangent lines and

The textbook offers several examples worked in detail. Slopes of tangent lines and instantaneous rates of change typically involve limits of the form  $\frac{0}{0}$ , and therefore the techniques presented here will be very important in Chapter 3.

Consider the graph of each of these not continuous functions.

### **Discussion Topics/Class Activities**

Revisit Example 2 of Section 2.1: The speed of sound (m/s) in dry air is related to the temperature T (degrees Kelvin) by the formula  $v = f(T) = 20\sqrt{T}$ . Find the instantaneous rate of change in v with respect to T when T = 273.

For  $T \neq 273$ , the average rate of change in velocity over the interval from T to 273 is

$$\frac{f(T) - f(273)}{T - 273} = \frac{20\sqrt{T} - 20\sqrt{273}}{T - 273}$$

Now argue as in Example 3 of the current section:

instantaneous rate of change = 
$$\lim_{T \to 273} \text{average ROC}$$
  
= 
$$\lim_{T \to 273} \frac{20\sqrt{T} - 20\sqrt{273}}{T - 273}$$
  
= 
$$\lim_{T \to 273} \frac{20(\sqrt{T} - \sqrt{273})((\sqrt{T} + \sqrt{273}))}{(T - 273)(\sqrt{T} + \sqrt{273})}$$
  
= 
$$\lim_{T \to 273} \frac{20(T - 273)}{(T - 273)((\sqrt{T} + \sqrt{273}))}$$
  
= 
$$\lim_{T \to 273} \frac{20}{\sqrt{T} + \sqrt{273}}$$
  
= 
$$\frac{10}{\sqrt{273}} \approx 0.605228 \text{ m/s per degree Kelvin}$$

# Suggested Problems

Exercises 1, 7, 9, 11, 17, 19, 26 (algebraic), 27, 31 (trigonometric), 35 (graphing calculator), 39, 46, 48 (algebraic)

# Worksheet 2.5. Evaluating Limits Algebraically

1. 
$$\lim_{x \to 8} \frac{x^2 - 64}{x - 8} =$$
2. 
$$\lim_{x \to 2} \frac{x^3 - 4x}{x - 2} =$$
3. 
$$\lim_{h \to 4} \frac{(h + 2)^2 - 9h}{h - 4} =$$

4. 
$$\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16} =$$

5. 
$$\lim_{t \to \frac{\pi}{2}} \frac{\cot t}{\csc t} =$$

$$6. \lim_{t \to \frac{\pi}{2}} (\sec t - \tan t) =$$

7. 
$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta} =$$

1. 
$$\lim_{x \to 8} \frac{x^2 - 64}{x - 8} = 16$$
  
2. 
$$\lim_{x \to 2} \frac{x^3 - 4x}{x - 2} = 8$$
  
3. 
$$\lim_{h \to 4} \frac{(h + 2)^2 - 9h}{h - 4} = 3$$
  
4. 
$$\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16} = \frac{1}{8}$$
  
5. 
$$\lim_{t \to \frac{\pi}{2}} \frac{\cot t}{\csc t} = 0$$
  
6. 
$$\lim_{t \to \frac{\pi}{2}} (\sec t - \tan t) = 0$$
  
7. 
$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta} = 0$$

#### 2.6. Trigonometric Limits.

**Class Time** AB and BC, 1 period. Essential.

### **Key Points**

• Squeeze Theorem. •  $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$  and  $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$ .

### Lecture Material

A function f(x) is squeezed at x = c if there exist functions  $\ell(x)$  and u(x) such that

- (1)  $\ell(x) \leq f(x) \leq u(x)$  for all  $x \neq c$  in an open interval I containing c and
- (2)  $\lim_{x \to c} \ell(x) = \lim_{x \to c} u(x) = L$  for some L

According to the Squeeze Theorem, it follows that  $\lim \ell(x) = L$  as well. Illustrate the Squeeze Theorem graphically and apply it to evaluate limits such as  $\lim x \sin(1/x)$ . There is also a one-sided version of the Squeeze Theorem: Consider  $\lim_{x\to 0^+} \sqrt{x} \cos(1/x)$ .

In Section 2.1, graphical and numerical evidence was used to argue that  $\frac{\sin \theta}{\theta} = 1$ as  $\theta \to 0$ . This fact will be very important in Chapter 3, and a formal argument is presented here. For  $0 < \theta < \pi/2$ , Figure 5 implies that  $\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$ , and since  $\cos \theta$  is continuous at 0 with  $\cos 0 = 1$ , the Squeeze Theorem implies that  $\lim_{\theta \to 0} \cos \theta = 0$  $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \lim_{\theta \to 0} 1 = 1.$ Applying a change of variables,  $\theta = ax$ , it follows that for any a,

$$\lim_{x \to 0} \frac{\sin(ax)}{x} = \lim_{\theta \to 0} a \frac{\sin\theta}{\theta} = a.$$

To show that  $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$ , apply the Quotient Law from the Basic Limit Laws of Section 2.3 and the half-angle formula from trigonometry:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \frac{-2\sin^2(\theta/2)}{\theta} = -\lim_{\theta \to 0} \sin(\theta/2) \lim_{\theta \to 0} \frac{\sin(\theta/2)}{\theta/2} = 0 \cdot 1 = 0$$

Discussion Topics/Class Activities Discuss Exercise 58.

### Suggested Problems

Exercises 3 (graphical), 5, 9, 12, 15 (basic), 17, 19, 25, 31, 35

## Worksheet 2.6. Trigonometric Limits

1. Use the Squeeze Theorem to evaluate  $\lim_{x \to 0} x^2 \sin \frac{1}{x}$ .

**2.** Use the Squeeze Theorem to evaluate  $\lim_{x \to 1} (x-1) \sin(\frac{\pi}{x-1})$ .

**3.** 
$$\lim_{h \to 0} \frac{\sin(5h)}{3h} =$$

4.  $\lim_{x \to 0} \frac{x^2}{\sin^2 x} =$ 

5. 
$$\lim_{t \to 0} \frac{\cos t - \cos^2 t}{t} =$$

1. Use the Squeeze Theorem to evaluate  $\lim_{x \to 0} x^2 \sin \frac{1}{x}$ . For  $x \neq 0, 0 \le |x^2 \sin \frac{1}{x}| \le x^2$ , so  $x^2 \sin \frac{1}{x} \to 0$  as  $x \to 0$ .

2. Use the Squeeze Theorem to evaluate  $\lim_{\substack{x \to 1 \\ \pi}} (x-1) \sin(\frac{\pi}{x-1})$ . For  $x \neq 1, -(x-1) \leq (x-1) \sin \frac{\pi}{x-1} \leq (x-1)$ , so  $\lim_{x \to 1} (x-1) \sin(\frac{\pi}{x-1}) = 0$ .

- 3.  $\lim_{h \to 0} \frac{\sin(5h)}{3h} = \frac{5}{3}$
- 4.  $\lim_{x \to 0} \frac{x^2}{\sin^2 x} = 1$
- 5.  $\lim_{t \to 0} \frac{\cos t \cos^2 t}{t} = 0$

### 2.7. Limits at Infinity.

Class Time AB and BC, 1 hour. Essential.

### Key Points

- The notation  $x \to \infty$  indicates that x increases without bound.
- The notation  $x \to -\infty$  indicates that x decreases (through negative values) without bound.
- Limits at infinity
- Limits at minity
  lim f(x) = L if f(x) approaches L as x → ∞
  lim f(x) = L if f(x) approaches L as x → -∞
  If n > 0, then lim x<sup>n</sup> = ∞ and lim x<sup>-n</sup> = 0.
  If n > 0 is a whole number, then lim x<sup>n</sup> = {∞ if n is even -∞ if n is odd.
- If  $f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + m_0}$  with  $a_n \neq 0$  and  $b_m \neq 0$ , then  $\lim_{x \to \pm \infty} f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + m_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + m_0}$  $\frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{n-m}.$ • A horizontal line y = L is a horizontal asymptote if  $\lim_{x \to \infty} f(x) = L$  and/or
- $\lim_{x \to -\infty} f(x) = L.$

### Lecture Material

Begin by defining limits at infinity and then introduce the appropriate notation. Example 1 provides a simple graphical representation of horizontal asymptotes and limits at infinity. Explain and illustrate Theorem 1 with graphs of various powers of x (such as the graphs in Figure 4).

We define a horizontal asymptote to be a horizontal line y = L such that at least one of the following is true:  $\lim_{x \to \infty} f(x) = L$  or  $\lim_{x \to -\infty} f(x) = L$ . Of course a vertical line x = L is a vertical asymptote if f(x) has an infinite limit as  $x \to L$  from either the left or the right (or both). The main technique for evaluating infinite limits (and so finding horizontal asymptotes) is to divide both the numerator and the denominator by the highest power of the variable that occurs in the denominator. For rational functions, this easily leads to Theorem 2:

$$\lim_{x \to \pm \infty} \left( \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0} \right) = \frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{n-m}$$

Work Exercises 8, 11, and 16 to illustrate the use of this technique. Then work through Examples 3 and 4 which involve roots and fractional powers.

### **Discussion Topics/Class Activities**

Have students work Exercise 43 at their desks.

### Suggested Problems

Exercises 1, 3 (basic), 7–15 every other odd (limits), 17–21 odd (horizontal asymptotes), 23, 27, 29, 30, 34, 35, 39, 41 (harder)

# Worksheet 2.7. Limits at Infinity

**1.** Evaluate the following limits.

a. 
$$\lim_{x \to \infty} \frac{4x^2 - 5x + 7}{10x + x^3}$$

b. 
$$\lim_{x \to -\infty} \frac{5x^5 + 3x^2 - 4x + 7}{3x^5 - 4x^4 + 5x^2 - 9}$$

c. 
$$\lim_{x \to \infty} \frac{4x^4}{2x^2}$$

d. 
$$\lim_{x \to \infty} \frac{4x^2 + 9}{\sqrt{x^7 + 6x^2 - 5}}$$

e. 
$$\lim_{x \to \infty} \frac{1}{x-4} - \frac{1}{2x}$$

f. 
$$\lim_{x \to \infty} \left( \ln(2x+5) - \ln\left(\sqrt{x^2+7}\right) \right)$$

2. Find the horizontal asymptotes

a. 
$$y = \frac{t^{2/3}}{2+5e^{|t|}}$$

b. 
$$y = \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8}$$

1. Evaluate the following limits.

a. 
$$\lim_{x \to \infty} \frac{4x^2 - 5x + 7}{10x + x^3} = 4 \lim_{x \to \infty} x^{2-3} = 0$$
  
b. 
$$\lim_{x \to -\infty} \frac{5x^5 + 3x^2 - 4x + 7}{3x^5 - 4x^4 + 5x^2 - 9} = \frac{5}{3} \lim_{x \to -\infty} x^{5-5} = \frac{5}{3}$$

c. 
$$\lim_{x \to \infty} \frac{4x}{2x^2} = \frac{4}{2} \lim_{x \to \infty} \frac{x}{x^2} = 2 \lim_{x \to \infty} x^2 = \infty$$

d.  $\lim_{x\to\infty} \frac{4x^2+9}{\sqrt{x^7+6x^2-5}}$ Divide both numerator and denominator by  $x^{7/2}$ , giving:

$$\frac{4x^2+9}{\sqrt{x^7+6x^2-5}} \cdot \frac{x^{-7/2}}{x^{-7/2}} = \frac{4x^{-3/2}+9x^{-7/2}}{\sqrt{1+6x^{-5}-5x^{-7}}} = \frac{\frac{4}{x^{3/2}}+\frac{9}{x^{7/2}}}{\sqrt{1+\frac{6}{x^5}-\frac{5}{x^7}}}$$

Thus, 
$$\lim_{x \to \infty} \frac{4x^2 + 9}{\sqrt{x^7 + 6x^2 - 5}} = \lim_{x \to \infty} \frac{\frac{4}{x^{3/2}} + \frac{9}{x^{7/2}}}{\sqrt{1 + \frac{6}{x^5} - \frac{5}{x^7}}} = \frac{0}{1} = 0$$

e. 
$$\lim_{x \to \infty} \frac{1}{x-4} - \frac{1}{2x} = \lim_{x \to \infty} \frac{2x}{2x(x-4)} - \frac{(x-4)}{2x(x-4)} = \lim_{x \to \infty} \frac{x+4}{2x(x-4)} = \frac{1}{2} \lim_{x \to \infty} x^{1-2} = 0$$
  
f. 
$$\lim_{x \to \infty} \left( \ln(2x+5) - \ln\left(\sqrt{x^2+7}\right) \right) = \lim_{x \to \infty} \ln\left(\frac{2x+5}{\sqrt{x^2+7}}\right) = \ln(2)$$

2. Find the horizontal asymptotes

a. 
$$y = \frac{t^{2/3}}{2 + 5e^{|t|}}$$
  
Since  $\lim_{t \to \infty} \frac{t^{2/3}}{2 + 5e^{|t|}} = 0$  and  $\lim_{t \to -\infty} \frac{t^{2/3}}{2 + 5e^{|t|}} = 0$ , the horiztonal asymptote is  $y = 0$ .  
b.  $y = \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8}$ 

To calculate,  $\lim_{t\to\infty} \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8}$ , one must first divide numerator and denominator by  $t^3 = \sqrt{t^6}$ .

$$\lim_{t \to \infty} \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8} = \lim_{t \to \infty} \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8} \cdot \frac{\frac{1}{\sqrt{t^6}}}{\frac{1}{t^3}}$$
$$= \lim_{t \to \infty} \frac{\sqrt{4 - \frac{8}{t^4} + \frac{9}{t^6}}}{1 - \frac{8}{t^3}} = 2$$

To calculate,  $\lim_{t\to-\infty} \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8}$ , one must first divide numerator and denominator by  $t^3 = -\sqrt{t^6}$ .

$$\lim_{t \to -\infty} \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8} = \lim_{t \to -\infty} \frac{\sqrt{4t^6 - 8t^2 + 9}}{t^3 - 8} \cdot \frac{-\frac{1}{\sqrt{t^6}}}{\frac{1}{t^3}}$$
$$= \lim_{t \to -\infty} \frac{-\sqrt{4 - \frac{8}{t^4} + \frac{9}{t^6}}}{1 - \frac{8}{t^3}} = -2$$

Therefore, the horizontal asymptopes are y = 2 and y = -2.

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Class Time AB 1 period; BC 1/2 period. Essential.

### **Key Points**

• Intermediate Value Theorem.

### Lecture Material

If f(x) is continuous on an interval I, then the graph of f has no breaks, so the range of f on I has no gaps. This is the content of the Intermediate Value Theorem: If f(x)is continuous on the interval [a, b] and if L lies between f(a) and f(b), then there is a cbetween a and b so that f(c) = L.

Give graphical illustrations of the theorem and examples showing that the conclusion may fail for functions that are not continuous at every point in the interval; for example, if  $f(x) = \frac{x^2 - 1}{x - 1}$ , if  $x \neq 1$  and f(1) = 3, then f(0) < 2 < f(2), but there is no c so that f(c) = 2. Also consider the graph of a noncontinuous piecewise-defined function such as  $f(x) = \frac{|x - 1|}{x - 1}$ .

Approximating solutions of equations is a long-standing important problem in mathematics. As a special case of the Intermediate Value Theorem, it follows that if f is continuous and if f(a) and f(b) have different signs, then the equation f(x) = 0 has a solution between a and b. This leads to the Bisection Method for approximating zeroes.

Illustrate the Bisection Method by approximating a solution of the equation  $\sin(\pi x) = 1 - x$  to within  $10^{-2}$ . The bisection method is not tested on the AP calculus exams. **Solution:** Let  $f(x) = \sin(\pi x) + x - 1$ . If  $x_0 = 1/4$  and  $x_1 = 1/3$ , then  $f(x_0) \cdot f(x_1) < 0$ , and f has a zero between  $x_0$  and  $x_1$ . Define  $x_2$  to be the midpoint of  $[x_0, x_1]$ :  $x_2 = 7/24$ . Since  $f(x_0) \cdot f(x_2) < 0$ , we know that the zero of f lies in the interval  $[x_0, x_2]$ ; set  $x_3 = (x_0 + x_2)/2 = 13/48$  and test. Since  $f(x_0) \cdot f(x_3) < 0$ , the solution lies in  $[x_0, x_3]$ ; set  $x_4 = (x_0 + x_3)/2 = 25/96$  Finally,  $f(x_0) \cdot f(x_4) > 0$ , so  $f(x_3) \cdot f(x_4) < 0$ . If  $x_5 = (x_3 + x_4)/2 = 17/64$ , then the solution lies in  $[x_5, x_3]$ , and since  $x_3 - x_5 \approx 0.005 < 10^{-2}$ , the solution of  $\sin(\pi x) = 1 - x$  is  $x \approx 0.27$ .

### **Discussion Topics/Class Activities**

Exercises 26 and 27 are good discussion problems.

#### Suggested Problems

Exercises 1, 3 (basic), 6 (harder), 7, 12 (abstract)

## Worksheet 2.8. Intermediate Value Theorem

1. Show that  $g(x) = \frac{x}{x+1}$  takes on the value 0.599 for some  $x \in [1, 2]$ .

- **2.** Show that  $\cos \theta = \theta$  has a solution in the interval. [0, 1]
- **3.** Using the Intermediate Value Theorem, show that  $f(x) = x^3 8x 1$  has a root in the interval [2.75,3]. Apply the Bisection Method twice to find an interval of length  $\frac{1}{16}$  containing this root.

**4.** Suppose that f(x) = x + 2 if x < -2 and that  $f(x) = \frac{1}{2}x + 3$  if  $x \ge -2$ . Show that there does not exist a number c such that f(c) = 1.

1. Show that  $g(x) = \frac{x}{x+1}$  takes on the value 0.599 for some  $x \in [1, 2]$ .

g is continuous on [1,2] and  $g(2) = \frac{1}{2} < .599 < \frac{2}{3} = g(2)$ . The Intermediate Value Theorem thus implies that .599 = g(c) for some c in (1,2).

- 2. Show that  $\cos \theta = \theta$  has a solution in the interval [0, 1].  $f(\theta) = \cos \theta - \theta$  is continuous on [0, 1] with f(0) > 0 and  $f(1) = \cos(1) - 1 < 0$ . The Intermediate Value Theorem applies.
- **3.** Using the Intermediate Value Theorem, show that  $f(x) = x^3 8x 1$  has a root in the interval [2.75, 3]. Apply the Bisection Method twice to find an interval of length  $\frac{1}{16}$  containing this root.

Since the sign of f changes between  $x = \frac{23}{8}$  and  $x = \frac{47}{16}$ , the function f has a root in the interval  $\left(\frac{23}{8}, \frac{47}{16}\right)$ .

4. Suppose that f(x) = x + 2 if x < -2 and that  $f(x) = \frac{1}{2}x + 3$  if  $x \ge -2$ . Show that there does not exist a number c such that f(c) = 1.

f has range  $(-\infty, 0) \cup [2, \infty)$ .

### Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS. Key Point

• Formal  $(\varepsilon - \delta)$  definition of a limit.

### Lecture Material

 $\lim_{x\to c} f(x) = L$  provided that for every  $\varepsilon > 0$ , there is a positive number  $\delta$  so that if  $0 < |x-c| < \delta$  then  $|f(x) - L| < \varepsilon$ . If students find the classical notation intimidating, rephrase the definition in terms of powers of 10 as in the text. Illustrate the definition graphically as in Figure 4.

If f(x) = 3x - 5, ask students, "How close must x be to 2 to ensure that the error |f(x) - 1| is less than  $10^{-4}$ ?" and illustrate the problem as in Figure 1.

To construct a formal proof that  $\lim_{x \to c} f(x) = L$ , the text suggests a general strategy:

**Step 1.** Express the gap |f(x) - L| in terms of the difference |x - c|.

**Step 2.** Choose  $\delta$  (or  $10^{-m}$ ) in terms of  $\varepsilon$  ( $10^{-n}$ ) (and also perhaps c).

Apply this strategy to limits as in Examples 1–3. Another example: Show that  $\lim_{x \to \infty} \sqrt{x} = 1$ .

Step 1. 
$$|\sqrt{x} - 1| = \frac{|(\sqrt{x} - 1)(\sqrt{x} + 1)|}{\sqrt{x} + 1} = \frac{|x - 1|}{\sqrt{x} + 1} \le |x - 1|$$
 since  $\sqrt{x} + 1 \ge 1$  for all  $x \ge 0$ 

 $x \ge 0.$ 

**Step 2.** Given an arbitrary  $\varepsilon > 0$ , let  $\delta = \min\{1, \varepsilon\}$ . Then, from the estimate in Step 1,  $0 < |x - 1| < \delta$  implies that  $|\sqrt{x} - 1| < \varepsilon$ .

### **Discussion Topics/Class Activities**

Negate the definition of the limit and use the negation to show that  $\lim_{x\to 0} \frac{x}{|x|}$  and  $\lim_{x\to 0} \sin \frac{1}{x}$  do not exist. The  $\lim_{x\to a} f(x) \neq L$  if and only if there exists  $\epsilon > 0$  such that for every  $\delta > 0, |f(x) - L| \geq \epsilon$  whenever  $|x - a| < \delta$ .

#### **Suggested Problems**

Exercises 1, 2 (basic), 4, 5 (harder), 6, 7, 9 (graphing calculator), 12 (graphical), 14 (numerical), 18, 27 (abstract)

### Worksheet 2.9. The Formal Definition of a Limit

- 1. Suppose f(x) = 3x 5. How close must x be to 2 to ensure that the error |f(x) 1| is less than  $10^{-4}$ ?
- 2. Find a number  $\delta$  such that  $|x^2 4|$  is less than  $10^{-4}$  if  $0 < |x 2| < \delta$ .

**3.** Prove rigorously that  $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ .

4. Prove rigorously that  $\lim_{x \to 2} \frac{1}{x^2} = \frac{1}{4}$ .

5. Using the negation of the definition of the limit, prove that  $\lim_{x\to 0} \frac{x}{|x|}$  does not exist.

6. Using the negation of the definition of the limit, prove that  $\lim_{x\to 0} \sin \frac{1}{x}$  does not exist.

1. Suppose f(x) = 3x - 5. How close must x be to 2 to ensure that the error |f(x) - 1| is less than  $10^{-4}$ ?

$$|f(x) - 1| = |(3x - 5) - 1| = 3|x - 2| < 10^{-4} \Leftrightarrow |x - 2| < \frac{1}{3}10^{-4}$$

**2.** Find a number  $\delta$  such that  $|x^2 - 4|$  is less than  $10^{-4}$  if  $0 < |x - 2| < \delta$ . If  $\delta > 0$  and  $|x - 2| < \delta$ , then

$$x^{2} - 4| = |x - 2| |(x - 2) + 4| < |x - 2|(|x - 2| + 4) < \delta(\delta + 4)$$

Any positive  $\delta < \frac{4 \, 10^{-4}}{1 - 10^4} = \frac{4}{9999} \approx 0.00040004$  will work.

**3.** Prove rigorously that  $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ . Let  $\varepsilon > 0$  and let  $\delta = \varepsilon$ . Then for every x such that  $0 < |x - 0| < \delta$ ,

$$\left|x\sin\frac{1}{x} - 0\right| \le |x| < \varepsilon$$

4. Prove rigorously that  $\lim_{x\to 2} \frac{1}{x^2} = \frac{1}{4}$ . Let  $\varepsilon > 0$  and choose  $\delta$  such that  $0 < \delta < \min\{4/5\varepsilon, 1\}$ . If  $0 < |x-2| < \delta$ , then  $\delta < 1$ 

Let  $\varepsilon > 0$  and choose  $\delta$  such that  $0 < \delta < \min\{4/5\varepsilon, 1\}$ . If  $0 < |x - 2| < \delta$ , then  $\delta < 1$  implies that 1 < x < 3, so

$$\left|\frac{1}{x^2} - \frac{1}{4}\right| = \frac{|x-2| \, |x+2|}{4x^2} \le \frac{|x-2|(3+2)}{4} < \frac{5}{4}\delta \le \varepsilon$$

5. Using the negation of the definition of the limit, prove that  $\lim_{x\to 0} \frac{x}{|x|}$  does not exist.

 $\lim_{x \to 0} f(x) \text{ does not exist provided that for every number } L \text{ there exists an } \varepsilon > 0 \text{ such that for every } \delta > 0 \text{ there is an } x \text{ satisfying } 0 < |x| < \delta \text{ and } |f(x) - L| \ge \varepsilon.$ Let  $f(x) = \frac{x}{|x|}$  for  $x \neq 0$  and fix a number L. If  $|L| \neq 1$ , then  $|f(x) - L| > \frac{|1 - |L||}{2}$ 

for all  $x \neq 0$ . If  $L = \pm 1$ , and  $\delta > 0$ , choose *n* so that  $10^{-n} < \delta$ . Then one of the two numbers  $x_1 = 10^{-n}$ ,  $x_2 = -10^{-n}$  satisfies |f(x) - L| > 1.

6. Using the negation of the definition of the limit, prove that  $\lim_{x\to 0} \sin \frac{1}{x}$  does not exist.

Let 
$$x_n = \frac{1}{n\pi}$$
 and  $y_n = \frac{2}{(4n+1)\pi}$ . Then  $0 \neq x_n \to 0$  and  $0 \neq y_n \to 0$  as  $n \to \infty$ , but  $\sin \frac{1}{x_n} = 0$  and  $\sin \frac{1}{y_n} = 1$  for every  $n$ .

# Chapter 2 AP Problems

- 1. Find the instantaneous rate of change at the point indicated.  $f(x) = 2\sqrt{x}$  at x = 4.
- A. 0 B.  $\frac{1}{4}$ C.  $\frac{1}{2}$ D. 4 E. 8 2.  $\lim_{x \to 1} \frac{|x-1|}{x-1} =$ A. -1 B. 0 C. 1 D. x-1
  - E. undefined

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3. 
$$\lim_{x \to 0} \frac{\sin(x + \frac{\pi}{2}) - 1}{x} =$$
A. -1
B. 0
C. 1
D.  $\frac{\pi}{2}$ 
E. undefined

4. Let 
$$f(x) = \begin{cases} x-3 & \text{if } x \le 2\\ 2x+1 & \text{if } x > 2 \end{cases}$$
  
I.  $f(2)$  exists  
II.  $f$  is continuous at 2  
III.  $\lim_{x \to 2} f(x)$  exists  
A. I only  
B. II only  
C. III only  
D. I and II only

E. I, II, and III

Which of the following statements are true about f?

5. 
$$\lim_{x \to \infty} \frac{x^4 - 3x^2 - 2x + 5}{3x^4 - 3x^2 + 3x - 1} =$$
  
A.  $-\frac{2}{3}$   
B. 0  
C.  $\frac{1}{3}$   
D. 1  
E. 3



E. does not exist

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# Solutions to Chapter 2 AP Problems

**1.** Find the instantaneous rate of change at the point indicated.  $f(x) = 2\sqrt{x}$  at x = 4.



2.  $\lim_{x \to 1} \frac{|x-1|}{x-1} =$ A. -1 B. 0 C. 1 D. x - 1

E. undefined

E [THIS QUESTION CORRESPONDS WITH SECTION 2.2]

3. 
$$\lim_{x \to 0} \frac{\sin(x + \frac{\pi}{2}) - 1}{x} =$$
  
A. -1  
B. 0  
C. 1  
D.  $\frac{\pi}{2}$   
E. undefined

### **B** [THIS QUESTION CORRESPONDS WITH SECTION 2.3]

4. Let 
$$f(x) = \begin{cases} x-3 & \text{if } x \le 2\\ 2x+1 & \text{if } x > 2 \end{cases}$$
 Which of the following statements are true about  $f$ ?

I. f(2) exists II. f is continuous at 2 III.  $\lim_{x \to 2} f(x)$  exists

A. I only

B. II only

C. III only

D. I and II only

E. I, II, and III

# A [THIS QUESTION CORRESPONDS WITH SECTION 2.4]

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5. 
$$\lim_{x \to \infty} \frac{x^4 - 3x^2 - 2x + 5}{3x^4 - 3x^2 + 3x - 1} =$$
  
A.  $-\frac{2}{3}$   
B. 0  
C.  $\frac{1}{3}$   
D. 1  
E. 3  
C [THIS QUESTION CORRESPONDS WITH SECTION 2.5]

6. 
$$\lim_{x \to 0} \frac{\sin 3x}{x} =$$
  
A. -3  
B. 0  
C. 1  
D. 3

E. does not exist

D [THIS QUESTION CORRESPONDS WITH SECTION 2.6]

### Ray Cannon's Chapter 3 Overview

Section 3.1 opens with a definition central to AP Calculus. Chapter 2 showed us that the same limit is used in geometry to compute the slope of a tangent line and in physics to compute instantaneous velocity. One of the glories of mathematics, as well as a feature that makes it difficult, is that when the same mathematics is used to solve two different problems, mathematicians like to abstract the machinery used from the application. This leads us to the notion of the derivative as a purely mathematical idea, and students should be able to use the definition in both its forms. Section 3.2 then makes the transition to the fact that the derivative can be computed for various values, and starts the treatment of the derivative of a function as another function. Section 3.2 also includes some "rules" for computing derivatives. By including the derivative of the exponential function as contrasted to the power rule, the text makes clear that not all derivatives follow the same rule. This section also includes the theorem that differentiability implies continuity with an example to show the converse is not true. This is a long and important section and probably should not be covered in just one day.

Section 3.3 continues developing computational rules (product and quotient). It should also be noted that by stating the rule, we are also saying, for example, that the product of differentiable functions is differentiable. Students tend to overlook this aspect of the theorem, and they should not. Sec 3.4 reinforces the conceptual idea that the derivative is the instantaneous rate of change for any application. AP students in particular need to have a command of the section on linear motion and familiarity with the terms velocity, speed, and acceleration. Section 3.5 develops the idea of higher order derivatives and makes the identification of acceleration with the second derivative of the position function. The second derivative also offers insight into how the graph is bending. The development of the techniques of differentiation must include general rules about how to combine functions, and also specific examples for a particular class of functions. Section 3.6 shows how to compute the derivatives of the trigonometric functions. Here the important limit  $\lim_{x\to 0} \frac{\sin x}{x} = 1$  is used; AP students need to know this limit.

Section 3.7 introduces the chain rule, which is extremely important. Section 3.8 shows how to compute the derivative of inverse functions, with the particular applications to the inverse trig functions; pay special attention to the inverse-sine and inverse-tangent. The same ideas are used in Section 3.9 to produce the derivative of lnx. This is a difficult derivative for students to use correctly with the chain rule, and so they may need lots of practice. There is always a question on the AP exam dealing with implicit differentiation, so Section 3.10 must be dealt with carefully. Many students do not use the chain rule

correctly with implicitly defined functions. AP students are not responsible for knowledge of the hyperbolic trig functions, so you need not cover those functions if time and pacing are a problem. Section 3.11 covers related rates, again a topic that is always on the AP exam. The language of rates of change and correct use of the chain rule, as in Section 3.7, are very important in this section, and both must be mastered.
### 3.1. Definition of the Derivative.

Class Time AB 2 periods; BC 1–2 periods. Essential.

## **Key Points**

• The definition of the derivative at *a*:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

• The equation of the tangent line at a point (a, f(a)) on the function:

$$y = f(a) + f'(a)(x - a)$$

- The derivative is the slope of the tangent line.
- The derivative is the instantaneous rate of change.
- Notations for the derivative.
- The derivative of a linear and constant function.
- Estimating the derivative.

### Lecture Material

Begin the lecture by reminding students of the motivational material given in Section 2.1 concerning the relationship between the slope of the secant line between two points P and Q and and the slope of the tangent line at P. Illustrate this relationship graphically using the slide provided, or illustrate using graphing or dynamic geometry software. If y = f(x), then the slope of the secant line between (a, f(a)) and (x, f(x)) is  $\frac{f(x) - f(a)}{x - a}$ . If we take the limit as x goes to a, we get the slope of the tangent line at x = a. This is also the instantaneous rate of change of f at x = a. If this limit exists, we call it the derivative of f at x = a. The notation f'(a) needs to be introduced and then the other formula for the derivative needs to be derived by letting x = a + h where  $h \neq 0$ . Then at least two examples of finding the derivative at a point should be shown in class, perhaps a quadratic function and  $f(x) = \frac{1}{x}$ .

Next derive the equation of the tangent line using the point-slope formula for the line, and then illustrate it using one of the examples used earlier.

Finally, the instructor should state and prove Theorem 1 involving linear and constant functions.

Show students how to find the value of a derivative at a point on their graphing calculator.

## **Discussion Topics/Class Activities**

Have students discuss examples similar to Examples 5 and 6 in the text. For Example 5, the students could fill in a table like table 1, but for  $\cos x$  at  $\frac{\pi}{6}$ . Discuss how Example 6 changes when using  $\cos x$  instead of  $\sin x$ . Illustrate using graphing or dynamic geometry software.

Suggested Problems (spread over 2 assignments)

Exercises 3, 5, 6, 7-14, 35, 47, 49, 51, 58, 72 (See Equation 4, p. 128), 73

# Worksheet 3.1. Definition of the Derivative

**1.** Given that  $f(x) = 3x^2 + 2x$ , use the definition of the derivative to find f'(-1).

2. Given that  $f(x) = \frac{1}{x}$ , use the definition of the derivative to find f'(2) and then find the equation of the tangent line at x = 2.

**3.** Given that  $f(t) = \sqrt{1+t}$ , use the definition of the derivative to find f'(0) and then find the equation of the tangent line at x = 0. Draw the graph of f(t) on the interval [-1, 1] and on this graph draw the tangent line at x = 0.



1. Given that  $f(x) = 3x^2 + 2x$ , use the definition of the derivative to find f'(-1). Let  $f(x) = 3x^2 + 2x$ . Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 + 2(x+h) - (3x^2 + 2x)}{h}$$
$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 3x^2 - 2x}{h}$$
$$= \lim_{h \to 0} \frac{3h^2 + 6xh + 2h}{h}$$
$$= \lim_{h \to 0} (3h + 6x + 2) = 6x + 2$$

Note f'(-1) = 6(-1) + 2 = -4 and f(-1) = 1. The tangent line at a = -1 is therefore y = f'(-1)(x+1) + f(-1) = -4(x+1) + 1 = -4x - 3.

2. Given that  $f(x) = \frac{1}{x}$ , use the definition of the derivative to find f'(3) and then find the equation of the tangent line at x = 3. Let  $f(x) = x^{-1}$ . Then

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{1}{3+h} - \left(\frac{1}{3}\right)}{h} = \lim_{h \to 0} \frac{\frac{3-3-h}{3(3+h)}}{h}$$
$$= \lim_{h \to 0} \frac{-h}{(9+3h)h} = -\frac{1}{9}$$

The tangent at a = 3 is

$$y = f'(3)(x-3) + f(3) = \left(\frac{1}{3}\right) - \left(\frac{1}{9}\right)(x-3).$$

**3.** Given that  $f(t) = \sqrt{1+t}$ , use the definition of the derivative to find f'(0) and then find the equation of the tangent line at x = 0. Draw the graph of f(t) on the interval [-1, 1] and on this graph draw the tangent line at x = 0.

Let  $f(t) = \sqrt{t+1}$ . Then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt{h+1} - \sqrt{1}}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{h+1} - 1}{h} \cdot \frac{\sqrt{h+1} + 1}{\sqrt{h+1} + 1}$$
$$= \lim_{h \to 0} \frac{h}{h(\sqrt{h+1} + 1)}$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{h+1} + 1} = \frac{1}{2}.$$

The tangent line at a = 0 is

$$y = f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

#### 3.2. The Derivative as a Function.

Class Time AB 2 periods; BC 1–2 periods. Essential.

### **Key Points**

- The derivative is a function of x with its own domain and range.
- Leibniz notation for the derivative.
- The Power Rule.
- The Sum and Constant Multiple Rule.
- Differentiability implies continuity.

• 
$$\frac{d}{dx}e^x = e^x$$

• Relationship between f and f'.

### Lecture Material

Begin by restating the definition of the derivative of a function at x = a, and then replace a with x. Next calculate f'(x) for  $f(x) = \sqrt{x}$ , and note that the domain of f'(x) is smaller than the domain of f(x).

Then introduce Leibniz notation and differentials.

Next state the Power Rule. It can be proved for n any positive integer or for a specific n, such as n = 3. It should be mentioned that the Power Rule applies for n equal to any real number, but we aren't in a position to prove it yet.

Develop  $\frac{d}{dx}e^x = e^x$ .

Now state Theorem 2 and prove at least the Sum Rule. Do an example that combines Theorems 1 and 2. Finally, state and prove Theorem 3 and illustrate that the converse is not true, using the example of f(x) = |x|. Discuss vertical tangents (Example 10). Stress graphical insight (p. 137) and local linearity.

### Discussion Topics/Class Activities

Have students discuss the relationship between the sign of the first derivative and whether the function is increasing or decreasing (Examples 5, 6). Graph  $f(x) = 2x^3 + 3x^2 - 36x$  using a graphing calculator and decide where the function is increasing and decreasing. Then graph the derivative and decide where it is positive and negative.

### Suggested Problems (spread over 2 assignments)

Exercises 3, 5, 7, 9, 21, 23, 25, 29, 33, 43, 45, 46, 51, 53, 66, 88 (see note above Exercise 85)

## Worksheet 3.2. The Derivative as a Function

1. Suppose that  $f(x) = \sqrt{x}$ . Find the domain and range of f and graph it. Using the definition of the derivative, find f'(x) at x. Find its domain and range and graph it together with f(x).



- 2. Using the derivative formulas, find the derivative of the following functions.
  a. g(z) = 7z<sup>-3</sup> + z<sup>2</sup> + 5.
  - b.  $f(x) = \sqrt[4]{x} + \sqrt[3]{x}$
  - c. P(z) = (3z 1)(2z + 1)
- **3.** Find the equation of the tangent line to  $f(x) = \frac{\sqrt{x}+1}{x}$  at x = 16.

1. Suppose  $f(x) = \sqrt{x}$ . Find the domain and range of f and graph it. Using the definition of the derivative, find f'(x) at x. Find its domain and range and graph it together with f(x).

The domain of f is  $\{x : x \ge 0\}$ . The range of f is  $\{z : z \ge 0\}$   $f'(x) = \frac{2}{x\sqrt{x}}$  Its domain is  $\{x : x > 0\}$  and its range is  $\{z : z \ge 0\}$ .



- 2. Using the derivative formulas, find the derivative of the following functions. a.  $g(z) = 7z^{-3} + z^2 + 5$ .
  - $\frac{d}{dz}\left(7z^{-3} + z^2 + 5\right) = 2z 21z^{-4}$
  - b.  $f(x) = \sqrt[4]{x} + \sqrt[3]{x}$   $f(s) = \sqrt[4]{s} + \sqrt[3]{s} = s^{\frac{1}{4}} + s^{\frac{1}{3}}$ . In this form, we can apply the sum and power rules.  $f'(s) = \frac{1}{4}(s^{\frac{1}{4}-1}) + \frac{1}{3}(s^{\frac{1}{3}-1}) = \frac{1}{4}s^{-\frac{3}{4}} + \frac{1}{3}s^{-\frac{2}{3}}$

c. 
$$P(z) = (3z - 1)(2z + 1)$$
  
 $\frac{d}{dz}((3z - 1)(2z + 1)) = \frac{d}{dz}(6z^2 + z - 1) = 12z + 1.$ 

**3.** Find the equation of the tangent line to  $f(x) = \frac{\sqrt{x}+1}{x}$  at x = 16. With  $y = \frac{x^{1/2}+1}{x} = x^{-1/2} + x^{-1}$ , we have  $y' = -\frac{1}{2}x^{-3/2} - x^{-2}$ . Therefore,  $y'(16) = -\frac{1}{2}(16)^{-3/2} - 16^{-2} = -\frac{3}{256}$ . Thus the equation of the tangent line is  $y = \frac{-3}{256}x - \frac{-2}{16}$ .

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Class Time AB 1 period; BC 1 period. Essential.

## **Key Points**

• The Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

• The Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

### Lecture Material

State the Product and Quotient rules. Emphasize that the derivative of a product is not the product of the derivatives and likewise with the quotient. Then do at least one example of each rule. End with an example like  $f(x) = (3x + 1)(x^2 + 2x)$ , where first the Product Rule is used to take the derivative and then the function is rewritten by multiplying the two factors, and the Power Rule is used to take the derivative. Show that the results are the same.

### Discussion Topics/Class Activities

Have students graph f(x) = 2x + 1,  $g(x) = x^2$ , and  $f(x)g(x) = 2x^3 + x^2$  and estimate the slopes of the tangent lines at x = 1 to see that  $(fg)'(1) \neq f'(1)g'(1)$ .

### Suggested Problems

Exercises 1, 3, 7, 9, 13, 15 (assign more if more practice is needed), 52, 62 (proof of the quotient rule)

# Worksheet 3.3. The Product and Quotient Rules

1. Using the Product Rule, find the derivative of  $f(x) = \sqrt{x(1 - x^4)}$ .

**2.** Using the Quotient Rule, find the derivative of  $f(x) = \frac{x+4}{x^2+x+1}$ .

**3.** Find the equation of the tangent line to  $f(x) = \frac{x^2}{\sqrt{x} + x}$  at x = 9.

1. Using the product rule find the derivative of  $f(x) = \sqrt{x(1-x^4)}$ Let  $f(x) = \sqrt{x(1-x^4)}$ . Then

$$f'(x) = \left(\frac{d}{dx}\sqrt{x}\right)(1-x^4) + \sqrt{x}\frac{d}{dx}(1-x^4)$$
$$= \left(\frac{1}{2}\right)x^{-1/2}(1-x^4) + \sqrt{x}(-4x^3) = \left(\frac{1}{2}\right)x^{-1/2} - \left(\frac{9}{2}\right)x^{7/2}.$$

2. Using the quotient rule find the derivative of  $f(x) = \frac{x+4}{x^2+x+1}$ 

Let 
$$f(x) = \frac{x+4}{x^2+x+1}$$
 Then  

$$f'(x) = \frac{(x^2+x+1)\frac{d}{dx}(x+4) - (x+4)\frac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{(x^2+x+1) - (x+4)(2x+1)}{(x^2+x+1)^2} = \frac{-x^2 - 8x - 3}{(x^2+x+1)^2}.$$

**3.** Find the equation of the tangent line to  $f(x) = \frac{x^2}{\sqrt{x} + x}$  at x = 9.

Let 
$$w(z) = \frac{z^2}{\sqrt{z+z}}$$
. Then  

$$\frac{dw}{dz} = \frac{(\sqrt{z}+z)\frac{d}{dz}z^2 - z^2\frac{d}{dz}(\sqrt{z}+z)}{(\sqrt{z}+z)^2}$$

$$= \frac{2z(\sqrt{z}+z) - z^2((1/2)z^{-1/2}+1)}{(\sqrt{z}+z)^2} = \frac{(3/2)z^{3/2}+z^2}{(\sqrt{z}+z)^2}.$$

Therefore,

$$\left. \frac{dw}{dz} \right|_{z=9} = \frac{(3/2)(9)^{3/2} + 9^2}{(\sqrt{9} + 9)^2} = \frac{27}{32}.$$

So the equation of the tangent line is  $y = \frac{27}{32}x - \frac{27}{32}$ .

#### 3.4. Rates of Change.

Class Time AB 1 period; BC 1 period. Essential.

### **Key Points**

• The average rate of change of a function over an interval:

Average ROC = 
$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

• The instantaneous rate of change of a function at a point:

Instantaneous ROC at 
$$x_0 = f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

• The marginal cost:

$$C(x_0+1) - C(x_0) \approx C'(x_0)$$

• The position and velocity functions of an object tossed vertically with no air resistance:

$$s(t) = s_0 + v_0 t - \frac{1}{2}gt^2, \ v(t) = v_0 - gt$$

where g is 32 ft/s<sup>2</sup> or 9.8 m/s<sup>2</sup>,  $s_0$  is initial position, and  $v_0$  is initial velocity.

### Lecture Material

Define the average rate of change of a function over an interval and the instantaneous rate of change of the function at a point. Be sure students understand the difference between the average and instantaneous rate of change, and that the instantaneous rate of change is the limit of the average rate of change. Then discuss Example 1 using the slide provided.

If time permits, discuss marginal cost. This is the average rate of change of the cost function over an interval of 1. It can be approximated by the derivative of the cost function. Marginal cost is not tested on the AP Calculus exams.

Galileo discovered that the height s(t) and the velocity v(t) of an object tossed vertically when air resistance is negligible are  $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$  and  $v(t) = v_0 - gt$ , where g is 32 ft/s<sup>2</sup> or 9.8 m/s<sup>2</sup>,  $s_0$  is initial position, and  $v_0$  is initial velocity. You can derive these formulas by starting with the acceleration due to gravity g and antidifferentiating twice and solving for the integration constants. Work Example 6.

#### Discussion Topics/Class Activities

Have students work Exercise 18. Provide guidance.

The AP exams almost always include problems concerning particles moving on a line (AB) or in the plane (BC). Do a problem like Exercise 22 (same as Worksheet 3.4 # 3) to begin work on this type of problem. This will be continued in later chapters.

## Suggested Problems

Exercises 1, 2, 5, 9, 10, 11, 30

## Worksheet 3.4. Rates of Change

- The population P(t) of Freedonia in 1933 was P(1933) = 5 million.
   a. What is the meaning of P'(1933)?
  - b. Estimate P(1934) if P'(1933) = .2. What if P'(1933) = 0?
- 2. Find the rate of change of the volume of a cube with respect to the length of its side s when s = 3 and s = 5.
- **3.** The height in feet of a helicopter at time t in minutes is  $s(t) = -3t^3 + 400t$  for  $0 \le t \le 10$ .
  - a. Plot the graphs of height s(t) and velocity v(t).



c. Find the maximum height of the helicopter.

- 4. It takes a stone 3 seconds to hit the ground when dropped from the top of a building. How high is the building and what is the stone's velocity on impact?
- 5. The demand for a commodity generally decreases as the price is raised. Suppose that the demand for oil (per capita per year) is  $D(p) = \frac{900}{p}$  barrels where p is the price per barrel in dollars. Find the demand when p =\$40. Estimate the decrease in demand if p is raised to \$41 and the increase if p is decreased to \$39.

- **1.** The population P(t) of Freedonia in 1933 was P(1933) = 5 million.
  - a. What is the meaning of P'(1933)?

Because P(t) measures the population of Freedonia as a function of time, the derivative P'(1933) measures the rate of change of the population of Freedonia in the year 1933.

- b. Estimate P(1934) if P'(1933) = .2. What if P'(1933) = 0?  $P(1934) \approx P(1933) + P'(1933)$ . Thus, if P'(1933) = 0.2, then  $P(1934) \approx 5.2$  million. On the other hand, if P'(1933) = 0, then  $P(1934) \approx 5$  million.
- 2. Find the rate of change of the volume of a cube with respect to the length of its side s when s = 3 and s = 5.

Let the area be  $A = f(s) = s^2$ . Then the rate of change of A with respect to s is A'(s) = 2s. When s = 3, the area changes at a rate of 6 square units per unit increase. When s = 5, the area changes at a rate of 10 square units per unit increase.

**3.** The height in feet of a helicopter at time t in minutes is  $s(t) = -3t^3 + 400t$  for  $0 \le t \le 10$ . a. Plot the graphs of height s(t) and velocity v(t). The height is  $s(t) = 400t - 3t^3$  and velocity is  $v(t) = s'(t) = 400 - 9t^2$ .

- b. Find the velocity at t = 6 and t = 7. We have v(6) = 76 ft/min and v(7) = -41 ft/min.
- c. Find the maximum height of the helicopter. The maximum height of the helicopter occurs when v(t) = 0. When  $v(t) = 400 - 9t^2 =$ 0, we have  $t = \pm \frac{20}{3}$ . Discarding the negative time (which occurred before measurements

started) leaves  $t = \frac{20}{3} \approx 6.67$  min. At this time the height is  $s(\frac{20}{3}) = 16000/9 \approx 1777.78$ ft.

4. It takes a stone 3 seconds to hit the ground when dropped from the top of a building. How high is the building and what is the stone's velocity upon impact?

Galileo's formula gives  $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = s_0 - 4.9t^2$ , where the time t is in seconds (s) and the height s is in meters (m). When the ball hits the ground after 3 seconds its height is 0. Solve  $0 = s(3) = s_0 - 4.9(3)^2$  to obtain  $s_0 = 44.1$  m. The velocity at impact is  $v(3) = -9.8t|_{t=3} = -29.4$  m/s.

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5. The demand for a commodity generally decreases as the price is raised. Suppose that the demand for oil (per capita per year) is  $D(p) = \frac{900}{p}$  barrels where p is the price per barrel in dollars. find the demand when p = 40 dollars. Estimate the decrease in demand if p is raised to 41 dollars and the increase if p is decreased to 39 dollars.

 $D(p) = 900p^{-1}$ , so  $D'(p) = -900p^{-2}$ . When the price is \$40 per barrel, the per capita demand is D(40) = 22.5 barrels per year. With an increase in price from \$40 to \$41 per barrel, the change in demand D(41) - D(40) is approximately  $D'(40) = -900(40^{-2}) = -.5625$  barrels per year. With a decrease in price from \$40 to \$39 per barrel, the change in demand D(39) - D(40) is approximately -D'(40) = +.5625. An increase in oil prices of \$1 leads to a decrease in demand of .5625 barrels per year.

### 3.5. Higher Derivatives.

## Class Time 1 period. Essential

## **Key Points**

- Higher derivatives.
- Notations for higher derivatives.

### Lecture Material

Discuss what is meant by higher derivatives and explain Leibnitz notation for higher derivatives. Do a couple of examples, and explain that for polynomials of degree n, the (n + 1)st derivative is 0. As an application of the second derivative, talk about the acceleration of an object tossed vertically in the air. Finally, use the slide provided to show the relationship between the behavior of the first derivative and the sign of the second derivative. Show students how to find the numerical value of higher derivatives on their graphing calculators and/or how to find the higher order derivative on their CAS.

### **Discussion Topics/Class Activities**

Use preliminary Questions 1 & 2 (p. 140) to explain the meaning of second derivatives in practical settings. Do variations on Exercises 35 and 36 where students emphasize how the shapes of f' and f'' give information about the graph of f.

## Suggested Problems

Exercises 1, 3, 9, 11, 15, 17, 29, 39, 40, 41, 51

# Worksheet 3.5. Higher Derivatives

**1.** Calculate the second and third derivatives of  $y = \sqrt{x}$ .

**2.** Find 
$$\frac{d^2}{dt^2}\left(\frac{1}{t^3+1}\right)$$
.

**3.** Find where the second derivative of  $F(x) = \frac{x^2}{x-3}$  is 0.

4. Find a general formula for  $f^{(n)}(x)$  if  $f(x) = x^{-2}$ .

- 1. Calculate the second and third derivatives of  $y = \sqrt{x}$ . Let  $y = \sqrt{x} = x^{\frac{1}{2}}$ . Then  $y' = \frac{1}{2}x^{-\frac{1}{2}}$ ,  $y'' = -\frac{1}{4}x^{-\frac{3}{2}}$ , and  $y''' = \frac{3}{8}x^{-\frac{5}{2}}$ .
- **2.** Find  $\frac{d^2}{dt^2} \left(\frac{1}{t^3+1}\right)$ . Let

$$f(t) = \frac{1}{t^3 + 1}.$$

Then

$$f'(t) = \frac{0 - 3t^2}{\left(t^3 + 1\right)^2}.$$

In order to find f''(t), we will have to take the derivative of  $(t^3 + 1)^2 = (t^3 + 1)(t^3 + 1)$ . We will use the product rule:

$$\frac{d}{dt}(t^3+1)(t^3+1) = 3t^2(t^3+1) + 3t^2(t^3+1) = 6t^2(t^3+1).$$

Applying this derivative inside the quotient rule we are about to take, we get

$$f''(t) = \frac{(t^3 + 1)^2(-6t) + 3t^2(6t^2(t^3 + 1))}{(t^3 + 1)^4}$$
$$= \frac{(t^3 + 1)((t^3 + 1)(-6t) + 18t^4)}{(t^3 + 1)^4}$$
$$= \frac{12t^4 - 6t}{(t^3 + 1)^3}$$

**3.** Find F''(2) if  $F(x) = \frac{x^2}{x-3}$  is 0. From the quotient rule,

$$F'(x) = \frac{(x-3)2x - (x^2)(1)}{(x-3)^2} = \frac{x^2 - 6x}{(x-3)^2}$$

To take the second derivative, we need the derivative of  $(x-3)^2$ . Using the product rule applied to  $(x-3)^2 = (x-3)(x-3)$ , we get:

$$(x-3)1 + (x-3)1 = 2(x-3).$$

Let F'(x) = (f(x)/g(x)), where  $f(x) = x^2 - 6x$  and  $g(x) = (x-3)^2$ . f'(x) = 2x - 6 and g'(x) = 2(x-3). From the quotient rule:

$$F''(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g(2)^2}.$$

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We compute f(2) = -8, g(2) = 1, f'(2) = -2 and g'(2) = -2. Hence  $F''(2) = \frac{1(-2) - (-8)(-2)}{1^2} = -18$ 

4. Find a general formula for 
$$f^{(n)}(x)$$
 if  $f(x) = x^{-2}$ .  
 $f'(x) = -2x^{-3}$ ,  $f''(x) = 6x^{-4}$ ,  $f'''(x) = -24x^{-5}$ ,  $f^{(4)}(x) = 5 \cdot 24x^{-6}$ ,.... We can conclude that the *n*th derivative can be written as  $f^{(n)}(x) = (-1)^n (n+1)! x^{-(n+2)}$ .

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## 3.6. Trigonometric Functions.

Class Time 1 period. Essential.

## **Key Points**

- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x$
- $(\cot x)' = -\csc^2 x$
- $(\sec x)' = \sec x \tan x$
- $(\csc x)' = -\csc x \cot x$

## Lecture Material

First prove that the derivative of  $\sin x$  is  $\cos x$ , and then say that a similar argument shows that the derivative of  $\cos x$  is  $-\sin x$ . From there, give the derivatives for the rest of the trigonometric functions and derive one or two of them using the Quotient Rule. Then do Example 3.

## **Discussion Topics/Class Activities**

Using a graphing calculator, compare the graphs of  $y = \sec x$  and its derivative  $y' = \sec x \tan x$  on the interval  $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ .

### Suggested Problems

Exercises 1, 3, 5, 7, 9, 11, 25, 31, 35, 37, 43, 47, 56

# Worksheet 3.6. Trigonometric Functions

- **1.** Find the derivative of  $f(x) = x^2 \cos x$ .
- **2.** Find the equation of the tangent line to  $g(\theta) = \frac{\theta}{\cos \theta}$  at  $\theta = 0$ .



**3.** Find the second derivative of  $f(x) = \tan x$ .

4. Find points at which  $f(x) = \cos^2 x$  has a horizontal tangent line.

- 1. Find the derivative of  $f(x) = x^2 \cos x$ . Let  $f(x) = x^2 \cos x$ . Then  $f'(x) = x^2 (-\sin x) + (\cos x) (2x) = 2x \cos x - x^2 \sin x$ .
- 2. Find the equation of the tangent line to  $g(\theta) = \frac{\theta}{\cos \theta}$  at  $\theta = 0$ . Let  $g(\theta) = \theta / \cos \theta = \theta \sec \theta$ . Then  $g'(\theta) = \theta \sec \theta \tan \theta + \sec \theta$ . Since g(0) = 0 and g'(0) = 1 the equation of the tangent line is  $y = \theta$ .
- 3. Find the second derivative of  $f(x) = \tan x$ . Let  $f(x) = \tan x$ . Then  $f'(x) = \sec^2 x = \sec x \sec x$  and  $f''(x) = \sec x(\sec x \tan x) + \sec x(\sec x \tan x)$  $= 2(\sec x)(\sec x \tan x) = 2\sec^2 x \tan x$ .
- 4. Find points at which  $f(x) = \cos^2 x$  has a horizontal tangent line.

Writing f as  $(\cos x)(\cos x)$  and using the product rule, we have  $f'(x) = -2\cos x \sin x$ . Then f'(x) = 0 if  $\cos x = 0$  or  $\sin x = 0$ . This happens at  $x = \frac{(2n+1)\pi}{2}$  and at  $x = n\pi$  for all integers n.

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## 3.7. The Chain Rule.

Class Time 1 period. Essential.

## **Key Points**

• Chain Rule:

 $(f \circ g)'(x) = f'(g(x))g'(x)$ 

• General Power Rule:

 $((u(x))^n)' = n(u(x))^{n-1}u'(x)$  for n any real number

## Lecture Material

State the Chain Rule and do several examples including examples using trigonometric functions. If time permits, prove the Chain Rule. Discuss examples of the General Power Rule.

Do Example 4 to foreshadow related rates problems.

### **Discussion Topics/Class Activities**

Have students discuss the salary illustration given in the conceptual insight paragraph in this section.

## Suggested Problems

Exercises 3, 5, 11, 13, 15, 17, 19, 25, 27, 35, 37, 45 (assign additional exercises as needed)

## Worksheet 3.7. The Chain Rule

- 1. Find the derivative of  $f(x) = (7x 9)^5$ .
- **2.** Use the Chain Rule to find the derivative of  $f(t) = \sin^2 t$ .
- **3.** Find the derivative of  $f(x) = \frac{\sqrt{x+1}}{x+2}$ .
- 4. Find the equation of the tangent line to  $f(\theta) = \sin(\cos \theta)$  at  $\theta = \frac{\pi}{2}$ .



5. Find the points at which  $f(x) = \left(\frac{x+1}{x-1}\right)^4$  has a horizontal horizontal tangent line.

1. Find the derivative of  $f(x) = (7x - 9)^5$ 

Let g(x) = 7x - 9. We apply the General Power Rule.

$$\frac{d}{dx}g(x)^5 = \frac{d}{dx}(7x-9)^5 = 5(7x-9)^4(7) = 35(7x-9)^4.$$

Alternately, let  $f(x) = x^5$  and apply the Shifting and Scaling Rule. Then

$$\frac{d}{dx}f(7x-9) = \frac{d}{dx}(7x-9)^5 = 7(5(7x-9)^4) = 35(7x-9)^4.$$

- 2. Use the Chain Rule to find the derivative of  $f(t) = \sin^2 t$ . Let  $y = \sin^2 x = (\sin x)^2$ . Then  $y' = 2 \sin x (\cos x)$ .
- **3.** Find the derivative of  $f(x) = \frac{\sqrt{x+1}}{x+2}$ .  $f'(x) = \frac{(x+2)(\frac{1}{2})(x+1)^{-1/2} - (x+1)^{1/2}}{(x+2)^2} = \frac{-x}{2(x+1)^{1/2}(x+2)^2}$

4. Find the equation of the tangent line to  $f(\theta) = \sin(\cos \theta)$  at  $\theta = \frac{\pi}{2}$ .

 $f'(\theta) = \cos(\cos\theta)(-\sin\theta)$ . Since  $f'(\frac{\pi}{2}) = -1$  and  $f(\frac{\pi}{2}) = 0$ , the equation of the tangent line is  $y = -(x - \frac{\pi}{2})$ .

5. Find the points at which  $f(x) = \left(\frac{x+1}{x-1}\right)^4$  has a horizontal tangent line. Let  $y = \left(\frac{x+1}{x-1}\right)^4$ . Then  $y' = 4\left(\frac{x+1}{x-1}\right)^3 \cdot \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2}$  $= -\frac{8(x+1)^3}{(x-1)^5} = \frac{8(1+x)^3}{(1-x)^5}$ 

Thus f has a horizontal tangent line at x = -1

Class Time AB 2 periods; BC 1–2 periods. Essential.

### **Key Points**

• The derivative of the inverse: If f(x) is differentiable and one-to-one with inverse g(x), then

$$g'(x) = \frac{1}{f'(g(x))}$$

• Derivative and integral formulas:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$$
$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \qquad \int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1}x + C$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{x^2+1}$$
$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

## Lecture Material

Reflect the graph of f(x) and the line tangent to the graph of f(x) at (a,b), y = b + f'(a)(x-a), across the line y = x to obtain the graph of  $g(x) = f^{-1}(x)$  and the tangent line at  $(b, a) = (b, f^{-1}(b))$  (see Figure 1):

$$y - a = \frac{1}{f'(a)}(x - b)$$

Extract from this Theorem 1: Assume that f(x) is differentiable and one-to-one with inverse g(x). If b belongs to the domain of g(x) and  $f'(g(b)) \neq 0$ , then

$$g'(b) = \frac{1}{f'(g(b))}$$

Illustrate the use of this result by working Exercise 4. Now apply Theorem 1 to the inverse trigonometric functions to obtain the formulas for the derivatives of  $\sin^{-1} x$  and  $\cos^{-1}(x)$  (the other inverse trigonometric functions are handled similarly, and they can now be simply stated). Work Exercises 25 and 27 to illustrate the use of these formulas.

Another way to derive these formulas is to use implicit differentiation (see Section 3.10). For example: To find the derivative of  $y = \sin^{-1} x$ , write  $x = \sin y$ . Then differentiate implicitly and solve for  $\frac{dy}{dx}$ :

$$1 = \cos y \frac{dy}{dx}$$
$$\frac{1}{\cos y} = \frac{dy}{dx}$$
$$\frac{1}{\cos(\sin^{-1} x)} = \frac{dy}{dx}$$
$$\frac{1}{\sqrt{1 - x^2}} = \frac{dy}{dx}$$

Suggested Problems

Exercises 3, 7, 11, 13, 23-37 odd

# Worksheet 3.8. Derivatives of Inverse Functions

**1.** If  $f(x) = \sqrt{x^3 + 8}$ , then

- a. the domain of f is \_\_\_\_\_ and the range of f is \_\_\_\_\_
- b. Find a formula for  $f^{-1}(x)$ . What are the domain and range of  $f^{-1}(x)$ ?

**2.** Sketch the graph of the inverse  $f^{-1}(x)$  for the function f(x).



**3.** If  $f(x) = e^x$ , sketch the graphs of f(x),  $-2 \le x \le 2$ , and the inverse  $f^{-1}(x)$ .



**4.** Suppose that f is an invertible function satisfying f(3) = 2 and f'(3) = -4. If  $g(x) = f^{-1}(x)$ , find g'(2).

5. Find 
$$g'\left(-\frac{1}{2}\right)$$
 if  $g(x)$  is the inverse of  $f(x) = \frac{x^3}{x^2+1}$ . (Do not try to find a formula for  $f^{-1}(x)$ .)

- **1.** If  $f(x) = \sqrt{x^3 + 8}$ , then
  - a. The domain of f is domain $(f) = [-2, \infty)$ , and the range of f is range $(f) = [0, \infty)$ .
  - b. A formula is  $f^{-1}(x) = \sqrt[3]{x^2 8}$ . The domain is domain $(f^{-1}) = [0, \infty)$ , and the range is range $(f^{-1}) = [-2, \infty)$ .
- **2.** Sketch the graph of the inverse  $f^{-1}(x)$  for the function f(x).



**3.** If  $f(x) = e^x$ , sketch the graphs of f(x),  $-2 \le x \le 2$ , and the inverse  $f^{-1}(x)$ .



4. Suppose that f is an invertible function satisfying f(3) = 2 and f'(3) = -4. If  $g(x) = f^{-1}(x)$ , then find g'(2).  $g'(2) = \frac{1}{1} = -\frac{1}{1}$ 

$$g'(2) = \frac{1}{f'(g(2))} = -\frac{1}{4}$$
  
5. Find  $g'\left(-\frac{1}{2}\right)$  if  $g(x)$  is the inverse of  $f(x) = \frac{x^3}{x^2 + 1}$ .  
Since  $f(-1) = -\frac{1}{2}$ ,  $g'\left(-\frac{1}{2}\right) = \frac{1}{f'(-1)} = \frac{1}{2}$ .

#### 3.9. Derivatives of Exponential and Logarithmic Functions.

Class Time 1 period. Essential.

## **Key Points**

• Derivative formulas:

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x}, \qquad \frac{d}{dx}b^x = (\ln b)b^x, \qquad \frac{d}{dx}\log_b x = \frac{1}{(\ln b)x}$$

The hyperbolic functions are not tested on the AB or BC exams and may be omitted.

### Lecture Material

Begin by deriving the formula for the derivatives of exponential functions:  $\frac{d}{dx}b^x = (\ln b)b^x$ , b > 0. Then, using the differentiation formula for inverse functions from Section 3.8 (Theorem 1) derive the formula for the derivative of  $\ln x$ :  $\frac{d}{dx}\ln x = \frac{1}{x}$  for x > 0. It may be useful to point out the chain rule form of this formula:  $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$ . Work Exercises 4, 16, and 18 to illustrate the use of these formulas.

Logarithmic differentiation is not tested on the AB or BC exams and may be omitted.

## Discussion Topics/Class Activities

Discuss Exercise 84, which gives an alternative derivation of the product rule.

### Suggested Problems

Exercises 1, 5, 9, 11, 13, 15, 17, 19, 21, 25, 31

# Worksheet 3.9. Derivatives of Exponential and Logarithmic Functions

**1.** Find the derivative of  $y = \ln(x^2)$ .

**2.** Find the derivative of 
$$y = \ln\left(\frac{x+1}{x^3+1}\right)$$
.

**3.** Find the derivative of  $y = 5^{x^2-x}$ .

4. Find the derivative of  $y = x^{x^2}$  using the methods of Example 9.

- 1. Find the derivative of  $y = \ln(x^2)$  $y' = \frac{2}{x}$  if  $x \neq 0$ .
- 2. Find the derivative of  $y = \ln\left(\frac{x+1}{x^3+1}\right)$  $y' = \left(\frac{x^3+1}{x+1}\right) \left(\frac{(x^3+1)-3x^3-3x^2}{(x^3+1)^2}\right) = \left(\frac{x^3+1}{x+1}\right) \left(\frac{-2x^3-3x^2+1}{(x^3+1)^2}\right)$   $= \left(\frac{x^3+1}{x+1}\right) \left(\frac{(1+x)^2(1-2x)}{(x^3+1)^2}\right) = \frac{(1+x)(1-2x)}{x^3+1} = \frac{(1+x)(1-2x)}{(x+1)(x^2-x+1)} = \frac{1-2x}{x^2-x+1}.$
- **3.** Find the derivative of  $y = 5^{x^2-x}$  $y' = \ln 5(2x-1)5^{x^2-x}$ .
- 4. Find the derivative of  $y = x^{x^2}$  using the methods of Example 9 First method:  $y(x) = x^{x^2} = (e^{\ln x})^{x^2} = e^{x^2 \ln x}$ . Thus  $y'(x) = (x + 2x \ln x)e^{x^2 \ln x} = (x + 2x \ln x)x^{x^2}$ .

Second method: Since  $\ln y = x^2 \ln x$ , we have  $\frac{y'(x)}{y(x)} = \frac{x^2}{x} + 2x \ln x = x + 2x \ln x$ . Thus  $y'(x) = y(x + 2x \ln x) = x^{x^2}(x + 2x \ln x)$ .

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### 3.10. Implicit Differentiation.

Class Time AB 2 periods; BC 1–2 periods. Essential.

### **Key Points**

- Implicit differentiation.
- Implicit differentiation for higher order derivatives.

### Lecture Material

Begin by asking students to find  $\frac{dy}{dx}$  for  $x^2 + y^2 = 1$  by first solving for y and differentiating. Then show them how to differentiate this relation implicitly and point out that the answers are really the same. Mention and show graphically that, since there is more than one point with the same x-coordinate, it is natural that the derivative should depend on both x and y. Use both forms to find the slope at  $\left(\frac{3}{5}, \frac{4}{5}\right)$  and  $\left(\frac{3}{5}, -\frac{4}{5}\right)$ . Then do several more complicated examples such as  $y^2 + 2xy = x^3$ . Finally, show how to find second derivatives for the same examples.

### **Discussion Topics/Class Activities**

Put the lemniscate slide up and have students work Exercise 56. If there is time, have students work Exercise 53.

Suggested Problems (spread over 2 assignments) Exercises 1, 3, 6, 9, 11, 21, 25, 31, 37, 46, 55, 57, 59, 61

# Worksheet 3.10. Implicit Differentiation

**1.** Find the derivative of y with respect to x if  $y^4 - y = x^3 + x$ .

**2.** Find the equation of the tangent line to the graph of  $x^2y^3 + 2y = 3x$  at the point (2, 1).



3. Show that there are no points on the graph of  $x^2 - 3xy + y^2 = 1$  where the tangent line is horizontal.

4. Find the equations of the tangent lines at the four points where x=1 on the folium  $(x^2+y^2)^2=\frac{25}{4}xy^2$ 

.



### Solutions to Worksheet 3.10

1. Find the derivative of y with respect to x if  $y^4 - y = x^3 + x$ 

$$\frac{d}{dx}(y^4 - y) = \frac{d}{dx}(x^3 + x)$$

$$\frac{4y^3y' - y'}{4y^3 - 1} = \frac{3x^2 + 1}{(3x^2 + 1)}$$

$$y' = \frac{3x^2 + 1}{4y^3 - 1}$$

2. Find the equation of the tangent line to the graph of  $x^2y^3 + 2y = 3x$  at the point (2, 1). Taking the derivative of both sides of  $x^2y^3 + 2y = 3x$  yields

$$3x^2y^2y' + 2xy^3 + 2y' = 3.$$

Substituting x = 2, y = 1 yields 12y' + 4 + 2y' = 3, and we solve:

$$12y' + 4 + 2y' = 3$$
  

$$14y' = -1$$
  

$$y' = -\frac{1}{14}.$$

Hence, the equation of the tangent line at (2,1) is  $y-1 = -\frac{1}{14}(x-2)$ , or  $y = \frac{8}{7} - \frac{1}{14}x$ .

3. Show that there are no points on the graph of  $x^2 - 3xy + y^2 = 1$  where the tangent line is horizontal.

Let the implicit curve  $x^2 - 3xy + y^2 = 1$  be given. Then

$$2x - 3xy' - 3y + 2yy' = 0$$

whence

$$y' = \frac{2x - 3y}{3x - 2y} = 0$$

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which implies  $y = \frac{2}{3}x$ . Substituting  $y = \frac{2}{3}x$  into the equation of the implicit curve gives  $-\frac{5}{9}x^2 = 1$ , which has *no* real solutions. Accordingly, there are *no* points on the implicit curve where the tangent line has slope zero.

4. Find the equations of the tangent lines at the 4 points where x = 1 on the folium

$$(x^2 + y^2)^2 = \frac{25}{4}xy^2$$

First, find the points (1, y) on the curve. Setting x = 1 in the equation  $(x^2 + y^2)^2 = \frac{25}{4}xy^2$  yields

$$(1+y^2)^2 = \frac{25}{4}y^2$$
$$y^4 + 2y^2 + 1 = \frac{25}{4}y^2$$
$$4y^4 + 8y^2 + 4 = 25y^2$$
$$4y^4 - 17y^2 + 4 = 0$$
$$(4y^2 - 1)(y^2 - 4) = 0$$
$$y^2 = \frac{1}{4} \text{ or } y^2 = 4$$

Hence  $y = \pm \frac{1}{2}$  or  $y = \pm 2$ . Taking the derivative of both sides of the original equation yields

$$2(x^{2} + y^{2})(2x + 2yy') = \frac{25}{4}y^{2} + \frac{25}{2}xyy'$$

$$4(x^{2} + y^{2})x + 4(x^{2} + y^{2})yy' = \frac{25}{4}y^{2} + \frac{25}{2}xyy'$$

$$(4(x^{2} + y^{2}) - \frac{25}{2}x)yy' = \frac{25}{4}y^{2} - 4(x^{2} + y^{2})x$$

$$y' = \frac{\frac{25}{4}y^{2} - 4(x^{2} + y^{2})x}{(y(4(x^{2} + y^{2}) - \frac{25}{2}x))}$$

• At (1,2),  $(x^2 + y^2) = 5$ , and

$$y' = \frac{\frac{25}{4}4 - 4(1+4)}{2(4(5) - \frac{25}{2})} = \frac{1}{3}$$

Hence, at (1,2), the equation of the tangent line is  $y = \frac{1}{3}(x-1) + 2$ .

• At (1, -2),  $(x^2 + y^2) = 5$  as well, and  $y' = \frac{5}{-15} = -\frac{1}{3}$ 

Hence, at (1, -2), the equation of the tangent line is  $y = -\frac{1}{3}(x-1) - 2$ . • At  $(1, \frac{1}{2})$ ,  $x^2 + y^2 = \frac{5}{4}$ , and  $y' = \frac{\frac{25}{16} - 5}{\frac{1}{2}(5 - \frac{25}{2})} = \frac{55}{16}\frac{4}{15} = \frac{11}{12}$ . Hence, at  $(1, \frac{1}{2})$ , the equation of the tangent line is  $y = \frac{11}{12}(x-1) + \frac{1}{2}$ .

Hence, at  $(1, \frac{1}{2})$ , the equation of the tangent line is  $y = \frac{12}{12}(x-1) + \frac{1}{2}$ . • At  $(1, -\frac{1}{2})$ ,  $x^2 + y^2 = \frac{5}{4}$ , and  $y' = \frac{\frac{25}{16} - 5}{-\frac{1}{2}(5 - \frac{25}{2})} = \frac{55}{16} \frac{4}{-15} = -\frac{11}{12}$ Hence, at  $(1, -\frac{1}{2})$ , the equation of the tangent line is  $y = -\frac{11}{12}(x-1) - \frac{1}{2}$ .

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### 3.11. Related Rates.

### Class Time 2 periods. Essential.

### Key Point

• Related Rates.

### Lecture Material

Explain that related rates are just an application of the Chain Rule. All the variables are functions of time, so when you take the derivative you have to use the Chain Rule. Provide examples using the Pythagorean Theorem, similar triangles, and trigonometric functions. The examples in the text are fine, or work Exercises 13, 20 and 21.

Solving related rate problems usually begins with finding a geometric relationship among the variables. This is often more difficult than doing the "calculus." Work Exercises 31 and 32 which have no geometry associated with them. This may help students better understand the calculus involved.

### Discussion Topics/Class Activities

Lead a discussion on how to solve Exercise 43 and have students work out the details at their desks. Students have difficulty with related rate problems so you should plan 2 periods on this topic. Supplement with questions from past AP Calculus exams.

Suggested Problems (spread over 2 assignments) Exercises 5, 7, 13, 16, 17, 19, 23, 25, 29, 30, 37, 39, 41

## Worksheet 3.11. Related Rates

- 1. The volume of a sphere of radius r is  $V = \frac{4}{3}\pi r^3$ . If the radius is expanding at a rate of 14 in/min, at what rate is the volume changing when r = 8 in.
- 2. Sonya and Isaac are in motorboats located at the center of a lake. At time t = 0, Sonya begins traveling south at a speed of 32 mph. At the same time Isaac takes off, heading east at a speed of 27 mph.
  - a. How far have Sonya and Issac traveled after 12 min?
  - b. At what rate are they separating after 12 min?

3. At a given moment, a plane passes directly above a radar station at an altitude of 6 miles and a speed of 500 mph. Suppose  $\theta$  is the angle between the line segment joining the plane and the radar station and the horizontal. How fast is  $\theta$  changing 10 minutes after the plane passes over the radar station?



4. As a man walks away from a 12-ft lamppost, the tip of his shadow moves twice as fast as he does. What is the man's height?

5. Calculate the rate in square centimeters per second at which area is swept out by the second-hand of a circular clock as a function of the clock's radius.

### Solutions to Worksheet 3.11

1. The volume of a sphere of radius r is  $V = \frac{4}{3}\pi r^3$ . If the radius is expanding at a rate of 14 in/min, at what rate is the volume changing when r = 8 in?

As the radius is expanding at 14 in/min, we know that  $\frac{dr}{dt} = 14$  in/min. Taking the derivative with respect to t of the equation  $V = \frac{4}{3}\pi r^3$  yields

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

Substituting r = 8 and  $\frac{dr}{dt} = 14$ ,  $\frac{dV}{dt} = 4\pi r^2 (14) = 56\pi (64) = 3584\pi \text{ in/min}$ 

- **2.** Sonya and Isaac are in motorboats located at the center of a lake. At time t = 0, Sonya begins traveling south at a speed of 32 mph. At the same time Isaac takes off, heading east at a speed of 27 mph.
  - a. How far have Sonya and Issac traveled after 12 minutes?

With Isaac x miles east of the center of the lake and Sonya y miles south of its center,

let *h* be the distance between them. After 12 minutes or  $\frac{12}{60} = \frac{1}{5}$  hour, Isaac has traveled  $\frac{1}{5} \times 27 = \frac{27}{5}$  miles and Sonya has traveled  $\frac{1}{5} \times 32 = \frac{32}{5}$  miles.

b. At what rate are they separating after 12 minutes? We have  $h^2 = x^2 + y^2$  and

$$2h\frac{dh}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

whence

$$\frac{dh}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{h} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}}$$

Substituting  $x = \frac{27}{5}$ ,  $\frac{dx}{dt} = 27$ ,  $y = \frac{32}{5}$ , and  $\frac{dy}{dt} = 32$ ,  $\frac{dh}{dt} = \frac{\left(\frac{27}{5}\right)(27) + \left(\frac{32}{5}\right)(32)}{\sqrt{\left(\frac{27}{5}\right)^2 + \left(\frac{32}{5}\right)^2}} = \sqrt{1753} \approx 41.87 \text{ mph}$  3. At a given moment, a plane passes directly above a radar station at an altitude of 6 miles and a speed of 500 mph. Suppose  $\theta$  is the angle between the line segment joining the plane and the radar station and the horizontal. How fast is  $\theta$  changing 10 minutes after the plane passes over the radar station?

Let x be the distance of the plane from the station along the ground and h the distance through the air.

(1) By the Pythagorean Theorem,

$$h^{2} = x^{2} + 6^{2} = x^{2} + 36$$
After an half hour,  $x = \frac{1}{2} \times 500 = 250$  miles. Thus  $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$ , whence  $\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}$ . With  $x = 250$ ,  $h = \sqrt{250^{2} + 36}$ , and  $\frac{dx}{dt} = 500$ ,  
 $\frac{dh}{dt} = \frac{250}{\sqrt{250^{2} + 36}} \times 500 \approx 499.86$  mph

(2) When the plane is directly above the station, x = 0, so the distance between the plane and the station is not changing, for at this instant

$$\frac{dh}{dt} = \frac{0}{6} \times 500 = 0 \text{ mph}$$

4. As a man walks away from a 12-ft lamppost, the tip of his shadow moves twice as fast as he does. What is the man's height?

Let L be the length from the base of the lamppost to the tip of the man's shadow. Let x be the distance from the base of the lamppost to the man's feet. Let h be the man's height. The right triangle with legs L - x, h (formed by the man and his shadow) and the right triangle with legs L, 12 (formed by the lamppost and the total length L) are similar. By this similarity,

$$\frac{L-x}{h} = \frac{L}{12}$$

h is constant, so taking the derivative of both sides of this equation yields

$$\frac{1}{h}(\frac{dL}{dt} - \frac{dx}{dt}) = \frac{1}{12}\frac{dL}{dt}$$

The problem statement states that L'(t) = 2x'(t), so 12 = 2h. Hence, h = 6 ft.

5. Calculate the rate in square centimeters per second at which area is swept out by the second-hand of a circular clock as a function of the clock's radius.

Let r be the radius of the circular clock in centimeters. In 60 seconds, the second hand sweeps out the full area of the circular clock face,  $A = \pi r^2$ . Therefore, the constant rate at which area is swept out by the second-hand is  $\frac{\pi r^2}{60}$  cm<sup>2</sup>/s.

# Chapter 3 AP Problems

1. 
$$\lim_{h \to 0} \frac{(2+h)^5 - 2^5}{h} =$$
A. 0
B. 1
C. 32
D. 80
E. 160

- 2. Given that  $f(x) = \sqrt{x} x^2$  for  $x \ge 0$ , a. Find f'(x).
  - b. For what value(s) of x, if any, does f(x) = 0?
  - c. For what value(s) of x, if any, does f'(x) = 0?
  - d. For what value(s) of x does f have a vertical tangent?

**3.** If 
$$f(x) = 2x\sqrt{x} - 3\sqrt{x} + \frac{1}{\sqrt{x}}$$
, then  $f'(x) =$   
A.  $2\sqrt{x} - \frac{3}{\sqrt{x}} + \frac{1}{x\sqrt{x}}$   
B.  $3\sqrt{x} - \frac{3}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$   
C.  $3\sqrt{x} - \frac{3}{\sqrt{x}} + \frac{1}{2x\sqrt{x}}$   
D.  $\frac{4x - 3}{\sqrt{x}}$   
E.  $2\sqrt{x} - 3 + \frac{1}{x\sqrt{x}}$ 

- 4. Given a function f such that f(1) = 2, f'(1) = -4 and  $h(x) = \frac{x^2}{f(x)}$ , what is the value of h'(1)?
  - A. -2 B. -1/2 C. 0 D. 1

E. 2

- 5. The position of a ball rolling down an inclined plane 8 meters long is given by the formula s = .2t<sup>2</sup> + .6t, where s is the number of meters traveled after t seconds.
  a. How far has the ball traveled after 2 seconds?
  - a. How far has the ban traveled after 2 seconds:
  - b. How fast is the ball traveling after 2 seconds? Indicate the units of measure.
  - c. What is the average velocity of the ball on the interval from t = 1 to t = 3 seconds?
  - d. Write an equation for v, the velocity of the ball at any time t, and use it to compute the velocity of the ball at the instant that it reaches the end of the inclined plane.

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- 6. The three curves labeled I, II, and III depicted in the graph below represent a function f and its first and second derivatives. If the functions were listed in the order f, f', f'', what would be the corresponding order of their labels?
  - A. I, II, III
  - B. II, III, I
  - C. II, I, III
  - D. I, III, II
  - E. III, II, I



- 7. If  $f(x) = x^2 \sin(x) + 2x \cos(x)$ , then f'(x) =
  - A.  $2x \cos(x) 2\sin(x)$
  - B.  $2x + \cos(x) + 2 \sin(x)$
  - C.  $x^{2}\cos(x) + 4x\sin(x) 2\cos(x)$
  - D.  $x^{2}\cos(x) + 2\cos(x)$
  - E.  $2x\sin(x) 2\cos(x)$
- 8. If  $h(x) = \sin(3x)$ , then  $h^{(99)}(x) =$ 
  - A.  $3^{99}\sin(3x)$
  - B.  $-3^{99}\sin(3x)$
  - C.  $3^{99}\cos(3x)$
  - D.  $-3^{99}\cos(3x)$
  - E.  $99\sin(3x)$
- 9. Consider the curve given by  $2x^2 xy y^2 + 18 = 0$ . a. Show that  $\frac{dy}{dx} = \frac{4x - y}{x + 2y}$ .
  - b. Find the x coordinates of all points on the curve where the tangent line is horizontal.

c. Show that there is no point on the curve where the slope of the tangent line is undefined.

- 10. A tank full of water in the shape of a cone with point facing down has a height of 12 meters and a radius of 4 meters. The tank is emptying at a rate of  $\frac{\pi h}{3}$  cubic meters per minute.  $(V = \frac{1}{3}\pi r^2 h)$ 
  - a. Write an equation that expresses the volume in terms of h only.

b. Find 
$$\frac{dh}{dt}$$
 in terms of  $h$ .

c. At what rate is the depth of the water changing when the volume is  $27\pi$  cubic meters?

## Solutions to Chapter 3 AP Problems



## **D** [THIS QUESTION CORRESPONDS WITH SECTION 3.1]. The limit formula is equivalent to evaluating the derivative of $y = x^5$ at x = 2.

2. Given that  $f(x) = \sqrt{x} - x^2$  for  $x \ge 0$ , a. Find f'(x).

$$f'(x) = \frac{1}{2}x^{-1/2} - 2x = \frac{1 - 4x^{3/2}}{2\sqrt{x}}$$

b. For what value(s) of x, if any, does f(x) = 0?

x = 0, x = 1

- c. For what value(s) of x, if any, does f'(x) = 0?
  - $x = \frac{1}{8}$

d. For what value(s) of x does f have a vertical tangent?

### x = 0[THIS QUESTION CORRESPONDS WITH SECTION 3.2]

3. If 
$$f(x) = 2x\sqrt{x} - 3\sqrt{x} + \frac{1}{\sqrt{x}}$$
, then  $f'(x) =$   
A.  $2\sqrt{x} - \frac{3}{\sqrt{x}} + \frac{1}{x\sqrt{x}}$   
B.  $3\sqrt{x} - \frac{3}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$   
C.  $3\sqrt{x} - \frac{3}{\sqrt{x}} + \frac{1}{2x\sqrt{x}}$   
D.  $\frac{4x - 3}{\sqrt{x}}$   
E.  $2\sqrt{x} - 3 + \frac{1}{x\sqrt{x}}$   
B [THIS QUESTION CORRESPONDS WITH SECTION 3.2]

- 4. Given a function f such that f(1) = 2, f'(1) = -4 and  $h(x) = \frac{x^2}{f(x)}$ , what is the value of h'(1)?
  - B. -1/2C. 0 D. 1 E. 2 E [THIS QUESTION CORRESPONDS WITH SECTION 3.3] We have  $h'(x) = \frac{f(x) \cdot 2x - x^2 \cdot f'(x)}{f(x)^2}$ , so  $h'(1) = \frac{2 \cdot 2 - 1 \cdot (-4)}{4} = 2$ .

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A. -2

- 5. The position of a ball rolling down an inclined plane 8 meters long is given by the formula  $s = .2t^2 + .6t$ , where s is the number of meters traveled after t seconds.
  - a. How far has the ball traveled after 2 seconds?

 $2~{\rm meters}$ 

b. How fast is the ball traveling after 2 seconds? Indicate the units of measure.

1.4 meters/second

c. What is the average velocity of the ball on the interval from t = 1 to t = 3 seconds?

1.4 meters/second

d. Write an equation for v, the velocity of the ball at any time t, and use it to compute the velocity of the ball at the instant that it reaches the end of the inclined plane.

v = .4t + .6; s = 8 when t = 5, so v(5) = 2.6 meters/second [THIS QUESTION CORRESPONDS WITH SECTION 3.4] 6. The three curves labeled I, II, and III depicted in the graph below represent a function f and its first and second derivatives. If the functions were listed in the order f, f', f'', what would be the corresponding order of their labels?

A. I, II, III

- B. II, III, I
- C. II, I, III
- D. I, III, II
- E. III, II, I



# E [THIS QUESTION CORRESPONDS WITH SECTION 3.5]

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7. If  $f(x) = x^2 \sin(x) + 2x \cos(x)$ , then f'(x) =A.  $2x \cos(x) - 2\sin(x)$ B.  $2x + \cos(x) + 2 - \sin(x)$ C.  $x^2 \cos(x) + 4x \sin(x) - 2\cos(x)$ D.  $x^2 \cos(x) + 2\cos(x)$ E.  $2x \sin(x) - 2\cos(x)$ 

## D [THIS QUESTION CORRESPONDS WITH SECTION 3.6]

8. If 
$$h(x) = \sin(3x)$$
, then  $h^{(99)}(x) =$ 

- A.  $3^{99}\sin(3x)$
- B.  $-3^{99}\sin(3x)$
- C.  $3^{99}\cos(3x)$
- D.  $-3^{99}\cos(3x)$
- E.  $99\sin(3x)$

## D [THIS QUESTION CORRESPONDS WITH SECTION 3.7]

9. Consider the curve given by  $2x^2 - xy - y^2 + 18 = 0$ . a. Show that  $\frac{dy}{dx} = \frac{4x - y}{x + 2y}$ .

We have 
$$4x - x\frac{dy}{dx} - y - 2y\frac{dy}{dx} = 0$$
, so  $4x - y = (x + 2y)\frac{dy}{dx}$ , i.e.  $\frac{dy}{dx} = \frac{4x - y}{x + 2y}$ .

b. Find the x coordinates of all points on the curve where the tangent line is horizontal.

$$x = \pm 1$$

c. Show that there is no point on the curve where the slope of the tangent line is undefined.

The slope is undefined when x + 2y = 0, so x = -2y. After substitution into the original equation, there is no real solution.

[THIS QUESTION CORRESPONDS WITH SECTION 3.10]

- 10. A tank full of water in the shape of a cone with point facing down has a height of 12 meters and a radius of 4 meters. The tank is emptying at a rate of  $\frac{\pi h}{3}$  cubic meters per minute.  $(V = \frac{1}{3}\pi r^2 h)$ 
  - a. Write an equation that expresses the volume in terms of h only.

$$V = \frac{1}{27}\pi h^3$$

b. Find 
$$\frac{dh}{dt}$$
 in terms of  $h$ .

$$\frac{dh}{dt} = \frac{-3}{h}$$

c. At what rate is the depth of the water changing when the volume is  $27\pi$  cubic meters?

# $\frac{dh}{dt} = -\frac{1}{3}$ [THIS QUESTION CORRESPONDS WITH SECTION 3.11]

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### Ray Cannon's Chapter 4 Overview

Chapter 2 provided the theoretical groundwork for the derivative, followed by Chapter 3, which gave us the theorems that allow us to compute derivatives easily. Chapter 4 now offers some applications of the derivative, and gives us a glimpse of how powerful Calculus is in problem solving. Section 4.1 returns to the idea of local linearization and approximating values of f(x) by corresponding values of the tangent line approximation. Section 4.2 states the theorem (Theorem 1) that guarantees the existence of maximum and minimum values of f(x) if f is continuous on an interval of the form [a, b]. AP students should know the statement of this theorem, but are not responsible for its proof. The students are also responsible for the language in this section (critical point, local (or relative) maximum, etc.) They should also know what AP Readers sometimes call "the Candidates Test" (Theorem 3) for finding max/min on [a, b]; that is, one need only look at the values of f(x) at the end points and critical points. This is an *extremely* important theorem.

Students should also be able to state and use the Mean Value Theorem. In Section 4.3, note that the definitions of *increasing* and *decreasing* are pre-calculus definitions. Having a positive derivative and increasing are not synonymous, as shown for example by  $f(x) = x^3$  which is increasing on  $(-\infty, \infty)$ . Note also that sign charts can be very helpful in deciding the behavior of f, but by themselves are not sufficient for justifying a local or absolute extreme value on the AP exam; a statement of the First Derivative Test, Second Derivative Test, or the Candidates Test is required.

Section 4.4 uses f'(x) increasing as the definition of *concave up*; some texts use f''(x) > 0. Either characterization is acceptable on the AP exam. Care must be taken to note that the Second Derivative Test is a local test, and by itself is not a justification for a global max or min. L'Hôpital's Rule is covered in Section 4.5, which is required for BC students, but is not part of the AB syllabus. Section 4.6 puts all the ideas of shape of the graph together with asymptotes to give a comprehensive view of how to graph a function.

Section 4.7 shows the power of calculus to solve problems. This is a difficult section for students, and many have difficulty "getting started" with these problems. Many teachers find it profitable to give just a very few of these problems in any one assignment, but then give many assignments that contain one or two of these problems even if the main focus of the assignment is something else. Newtons Method is the focus of Section 4.8, and is a nice example of using the tangent line, this time to approximate the zeros of a function, but is not required of either AB or BC students. The last section of Chapter 4, Section 4.9, deals with finding antiderivatives, a familiar topic on the AP exams, especially with problems dealing with linear motion.

# 4. Applications of the Derivative

### 4.1. Linear Approximation and Applications.

### Class Time AB 2 periods; BC 1 period. Essential

### **Key Points**

- The Linear Approximation is an estimate of the change  $\Delta f = f(a + \Delta x) f(a)$ of a function f at a and is given by  $\Delta f \approx f'(a)\Delta x$ .
- The linearization L(x) of f at x = a is given by L(x) = f'(a)(x a) + f(a); L(x) is the line tangent to the curve y = f(x) at x = a. L(x) can be used to approximate f(x) provided that a is close to x.
- The error in Linear Approximation is

$$\text{Error} = |\Delta f - f'(a)\Delta x|$$

### Lecture Material

Linear Approximation and linearization are conceptually straightforward applications of the derivative. Point out that by definition  $f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ , so if  $\Delta x$  is small,  $\Delta f \approx f'(a)\Delta x$  (illustrated graphically in Figure 1). Now work Exercises 6 and 11. Conceptually, the linearization L(x) at x = a approximates f(x) for x close to a is easiest seen by using a graphing calculator to "zoom" in on the graph of f(x) at a, or by Figure 4. Eventually, if and only if the function is differentiable, the graph will resemble a line, in particular, the tangent line. Observe that away from the point, linearization is not a good approximation, as in Figure 5. Now work Exercise 44.

Percentage error and differentials are not tested on the AB and BC exams.

#### **Discussion Topics/Class Activities**

Have the class graph a function together with a tangent line at some point, and then see how long it takes students to zoom with their calculators so that the function and the tangent line are indistinguishable.

### Suggested Problems

Exercises 1, 3, 7, 17, 19, 21, 25, 30, 35, 39

# Worksheet 4.1. Linear Approximation and Applications

1. Let  $f(x) = \sqrt{1+x}$ , a = 8, and  $\Delta x = 1$ . Estimate  $\Delta f$  using the Linear Approximation and use a calculator to compute the error.

2. Estimate  $16.5^{1/4} - 16^{1/4}$  using the Linear Approximation and find the error using a calculator.

**3.** Approximate  $(27.03)^{1/3}$  using linearization and use a calculator to compute the error.

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- 4. When the price of a bus pass from Albuquerque to Los Alamos is set at x dollars, the bus company takes in a monthly revenue of  $R(x) = 1.5x 01.x^2$  (in thousands of dollars).
  - (a) Estimate the change in revenue if the price is raised from \$50 to \$53.
  - (b) Suppose that x = 80. How will revenue be affected by a small increase in price? Explain using the Linear Approximation.

### Solutions to Worksheet 4.1

1. Let  $f(x) = \sqrt{1+x}$ , a = 8, and  $\Delta x = 1$ . Estimate  $\Delta f$  using the Linear Approximation and use a calculator to compute the error.

Let  $f(x) = (1+x)^{1/2}$ , a = 8, and  $\Delta x = 1$ . Then  $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ ,  $f'(a) = f'(8) = \frac{1}{6}$ and  $\Delta f \approx f'(a)\Delta x = \frac{1}{6}(1) = \frac{1}{6} \approx .166667$ . The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(9) - f(8) = \sqrt{10 - 3} \approx .162278.$$

The error in the Linear Approximation is therefore |.162278 - .166667| = .004389.

2. Estimate  $16.5^{1/4} - 16^{1/4}$  using the Linear Approximation and find the error using a calculator.

Let 
$$f(x) = x^{1/4}$$
,  $a = 16$ , and  $\Delta x = .5$ . Then  $f'(x) = \frac{1}{4}x^{-3/4}$  and  $f'(a) = f'(16) = \frac{1}{32}$ .

- The Linear Approximation is  $\Delta f \approx f'(a)\Delta x = \frac{1}{32}(.5) = .015625.$
- The actual change is

$$\Delta f = f(a + \Delta x) - f(a) = f(16.5) - f(16) \approx 2.015445 - 2 = .015445$$

• The error in this estimate is  $|.015625 - .015445| \approx .00018$ .

3. Approximate  $(27.03)^{1/3}$  using linearization and use a calculator to compute the error.

Let  $f(x) = x^{1/3}$ , a = 27, and  $\Delta x = .03$ . Then  $f'(x) = \frac{1}{3}x^{-2/3}$ ,  $f'(a) = f'(27) = \frac{1}{27}$ and the linearization to f(x) is

$$L(x) = f'(a)(x-a) + f(a) = \frac{1}{27}(x-27) + 3 = \frac{1}{27}x + 2.$$

Thus, we have  $(27.03)^{1/3} \approx L(27.03) \approx 3.0011111$ .

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- 4. When the price of a bus pass from Albuquerque to Los Alamos is set at x dollars, the bus company takes in a monthly revenue of  $R(x) = 1.5x 01.x^2$  (in thousands of dollars).
  - (a) Estimate the change in revenue if the price is raised from \$50 to \$53.
  - (b) Suppose that x = 80. How will revenue be affected by a small increase in price? Explain using the Linear Approximation.
  - (1) If the price is raised from \$50 to \$53, then  $\Delta x = 3$  and

$$\Delta R \approx R'(50)\Delta x = (1.5 - 0.02(50))(3) = 1.5.$$

We therefore estimate an increase of \$1500 in revenue.

(2) Because R'(80) = 1.5 - 0.02(80) = -0.1, the Linear Approximation gives  $\Delta R \approx -0.1\Delta x$ . A small increase in price would thus result in a decrease in revenue.

### 4.2. Extreme Values.

### Class Time 2 periods. Essential.

### Key Points

- The extreme values of a function f(x) on an interval I are the minimum and maximum values of f(x) for  $x \in I$ .
- If f(x) is continuous on [a, b], then f(x) has a minimum and a maximum value on [a, b].
- f(c) is a local maximum if  $f(x) \leq f(c)$  for all x in some open interval around c. Similarly, f(c) is a local minimum if  $f(x) \geq f(c)$  for all x in some open interval around c.
- c is a critical point of the function f(x) if either f'(c) = 0 or f'(c) does not exist.
- Fermat's Theorem: If f(c) is a local minimum or maximum, then c is a critical point.
- Extreme values of a function on a closed interval occur at the critical points or the endpoints of the interval. To find the extreme values of a continuous function f(x) on a closed interval [a, b]:
  - (1) Find the critical points of f(x).
  - (2) Calculate f(x) at the critical points in [a, b] as well as at the endpoints a and b.

The minimum and maximum values on [a, b] are the smallest and largest values computed in (2).

• Rolle's Theorem: If f(x) is continuous on [a, b], and differentiable on (a, b), and f(a) = f(b), then there exists  $c \in (a, b)$  such that f'(c) = 0.

### Lecture Material

This section is crucial to most of what follows as critical points are defined. It is essential that students are adept at finding critical points. Begin by defining extreme values, and observe that extreme values need not exist for functions that are not continuous or are defined on an open interval (as in Figure 2). This naturally leads to the statement of Theorem 1, that every continuous function on a closed interval attains a minimum and a maximum value. Then define local extrema and graphically show that if the graph of f(x) has a local minimum or maximum, then the tangent line is horizontal or the tangent line does not exist (that is, f(x) has a cusp, see Figure 4). This gives rise to the definition of a critical point, as well as to the statement of Fermat's Theorem on Local Extrema (Theorem 2). Work a couple of problems finding critical points, for example Exercises 6 and 9. Now all the tools for finding minimum and maximum values of continuous functions on closed intervals are in hand and should be presented. Work Exercise 18. Finally, discuss Rolle's Theorem and illustrate its use by working Exercise 54 or Exercise

58, as Rolle's Theorem is a special case of the Mean Value Theorem presented in the next section.

The *written* justification of an extreme value (local or absolute) is a common task on the AP Calculus exams. Here and in the next several sections, be sure to do likewise. (A "sign chart" is *not* considered sufficient justification; a sentence is required explaining the sign changes or showing that the critical points and end points were evaluated.)

### **Discussion Topics/Class Activities**

Work Exercise 80 with the class. This exercise is a quite interesting example of optimization in the real world dealing with the honeycomb structure in a beehive.

**Suggested Problems** (spread over 2 assignments) Exercises 1, 2, 3, 5, 13, 15, 17, 25, 29, 33, 39, 47, 55, 65, 67, 75 **1.** Find all critical points of the function  $f(t) = 8t^3 - t^2$ .

**2.** Find all critical points of the function  $f(x) = x^{1/3}$ .

**3.** Find the maximum and minimum values of the function  $y = -x^2 + 10x + 43$  on the interval [3,8].

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4. Find the maximum and minimum values of the function  $y = \frac{x^2 + 1}{x - 4}$  on the interval [5, 6].

5. Verify Rolle's Theorem for the function  $f(x) = \sin x$  on the interval  $[\frac{\pi}{4}, \frac{9\pi}{4}]$ .

6. Use Rolle's Theorem to prove that  $f(x) = \frac{x^3}{6} + \frac{x^2}{2} + x + 1$  has at most one real root.

### Solutions to Worksheet 4.2

- 1. Find all critical points of the function  $f(t) = 8t^3 t^2$ . Let  $f(t) = 8t^3 - t^2$ . Then  $f'(t) = 24t^2 - 2t = 2t(12t - 1) = 0$  implies that t = 0 and  $t = \frac{1}{12}$  are the critical points of f.
- **2.** Find all critical points of the function  $f(x) = x^{1/3}$ .

Let  $f(x) = x^{1/3}$ . Then  $f'(x) = \frac{1}{3}x^{-2/3}$ . The derivative is never zero, but does not exist at x = 0. Thus, x = 0 is the only critical point of f.

**3.** Find the maximum and minimum values of the function  $y = -x^2 + 10x + 43$  on the interval [3, 8].

Let  $f(x) = 43 + 10x - x^2$ . Then f'(x) = 10 - 2x = 0, whence x = 5 is a critical point of f. The minimum of f on the interval [3, 8] is f(8) = 59, whereas its maximum is f(5) = 68. (Note: f(3) = 64.)

4. Find the maximum and minimum values of the function  $y = \frac{x^2 + 1}{x - 4}$  on the interval [5, 6]. Let  $f(x) = \frac{x^2 + 1}{x - 4}$ . Then  $f'(x) = \frac{(x - 4) \cdot 2x - (x^2 + 1) \cdot 1}{(x - 4)^2} = \frac{x^2 - 8x - 1}{(x - 4)^2} = 0$ 

implies  $x = 4 \pm \sqrt{17}$ . Neither critical point lies in the interval [5, 6]. On this interval, the minimum of f is  $f(6) = \frac{37}{2} = 18.5$ , while its maximum is f(5) = 25.

5. Verify Rolle's Theorem for the function  $f(x) = \sin x$  on the interval  $[\frac{\pi}{4}, \frac{9\pi}{4}]$ . Because f is continuous on  $[\frac{\pi}{4}, \frac{3\pi}{4}]$ , differentiable on  $(\frac{\pi}{4}, \frac{3\pi}{4})$  and  $f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$ ,

we may conclude from Rolle's Theorem that there exists a  $c \in (\frac{\pi}{4}, \frac{3\pi}{4})$  at which f'(c) = 0. Here,  $f'(x) = \cos x$ , so we may take  $c = \frac{\pi}{2}$ . 156

6. Use Rolle's Theorem to prove that  $f(x) = \frac{x^3}{6} + \frac{x^2}{2} + x + 1$  has at most one real root.

We use proof by contradiction. Suppose  $f(x) = x^3/6 + x^2/2 + x + 1$  has two real roots, x = a and x = b. Then f(a) = f(b) = 0 and Rolle's Theorem guarantees that there exists a  $c \in (a, b)$  at which f'(c) = 0. However,

$$f'(x) = x^2/2 + x + 1 = \frac{1}{2}(x+1)^2 + 1 \ge 1$$

for all x, so there is no  $c \in (a, b)$  at which f'(c) = 0. Based on this contradiction, we conclude that  $f(x) = x^3/6 + x^2/2 + x + 1$  cannot have more than one real root.
### 4.3. The Mean Value Theorem and Monotonicity.

Class Time 2 periods. Essential.

### **Key Points**

• Mean Value Theorem: If f is continuous on [a, b] and differentiable on (a, b), then there exists  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- If f'(x) > 0 for all  $x \in (a, b)$ , then f is increasing on (a, b), while if f'(x) < 0 for all  $x \in (a, b)$ , then f is decreasing on (a, b).
- f'(x) can change sign only at a critical point. In particular, f(x) is monotonic on the intervals between critical points.
- To determine the sign of f'(x) on an interval between two critical points, find the sign of  $f'(x_0)$  at any point in the interval.
- First Derivative Test: If c is a critical point, then
  - If f' changes sign from positive to negative at c, then f has a local maximum at c.
  - If f' changes sign from negative to positive at c, then f has a local minimum at c.
  - If f' does not change sign at c, then f has no local extrema at c.

#### Lecture Material

The First Derivative Test is extremely important and should be stressed. Begin, though, with the Mean Value Theorem, as it is a generalization of Rolle's Theorem from the previous section. Graphically show the meaning of the Mean Value Theorem, namely, that there is some point c in (a, b) such that f'(c) (the slope of the line tangent to the curve y = f(x) at c) is the same as the slope of the line between (a, f(a)) and (b, f(b)) (for example with Figure 1), and point out that Rolle's Theorem is just a special case. Then work Exercise 2. Also point out Corollary 2.

Graphically show that an increasing function has positive derivative and that a decreasing function has negative derivative (for example, with Figure 3). Point out that the sign of the derivative can change only at critical points or points where the function is undefined, so a function is monotonic on an interval between two such points. Thus to determine whether f'(x) is positive or negative on such an interval, one only needs to determine whether f'(c) is positive or negative for any c in the interval. It might be worthwhile to point out that for rational functions that can be factored into a product of linear factors, a sign chart also provides the same information, with perhaps less work. Graphically illustrate the First Derivative Test (for example with Figure 6). Work Exercises 24, 30, and 38.

In Theorem 2, if f is also assumed to be continuous on the closed interval [a, b] (where f'(a) or f'(b) may or may not be zero), then the intervals in the conclusion are also closed. Thus  $f(x) = \sin(x)$  is increasing on  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  and decreasing on  $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ . Having  $\frac{\pi}{2}$  in both intervals may concern some students. Point out that functions increase and

decrease on intervals, not at points.

### **Discussion Topics/Class Activities**

The *written* justification of an extreme value (local or absolute) is a common task on the AP Calculus exams. Be sure to do likewise. (A "sign chart" is *not* considered sufficient justification; a sentence is required explaining the sign changes or showing that the critical points and end points were evaluated.)

Work Exercise 57. This exercise demonstrates some of the uses of the Mean Value Theorem by showing that given some information about a function and it's derivative, specific information about the function can be obtained using the Mean Value Theorem.

#### Suggested Problems (spread over 2 assignments)

Exercises 3, 5, 9, 11, 12, 13, 14, 17, 19, 21, 25, 29, 39 (19–52 are good places for students to practice writing justifications of extreme values), 56, 58

# Worksheet 4.3. The Mean Value Theorem and Monotonicity

1. Find a point c satisfying the conclusion of the Mean Value Theorem for the function  $y = \sqrt{x}$  and the interval [4,9].

2. Determine the intervals on which the function  $f(x) = x(x+1)^3$  is monotonically increasing or decreasing, and use the First Derivative Test to determine whether the local extrema are local minima or maxima (or neither).

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**3.** Determine the intervals on which the function  $f(x) = x^2 - x^4$  is monotonically increasing or decreasing, and use the First Derivative Test to determine whether the local extrema are local minima or maxima (or neither).

4. Determine the intervals on which the function  $f(x) = \cos \theta + \sin \theta$  on the interval  $[0, 2\pi]$  is monotonically increasing or decreasing, and use the First Derivative Test to determine whether the local extrema are local minima or maxima (or neither).

### Solutions to Worksheet 4.3

1. Find a point c satisfying the conclusion of the Mean Value Theorem for the function  $y = \sqrt{x}$  and the interval [4,9].

Let  $f(x) = x^{1/2}$ , a = 4, b = 9. By the MVT, there exists a  $c \in (4, 9)$  such that  $\frac{1}{2}c^{-1/2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{3 - 2}{9 - 4} = \frac{1}{5}$ Thus  $\frac{1}{\sqrt{c}} = \frac{2}{5}$ , whence  $c = \frac{25}{4} = 6.25 \in (4, 9)$ .

2. Determine the intervals on which the function  $f(x) = x(x+1)^3$  is monotonically increasing or decreasing, and use the First Derivative Test to determine whether the local extrema are local minima or maxima (or neither).

Let  $f(x) = x (x+1)^{3}$ . Then

$$f'(x) = x \cdot 3 (x+1)^2 + (x+1)^3 \cdot 1 = (4x+1) (x+1)^2 = 0$$

yields critical points  $c = -1, -\frac{1}{4}$ . By plugging in, we see that the first derivative is negative on  $[-\infty, -\frac{1}{4}]$  and positive on  $[-\frac{1}{4}, \infty]$ . Hence f has a local minima at  $x = -\frac{1}{4}$ .

3. Determine the intervals on which the function  $f(x) = x^2 - x^4$  is monotonically increasing or decreasing, and use the First Derivative Test to determine whether the local extrema are local minima or maxima (or neither).

Let  $f(x) = x^2 - x^4$ . Then  $f'(x) = 2x - 4x^3 = 2x(1 - 2x^2) = 0$  yields critical points  $c = 0, \pm \frac{1}{\sqrt{2}}$ . By plugging in, we see that the first derivative is negative on  $[-\frac{1}{\sqrt{2}}, 0] \cup [\frac{1}{\sqrt{2}}, \infty]$  and positive on  $[-\infty, -\frac{1}{\sqrt{2}}] \cup [0, \frac{1}{\sqrt{2}}]$ . Hence f has a local minima at  $x = \pm \frac{1}{\sqrt{2}}$ .

4. Determine the intervals on which the function  $f(x) = \cos \theta + \sin \theta$  on the interval  $[0, 2\pi]$  is monotonically increasing or decreasing, and use the First Derivative Test to determine whether the local extrema are local minima or maxima (or neither).

Let  $f(\theta) = \cos \theta + \sin \theta$ . Then  $f'(\theta) = \cos \theta - \sin \theta$ , which yields  $c = \frac{\pi}{4}, \frac{5\pi}{4}$  on the interval  $[0, 2\pi]$ . By plugging in the appropriate points we see that the first derivative is negative on  $[\frac{\pi}{4}, \frac{5\pi}{4}]$  and positive on  $[0, \frac{\pi}{4}] \cup [\frac{5\pi}{4}, 2\pi]$ . Hence f has a local minima at  $x = \frac{5\pi}{4}$  and local maxima at  $x = \frac{\pi}{4}$ .

#### 4.4. The Shape of a Graph.

### Class Time 2 periods. Essential.

### Key Points

- A differentiable function f(x) is concave up on (a, b) if f''(x) > 0 for all  $x \in (a, b)$ and concave down if f''(x) < 0 for all  $x \in (a, b)$ .
- The signs of the first two derivatives give the following information about the shape of the graph of f:
  - (1) If f' > 0, then f is increasing, while if f' < 0 then f is decreasing.
  - (2) If f'' > 0, then f is concave up, while if f'' < 0, then f is concave down.
- A point of inflection of f is a point where the concavity of f changes from concave up to concave down or from concave down to concave up.
- Second Derivative Test: If f'(c) = 0, then
  - -f(c) is a local maximum if f''(c) < 0.
  - -f(c) is a local minimum if f''(c) > 0.
  - the test fails and says nothing if f''(c) = 0 or if f''(c) does not exist.

### Lecture Material

Begin with the definition of concavity: A differentiable function on (a, b) is concave up on (a, b) if f'(x) is increasing on (a, b) and concave down on (a, b) if f'(x) is decreasing on (a, b). Point out that if f is concave up on an interval, then the tangent lines will be below the graph on the interval, while if f is concave down on an interval, the tangent lines will be above the graph on the interval (perhaps using Figures 1 and 2, as examples). Observe that the students already know how to do this computationally, as the procedure is precisely the same as determining where a function is increasing or decreasing, but f'(x)is used instead of f(x). Also, x = c is an inflection point for y = f(x) if the concavity changes from up to down or down to up at x = c. That is, x = c is an inflection point of y = f(x) if f'(x) has a local minimum or maximum at x = c. Work Exercise 4 and Example 4. Now state the Second Derivative Test: Let f(x) be differentiable with critical point c. If f''(c) exists, then

- (1) f''(c) > 0 implies f(c) is a local minimum.
- (2) f''(c) < 0 implies f(c) is a local maximum.

(3) f''(c) = 0 is inconclusive: f(c) may be a local minimum, or maximum, or neither. Illustrate graphically by showing that at a local maximum, the tangent line is above the graph (and so concave down), and at a local minimum, the tangent line is below the graph (and so concave up). Figure 10 may be useful in illustration. Now work Exercises 24, 32 and 40.

### **Discussion Topics/Class Activities**

Discuss Exercise 57 with the class. Note that the graph will be concave up at the beginning of the epidemic and concave down at the end of the epidemic (and so will have an inflection point). For part (b), note that the inflection point is where the rate of change changes from positive to negative, and so the number of new infections per day will decrease.

**Suggested Problems** (spread over 2 assignments) Exercises 1, 2, 3, 5, 7, 15, 19, 20–22, 24, 39, 45, 53, 62

# Worksheet 4.4. The Shape of a Graph

1. Determine the intervals on which the function  $y = t^3 - 3t^2 + 1$  is concave up and concave down and find the points of inflection.

2. Find the critical points of  $f(x) = x^4 - 8x^2 + 1$  and apply the Second Derivative Test (if possible) to determine whether they are local minima or maxima.

**3.** Determine the intervals on which the function  $f(x) = 2x^4 - 3x^2 + 2$  is concave up and concave down, find the points of inflection, and determine whether the critical points are local minima or maxima.

4. Determine the intervals on which the function  $f(t) = \sin^2 t$  is concave up and concave down, find the points of inflection, and determine whether the critical points are local minima or maxima.

### Solutions to Worksheet 4.4

1. Determine the intervals on which the function  $y = t^3 - 3t^2 + 1$  is concave up and concave down and find the points of inflection.

Let  $f(t) = t^3 - 3t^2 + 1$ . Then  $f'(t) = 3t^2 - 6t$  and f''(t) = 6t - 6 = 0 at t = 1. Now, f is concave up on  $(1, \infty)$ , since f''(t) > 0 there. Moreover, f is concave down on  $(-\infty, 1)$ , since f''(t) < 0 there. Finally, because f''(t) changes sign at t = 1, t = 1 is a point of inflection.

2. Find the critical points of  $f(x) = x^4 - 8x^2 + 1$  and apply the Second Derivative Test (if possible) to determine whether they are local minima or maxima.

Let  $f(x) = x^4 - 8x^2 + 1$ . Then  $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$ , and the critical points are x = 0 and  $x = \pm 2$ . Moreover,  $f''(x) = 12x^2 - 16$ , so f''(-2) = f''(2) = 32 > 0and f''(0) = -16 < 0. Therefore, by the Second Derivative Test, f(-2) = -15 and f(2) = -15 are local minima, and f(0) = 1 is a local maximum.

3. Determine the intervals on which the function  $f(x) = 2x^4 - 3x^2 + 2$  is concave up and concave down, find the points of inflection, and determine if the critical points are local minima or maxima.

Let  $f(x) = 2x^4 - 3x^2 + 2$ . Then  $f'(x) = 8x^3 - 6x = 2x(4x^2 - 3) = 0$  yields  $x = 0, \pm \frac{\sqrt{3}}{2}$ as candidates for extrema. Moreover,  $f''(x) = 24x^2 - 6 = 6(4x^2 - 1) = 0$  gives  $x = \pm \frac{1}{2}$ as inflection point candidates. Since f''(x) changes sign at  $x = \pm \frac{1}{2}$ , they are actual inflection points. Also, since f''(0) = -6 and  $f''(\pm \frac{\sqrt{3}}{2}) = 12$ , the Second Derivative Test implies that f(x) has a local maximum at x = 0 and local minima at  $x = \pm \frac{\sqrt{3}}{2}$ 

4. Determine the intervals on which the function  $f(t) = \sin^2 t$  is concave up and concave down, find the points of inflection, and determine if the critical points are local minima or maxima.

Let  $f(t) = \sin^2 t$  on  $[0, \pi]$ . Then  $f'(t) = 2 \sin t \cos t = \sin 2t = 0$  yields  $t = \frac{\pi}{2}$  as a candidate for an extremum. Moreover,  $f''(t) = 2 \cos 2t = 0$  gives  $t = \frac{\pi}{4}, \frac{3\pi}{4}$  as inflection point candidates. Since f''(x) changes sign at  $t = \frac{\pi}{4}, \frac{3\pi}{4}$ , they are actual inflection points. Also, since  $f''(\frac{\pi}{2}) = -2$ , the Second Derivative Test implies that f(x) has a local maximum at  $x = \frac{\pi}{2}$ .

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### 4.5. L'Hôpital's Rule.

Class Time AB 0 periods, not tested on the AB exam; BC 1 period, essential.

L'Hôpital's Rule is not tested on the AB Calculus exams. Many teachers choose to include it because it makes so many limits easy to find.

L'Hôpital's Rule is tested on the BC Calculus exams in the context of determining the convergence of improper integrals (Section 7.6).

### **Key Points**

- L'Hôpital's rule for indeterminate forms  $\frac{0}{0}$  and  $\frac{\infty}{\infty}$ .
- Indeterminate forms  $0 \cdot \infty$ ,  $\infty \infty$ ,  $0^0$  and  $1^\infty$ .
- Comparing rates of growth of two functions.

#### Lecture Material

L'Hôpital's Rule provides a method to evaluate limits  $\lim_{x \to a} \frac{f(x)}{g(x)}$  where  $\frac{f(x)}{g(x)}$  has indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  at x = a: If  $g'(x) \neq 0$  for all x near  $a, x \neq a$  and if  $\frac{f'(x)}{g'(x)} \to L$ 

(finite or infinite) as  $x \to a$ , then  $\lim_{x\to a} \frac{f(x)}{g(x)} = L$  too. The same result holds for one-sided limits and for limits at  $\pm \infty$ . In the case that f(a) = g(a) = 0, and both f'(x) and g'(x) are continuous at x = a with  $g'(a) \neq 0$ , this is easy to see, as shown at the end of the section.

Examples 1–5 are straightforward applications of L'Hôpital's Rule. Example 6 illustrates the importance of checking that f(x)/g(x) has appropriate indeterminate form. Examples 7 and 8 show how to rewrite forms such as  $\infty - \infty$  and  $0^0$ . For an example of

1<sup> $\infty$ </sup>, apply L'Hôpital's Rule to show that  $\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^t = e$ .

Functions f(x) and g(x) have comparable rates of growth as  $x \to \infty$  if  $0 < \lim_{x \to \infty} \frac{f(x)}{q(x)} < 0$ 

 $\infty$ . If, on the other hand, both  $f(x), g(x) \to \infty$  but  $f(x)/g(x) \to 0$  as  $x \to \infty$ , then it must be the case that  $g(x) \to \infty$  faster than  $f(x) \to \infty$ . We write  $f(x) \ll g(x)$  as  $x \to \infty$ . See Examples 8 and 9.

Rates of growth of polynomials are determined by their degree. (L'Hôpital's Rule is not needed to compare rates of growth of two polynomials—simply divide.) A very important application of L'Hôpital's Rule is that for any r > 0,

$$\ln x \ll x^r \ll e^x \quad \text{as } x \to \infty$$

In particular, rates of growth (decay) and L'Hôpital's Rule are important in determining convergence of infinite series.

## **Discussion Topics/Class Activities**

An informal proof of L'Hôpital's Rule is to graph the two functions (numerator and denominator) at once, and zoom in on their point of intersection (the limit value) which is on the x-axis (why?). Since the functions are differentiable, they are locally linear, so the graphs appear to be lines. The ratio of the y-coordinates of two lines that intersect on the x-axis is the ratio of the slopes (i.e. their derivatives). Thus the limit of the ratio of the two functions is the ratio of their derivatives.

## Suggested Problems

Exercises 1, 5, 9, 13, 17, 25, 27, 47, 57, 58

# Worksheet 4.5. L'Hôpital's Rule

1. Evaluate the limit. a.  $\lim_{x \to 0} \frac{\sin(4x)}{\sin(3x)}$ 

b. 
$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

c. 
$$\lim_{x \to \infty} x^2 e^{-x/2}$$

d. 
$$\lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{\sin x} \right)$$

e. 
$$\lim_{x \to 1} (1 + \ln x)^{1/(x-1)}$$

**2.** Let  $f(x) = x^{1/x}$  in the domain  $0 < x < \infty$ . a. Calculate  $\lim_{x \to 0^+} f(x)$  and  $\lim_{x \to \infty} f(x)$ .

b. Find the intervals on which f is increasing or decreasing and sketch the graph of f(x).



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## Solutions to Worksheet 4.5

1. Evaluate the following limits.  $\sin(4x) = 4\cos(4x) = 4$ 

a. 
$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(3x)} = \lim_{x \to 0} \frac{4\cos(4x)}{3\cos(3x)} = \frac{4}{3}$$
  
b. 
$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0$$
  
c. 
$$\lim_{x \to \infty} x^2 e^{-x/2} = \lim_{x \to \infty} \frac{x^2}{e^{x/2}} = \lim_{x \to \infty} \frac{2x}{\frac{e^{x/2}}{2}} = \lim_{x \to \infty} \frac{4}{\frac{e^{x/2}}{2}} = \lim_{x \to \infty} \frac{8}{e^{x/2}} = 0$$
  
d. 
$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0} \frac{\cos x - 1}{x \cos x + \sin x} = \lim_{x \to 0} \frac{-\sin x}{-x \sin x + 2\cos x} = 0$$
  
e. 
$$\lim_{x \to 1} (1 + \ln x)^{1/(x-1)}$$
  
Let  $y = (1 + \ln x)^{1/(x-1)}$ . Then 
$$\lim_{x \to 1} \ln y = \lim_{x \to 1} \frac{\ln(1 + \ln x)}{x - 1} = \frac{1}{1 + \ln x} = 1$$
. Thus 
$$\lim_{x \to 1} y = e.$$

2. Let  $f(x) = x^{1/x}$  in the domain  $0 < x < \infty$ .

a. Calculate  $\lim_{x \to 0^+} f(x)$  and  $\lim_{x \to \infty} f(x)$ . The first limit is not an indeterminate form.  $\lim_{x \to 0^+} f(x) = 0$ . The second limit is indeterminate. Let  $y = x^{1/x}$ . Then  $\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0$ . Hence  $\lim_{x \to \infty} x^{1/x} = e^0 = 1$ .

b. Find the intervals on which f is increasing or decreasing.

Let  $y = x^{1/x}$ . Then  $\ln y = \frac{\ln x}{x}$ . Taking the derivative of both sides with respect to x and solving for y' gives  $y'(x) = x^{\frac{1-2x}{x}}(1-\ln x)$ . This result is positive for all  $x \in [0, e]$  and negative for all  $x \in [e, \infty]$ , which implies that f(x) is increasing on (0, e] and decreasing on  $[e, \infty)$ . Thus f(x) has a local maximum at  $(e, e^{1/e})$ .



### 4.6. Graph Sketching and Asymptotes.

Class Time AB 3 periods; BC 2 periods. Essential.

### Key Points

- The graph of a function f is made up of arcs that have one of the following four basic shapes:
  - (1) ++ f' > 0, f'' > 0 Increasing and concave up
  - (2) +- f' > 0, f'' < 0 Increasing and concave down
  - (3) -+ f' < 0, f'' > 0 Decreasing and concave up
  - (4) -- f' < 0, f'' < 0 Decreasing and concave down
- A transition point of f is a point in the domain of f at which either f' changes sign (a local minimum or maximum) or f'' changes sign (an inflection point).
- For convenience, we break the curve sketching process into the following steps: (1) Determine the domain of f.
  - (2) Determine the signs of f' and f''.
  - (3) Note transition points and sign combinations.
  - (4) Draw arcs of appropriate shape and asymptotic behavior.
- A horizontal line y = L is a horizontal asymptote if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

• A vertical line x = L is a vertical asymptote if

$$\lim_{x \to L^+} f(x) = \pm \infty \quad \text{and/or} \quad \lim_{x \to L^-} f(x) = \pm \infty$$

• For a rational function 
$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0},$$
$$\lim_{x \to \pm \infty} f(x) = \frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{n-m}$$

### Lecture Material

Note that there are four possible combinations of nonzero signs of f'(x) and f''(x), namely ++, +-, -+, -+, -+, and --. Show students the "basic" shape of a graph for each of these possibilities, as in Figure 1. A transition point is a point where the basic shape of the graph changes (from one of the four possibilities to another) because of a sign change in either f'(x) or f''(x). Point out that that for a polynomial f(x),  $f(x) \to \pm \infty$  as  $x \to \pm \infty$ . These observations lead to an algorithm to obtain the graph of a function:

(1) Determine the signs of f' and f'' in the intervals between the transition points. (To perform this step, note that one must find the critical points of f as well as possible inflection points).

- (2) Note transition points and sign combinations. A sign diagram as in Figure 4 may be useful in summarizing this information as well as noting local extrema and inflection points. Also at this step, it would be useful to substitute the x-coordinates of the transitive points into f(x) to obtain their y-coordinates.
- (3) Draw arcs of appropriate shapes. Draw arcs of appropriate shapes and asymptotic behavior to connect transition points. Also, x and y-intercepts can be plotted.

Now work a couple of examples, for example, Exercises 14 and 32.

Turn to asymptotic behavior. For rational (and other types of functions), limits at  $\pm \infty$  can exist, and we define a horizontal asymptote to be a horizontal line y = L such that at least one of the following is true:  $\lim_{x\to\infty} f(x) = L$  or  $\lim_{x\to-\infty} f(x) = L$ . Of course a vertical line x = L is a vertical asymptote if f(x) has an infinite limit as  $x \to L$  from either the left or the right (or both). The main technique for evaluating infinite limits (and so finding horizontal asymptotes) is to divide both the numerator and the denominator by the highest power of the variable that occurs in the denominator. For rational functions, this easily leads to Theorem 1:

$$\lim_{x \to \pm \infty} \left( \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0} \right) = \frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{n-m}$$

Work Exercises 55, 60 and 62 to illustrate the use of this technique and Theorem 1.

### **Discussion Topics/Class Activities**

Slant asymptotes are not tested on the AP Calculus exams. This part may be omitted.

Work Exercise 72, discussing slant asymptotes with the class. Also show students how the slant asymptote y = x + 1 was found. Point out that other polynomial functions can act as "asymptotes" of f(x) = P(x)/Q(x) if the degree of P(x) is at least two more than the degree of Q(x). Can the graph of the function cross a slant asymptote (as the graph of a function cannot cross a vertical asymptote)?

Slant asymptotes are not tested on the AB or BC exams.

Suggested Problems (spread over 2 or 3 assignments) Exercises 1, 10, 13, 17, 21, 37, 41, 47, 51, 57, 63

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# Worksheet 4.6. Graph Sketching and Asymptotes

1. Sketch the graph of the function  $y = x^3 - 3x + 5$ . Indicate the transition points (local extrema and points of inflection).



2. Sketch the graph of the function  $y = 2\sin x - \cos^2 x$ ;  $[0, 2\pi]$ . Indicate the transition points.



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In Exercises 3 and 4, calculate the limits by dividing the numerator and denominator by the highest power of x appearing in the denominator.

3. 
$$\lim_{x \to \infty} \frac{3x^2 + 20x}{4x^2 + 9}$$

$$4. \lim_{x \to \infty} \frac{4}{x+5}$$

5. Calculate 
$$\lim_{t \to -\infty} \frac{\sqrt{x^4 + 1}}{x^3 + 1}$$
.





# Solutions to Worksheet 4.6

1. Sketch the graph of the function  $y = x^3 - 3x + 5$ . Indicate the transition points (local extrema and points of inflection).



2. Sketch the graph of the function  $y = 2\sin x - \cos^2 x$ ;  $[0, 2\pi]$ . Indicate the transition points.



In Exercises 3 and 4, calculate the limits by dividing the numerator and denominator by the highest power of x appearing in the denominator.

3. 
$$\lim_{x \to \infty} \frac{3x^2 + 20x}{4x^2 + 9} = \frac{3}{4}$$

- 4.  $\lim_{x \to \infty} \frac{1}{x+5} = 0$
- **5.** Calculate  $\lim_{t \to -\infty} \frac{\sqrt{x^4 + 1}}{x^3 + 1} = 0.$





### 4.7. Applied Optimization.

Class Time 2 periods. Essential.

### **Key Points**

- There are normally three main steps in solving an applied optimization problem: (1) Choose the variables.
  - Determine which quantities are relevant, perhaps by drawing a diagram, and assign to each an appropriate variable name.
  - (2) Find the function and the interval.

Restate the question as an optimization problem for a function f over some interval I. If the function depends on more than one variable, use a constraint equation to write f as a function of just one variable.

- (3) Optimize the function.
- If the interval I is open, f does not necessarily take on a minimum or maximum value on I. If it does, though, the values must occur at critical points within the interval. To determine whether a minimum or maximum exists, analyze the behavior of f as x approaches the endpoints of the interval.

### Lecture Material

This is an important section because it shows the practical applications of the derivative and hence the value of calculus. Go over the three steps that are usually encountered in an applied optimization problem. This might best be accomplished in the context of an example, say, Exercise 2. Most of the examples involve optimizing a continuous function f on a closed interval [a, b], and so by Theorem 1 of Section 4.2, the Existence of Extrema on a Closed Interval, a minimum value and a maximum value of f exist on [a, b], and they occur either at a critical point or at an endpoint of the interval. Following are the steps in determining the extrema:

- (1) Find the critical points of f in [a, b].
- (2) Evaluate f(x) at the critical points and at the endpoints a and b.
- (3) The largest and smallest values obtained in the previous step are the maximum and minimum values, respectively.

Occasionally, the function that needs to be optimized is defined on an open interval, so no maximum or minimum value need exist. In this case, any maximum or minimum value will occur at a critical point in the interval. Additionally, an appropriate limit of the function at each endpoint should be determined to verify that a minimum or maximum value exists. Work Exercises 2, 4, 14 (an appropriate example for discussing maximums and minimums on open intervals), and 22. Students will have the most trouble in writing the model (equation) to be optimized. Spend your time teaching how to do this.

While optimization is an important application of derivatives, questions of this type rarely appear on the AP Calculus exams because of the difficulty students have in writing the model. For this reason, do not spend too much time on this section.

### **Discussion Topics/Class Activities**

Derive Snell's Law (Exercise 44). Exercise 59 also deals with Snell's Law, so it might be good to work it in class or assign as homework.

Suggested Problems (2 assignments)

Exercises 1–13 odd, 16, 18, 21, 35, 39

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# Worksheet 4.7. Applied Optimization

- 1. A 100-inch piece of wire is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares?
  - (a) Express the sum of the areas of the squares in terms of the lengths x and y of the two pieces.
  - (b) What is the constraint equation relating x and y?
  - (c) Does this problem require optimization over an open interval or a closed interval?
  - (d) Solve the optimization problem.

2. The legs of a right triangle have lengths a and b satisfying a + b = 10. Which values of a and b maximize the area of the triangle?

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**3.** A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$1 per square foot, and the metal for the sides costs \$2 per square foot. Find the dimensions that minimize cost if the box has to have a volume of 20 ft<sup>3</sup>.

4. Find the angle  $\theta$  that maximizes the area of the trapezoid with base of length 4 and sides of length 2.



## Solutions to Worksheet 4.7

- A 100-inch piece of wire is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares? Let x and y be the lengths of the pieces.
  - (a) Express the sum of the areas of the squares in terms of the lengths x and y of the two pieces.

The perimeter of the first square is x, which implies the length of each side is  $\frac{x}{4}$  and the area is  $\left(\frac{x}{4}\right)^2$ . Similarly, the area of the second square is  $\left(\frac{y}{4}\right)^2$ . Then the sum of the areas is given by  $A = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2$ 

the sum of the areas is given by 
$$A = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2$$
  
) What is the constraint equation relating x and  $u^2$ 

(b) What is the constraint equation relating x and y? x + y = 100, whence y = 100 - x. Then

$$A(x) = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{100 - x}{4}\right)^2 = \frac{1}{8}x^2 - \frac{25}{2}x + 625$$

- (c) Does this problem require optimization over an open interval or a closed interval? Since it is possible for the minimum total area to be realized by not cutting the wire at all, optimization over the closed interval [0, 100] suffices.
- (d) Solve the optimization problem.

Solve  $A'(x) = \frac{1}{4}x - \frac{25}{2} = 0$  to obtain x = 50. Now A(0) = A(100) = 625, whereas A(50) = 312.5. Accordingly, the sum of the areas of the squares is minimized if the wire is cut in half.

2. The legs of a right triangle have lengths a and b satisfying a + b = 10. Which values of a and b maximize the area of the triangle?

Let the side lengths be a, b > 0. Now a + b = 10, whence b = 10 - a. Let  $A(a) = \frac{1}{2}ab = \frac{1}{2}a(10 - a) = 5a - \frac{1}{2}a^2$  and note that a is restricted to the closed interval [0, 10]. Solve A'(a) = 5 - a = 0 to obtain a = 5. Since A(0) = A(10) = 0 and  $A(5) = \frac{25}{2}$ , the maximum area is  $A(5) = \frac{25}{2}$  when a = b = 5. **3.** A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$1 per square foot, and the metal for the sides costs \$2 per square foot. Find the dimensions that minimize cost if the box has to have a volume of 20 ft<sup>3</sup>.

Let x > 0 be the length of a side of the square base and z > 0 the height of the box. With volume  $x^2 z = 20$ , we have  $z = 20/x^2$  and cost

$$C(x) = 1 \cdot 2 \cdot x^2 + 2 \cdot 4 \cdot xz = 2x^2 + 160x^{-1}$$

Solve  $C'(x) = 4x - 160x^{-2} = 0$  to obtain  $x = 2(5)^{1/3}$ . Since  $C(x) \to \infty$  as  $x \to 0+$  and as  $x \to \infty$ , the minimum cost is  $C(2(5)^{1/3}) = 24(5)^{2/3} \approx \$70.18$  when  $x = 2(5)^{1/3} \approx 3.42$  ft and  $z = 5^{1/3} \approx 1.71$  ft.

4. Find the angle  $\theta$  that maximizes the area of the trapezoid with base of length 4 and sides of length 2.



Allowing for degenerate trapezoids,  $0 \le \theta \le \pi$ . Via trigonometry and surgery (slice off a right triangle and rearrange the trapezoid into a rectangle), the area of the trapezoid is equivalent to the area of a rectangle of base  $4 - 2\cos\theta$  and height  $2\sin\theta$ ; in other words,

$$A(\theta) = (4 - 2\cos\theta) \cdot 2\sin\theta = 8\sin\theta - 4\sin\theta\cos\theta = 8\sin\theta - 2\sin2\theta$$

where  $0 \leq \theta \leq \pi$ . Solve

$$A'(\theta) = 8\cos\theta - 4\cos 2\theta = 4 + 8\cos\theta - 8\cos^2\theta = 0$$

for  $0 \leq \theta \leq \pi$  to obtain

$$\theta = \theta_0 = \cos^{-1}\left(\frac{1-\sqrt{3}}{2}\right) \approx 1.94553$$

Since  $A(0) = A(\pi) = 0$  and  $A(\theta_0) = 3^{1/4}(3 + \sqrt{3})\sqrt{2}$ , the area of the trapezoid is maximized when  $\theta = \cos^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$ .

### 4.8. Newton's Method.

# Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS. Key Points

• Newton's Method: To find a sequence of numerical approximations to a solution of f(x) = 0, begin with an initial guess  $x_0$ , chosen as close as possible to an actual solution, possibly by referring to a graph. Then construct  $x_0, x_1, \ldots$  using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In favorable cases, the sequence will converge to a solution.

• Usually, the sequence  $x_0, x_1, \ldots$  converges quickly to a solution. If  $x_n$  and  $x_{n+1}$  agree to *m* decimal places, one may be reasonably certain that  $x_n$  agrees with a true solution to *m* decimal places.

### Lecture Material

Newton's Method is a technique for approximating the zeros of functions. The basic idea is that the x-intercept of the line tangent to the curve y = f(x) at a point c is closer to a zero of f than c is to the same zero. This procedure is then iterated, as graphically depicted in Figure 2. Using straightforward algebra, derive the formula  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ . Now work Exercises 2 and 5. Point out that there are problems with Newton's Method, namely, that one cannot begin at a local extrema, the zero that is found may not necessarily be the root one would like to find, and it is not clear how many iterations must be done to find a zero to the required accuracy.

### **Discussion Topics/Class Activities**

Work Exercises 28 and 29 to demonstrate that Newton's Method will not always find zeros of functions. With Exercise 28, do not make an initial guess  $x_0$ , but instead show that  $x_{n+1} = -2x_n$ .

### Suggested Problems

Exercises 1, 3 (basic, computational), 5 (basic, computational and graphical), 9, 11 (intermediate, computational), 11, 13 (intermediate, computational and graphical)

# Worksheet 4.8. Newton's Method

Use Newton's Method with the given function and initial value  $x_0$  to calculate  $x_1, x_2, x_3$ .

1.  $f(x) = x^2 - 7, x_0 = 2.5$ 

**2.**  $f(x) = \cos x - x, x_0 = .8$ 

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**3.** Sketch the graph of  $x^3 - 4x + 1$  and use Newton's Method to approximate the largest positive root to within an error of at most  $10^{-3}$ .



4. The first positive solution of  $\sin x = 0$  is  $x = \pi$ . Use Newton's Method to calculate  $\pi$  to four decimal places.

### Solutions to Worksheet 4.8

Use Newton's Method with the given function and initial value  $x_0$  to calculate  $x_1, x_2, x_3$ .

1.  $f(x) = x^2 - 7, x_0 = 2.5$ Let  $f(x) = x^2 - 7$  and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 7}{2x_n}$$

With  $x_0 = 2.5$ , we compute  $x_1 = 2.65$ ,  $x_2 = 2.645754717$  and  $x_3 = 2.645751311$ .

2.  $f(x) = \cos x - x$ ,  $x_0 = .8$ Let  $f(x) = \cos x - x$  and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1}$$

With  $x_0 = 0.8$ , we compute  $x_1 = 0.739853306$ ,  $x_2 = 0.739085263$ , and  $x_3 = 0.739085133$ .

**3.** Sketch the graph of  $x^3 - 4x + 1$  and use Newton's Method to approximate the largest positive root to within an error of at most  $10^{-3}$ .



If  $f(x) = x^3 - 4x + 1$ , then the graph of f(x) shown suggests that the largest positive root is near x = 2. Taking  $x_0 = 2$ , Newton's Method gives  $x_1 = 1.875$ ,  $x_2 = 1.86097852$ , and  $x_3 = 1.860805879$ . The largest positive root of  $x^3 - 4x + 1$  is approximately 1.8608.

4. The first positive solution of  $\sin x = 0$  is  $x = \pi$ . Use Newton's Method to calculate  $\pi$  to four decimal places.

Let  $f(x) = \sin x$ . Taking  $x_0 = 3$ , we have  $x_1 = 3.142546543$ ,  $x_2 = 3.141592653$ , and  $x_3 = 3.141592654$ . Hence,  $\pi \approx 3.1416$  to four decimal places.

### 4.9. Antiderivatives.

### Class Time AB 3 periods; BC 2 periods. Essential.

Antiderivatives and Initial Value Problems may be considered here, or with the material in Chapter 5. If placed after 5.3 and 5.4 (FTC), the students will have a reason to need to know about antiderivatives.

### **Key Points**

- F(x) is an antiderivative of f(x) if F'(x) = f(x).
- Two antiderivatives of f(x) differ only by a constant.
- The general antiderivative of f(x) is denoted by the indefinite integral

$$\int f(x)dx = F(x) + C$$

- Indefinite integrals should be checked by differentiation. If F(x) + C is the correct antiderivative, then  $\frac{d}{dx}(F(x)) = f(x)$ .
- Integration formulas (see also Theorems 5 and 6):

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ provided that } n \neq 1$$
$$\int x^{-1} dx = \int \frac{dx}{x} = \ln|x| + C$$
$$\int \sin(kx+b) dx = -\frac{1}{k}\cos(kx+b) + C \text{ provided that } k \neq 0$$
$$\int \cos(kx+b) dx = \frac{1}{k}\sin(kx+b) + C \text{ provided that } k \neq 0$$
$$\int e^{kx+b} dx = \frac{1}{k}e^{kx+b} + C, \text{ provided that } k \neq 0$$
$$\int e^x dx = e^x + C$$
$$\int e^{kx+b} dx = \frac{1}{k}e^{kx+b} + C$$

• To solve the differential equation  $\frac{dy}{dx} = f(x)$  with initial condition  $y(x_0) = y_0$ , first find the general antiderivative y = F(x) + C of f(x). Then find C using the initial condition  $F(x_0) + C = y_0$ .

### Lecture Material

Begin by defining an antiderivative: F(x) is an antiderivative of f(x) on (a, b) if F'(x) = f(x) for all  $x \in (a, b)$ . Note that if F(x) is an antiderivative of f(x), then every other antiderivative of f(x) is of the form F(x) + C, for some constant C (this is Theorem 1).

Also observe that the indefinite integral is defined as  $\int f(x)dx = F(x) + C$ , where F(x)

is an antiderivative of f(x) and C is a constant. Then show the various integral formulas: the Power Rule (Theorem 2), integrals of the natural log (Theorem 3),  $e^x$ , Sum Rule, Multiples Rule (both are contained in Theorem 4), and the basic trigonometric integrals. As this is an extremely important topic, work several examples, such as Exercises 18, 28, 32, 34, 38, and 40. Then point out that if one knows the value of f(x) at one point, that is, one knows appropriate initial conditions, then the constant the specific antiderivative can be obtained. Work Exercises 50 and 65.

### **Discussion Topics/Class Activities**

Find the general antiderivative of  $(2x + 9)^{10}$ . First remind students that if they were differentiating  $(2x + 9)^{10}$ , then they would use the Chain Rule and the Power Rule to obtain  $10 \cdot (2x + 9)^{10-1} \cdot 2$ . Using the Power Rule for Integrals, one can make an initial guess that an antiderivative of  $(2x + 9)^{10}$  is  $\frac{(2x + 9)^{10+1}}{10 + 1} = \frac{(2x + 9)^{11}}{11}$ . Checking this guess by differentiating, it is easy to see that the initial guess is off by a factor of 2, which happens to be the derivative of 2x + 9.

### Suggested Problems (3 assignments)

Exercises 1–7 odd, 13–37 every other odd, 40, 41, 47–57 odd, 63–67 odd

# Worksheet 4.9. Antiderivatives

Evaluate the indefinite integral.

1. 
$$\int (5x^3 - x^{-2} - x^{3/5}) dx$$

$$2. \quad \int \frac{3}{x^{3/2}} dx$$

3. 
$$\int \frac{x^2 + 2x - 3}{x^4} dx$$

$$4. \quad \int \sin 9x \, dx$$

5. 
$$\int 18\sin(3z+8)dz$$

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$$6. \quad \int \left(\frac{8}{x} + 3e^x\right) \, dx$$

7. Solve the differential equation  $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$  with initial condition y(1) = 1.

8. Given that  $f''(x) = x^3 - 2x + 1$ , f'(0) = 1, and f(0) = 0, first find f' and then find f.

# Solutions to Worksheet 4.9

Evaluate the indefinite integral.

1. 
$$\int (5x^3 - x^{-2} - x^{3/5}) dx$$
$$\int (5x^3 - x^{-2} - x^{3/5}) dx = \frac{5}{4}x^4 + x^{-1} - \frac{5}{8}x^{8/5} + C$$
  
2. 
$$\int \frac{3}{x^{3/2}} dx$$
 Since
$$\frac{3}{x^{-3/2}} = 3x^{-3/2}$$

we get

$$\frac{1}{x^{3/2}} = 5x^{-1}$$

$$\int \frac{3}{x^{3/2}} dx = \int 3x^{-3/2} dx$$
$$= 3\left(\frac{1}{(-1/2)}x^{-1/2}\right) + C$$
$$= -6x^{-1/2} + C$$

**3.**  $\int \frac{x^2 + 2x - 3}{x^4} dx$ We don't know how to take the antiderivative of a quotient, so we will have to distribute the  $\frac{1}{x^4}$  over  $x^2 + 2x - 3$ :

$$\frac{x^2 + 2x - 3}{x^4} = x^{-4}(x^2 + 2x - 3)$$
$$= x^{-2} + 2x^{-3} - 3x^{-4}.$$

From this formula, we can calculate

$$\int \frac{x^2 + 2x - 3}{x^4} dx = \int \left(x^{-2} + 2x^{-3} - 3x^{-4}\right) dx$$
$$= \frac{1}{-1}x^{-1} + 2\left(\frac{1}{-2}x^{-2}\right) - 3\left(\frac{1}{-3}x^{-3}\right) + C$$
$$= -x^{-1} - x^{-2} + x^{-3} + C$$

# 4. $\int \sin 9x dx$

Using the indefinite integral formula for  $\sin(kx + b)$ ,

$$\int \sin 9x \, dx = \frac{1}{9}(-\cos 9x) + C = -\frac{1}{9}\cos 9x + C.$$

# 5. $\int 18\sin(3z+8)dz$

From the integral formula for  $\cos(kt+b)$  with k = -4, b = 3,

$$\int \cos(3-4t) \, dt = \frac{1}{-4} \sin(3-4t) + C = -\frac{1}{4} \sin(3-4t) + C$$

6. 
$$\int \left(\frac{8}{x} + 3e^x\right) dx$$
$$\int \left(\frac{8}{x} + 3e^x\right) dx = 8\ln|x| + 3e^x$$

7. Solve the differential equation  $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$  with initial condition y(1) = 1. Since  $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$ ,

$$\frac{y}{x} = 8x^3 + 3x^2 - 3,$$
  
$$y = \int (8x^3 + 3x^2 - 3) \, dx = 2x^4 + x^3 - 3x + C$$

Thus 1 = y(1) = 0 + C, whence C = 1. Therefore,  $y = 2x^4 + x^3 - 3x + 1$ .

8. Given that  $f''(x) = x^3 - 2x + 1$ , f'(0) = 1, and f(0) = 0, first find f' and then find f. Let g(x) = f'(x). The statement gives  $g'(x) = x^3 - 2x + 1$ , g(0) = 1. From it, we get  $g(x) = \frac{1}{4}x^4 - x^2 + x + C$ . Then g(0) = 1 gives 1 = C, so  $f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x + 1$ .  $f'(x) = \frac{1}{4}x^4 - x^2 + x + 1$ , so  $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$ . Next, f(0) = 0 gives C = 0, so  $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$ 

## Chapter 4 AP Problems

For 1, 2, 5 and 7, a calculator may be used. For 3, 4, and 6, no calculator allowed.

- 1. A basketball has a radius of 12 centimeters, measured on the outside, when properly inflated.  $(V = \frac{4}{3}\pi r^3)$ 
  - a. Write a formula for the linearization L(r) of the volume formula when the basketball is properly inflated.
  - b. Using a linear approximation, estimate the change in the volume when the radius is increased by .1 centimeters from its proper inflation. Show the work that leads to your answer.
  - c. The basketball is considered to be in danger of bursting from inflation when its volume is more than 6% above the volume when its properly inflated. What is its radius at the point when it becomes in danger of bursting? Give your answer correct to three decimal places.
- 2. How many critical points does the function  $f(x) = |x^3 2x|$  have over its entire domain?
  - A. 2
  - B. 3
  - C. 4
  - D. 5
  - E. Infinitely many

- **3.** The function g is continuous on [-1, 2] and differentiable on (-1, 2). If g(-1) = 2 and g(2) = -4, which of the following statements is not necessarily true?
  - A. There exists a value c on (-1, 2) such that f(c) = 0.
  - B. There exists a value c on (-1, 2) such that f'(c) = 0.
  - C. There exists a value c on (-1, 2) such that f(c) = -3.
  - D. There exists a value c on (-1, 2) such that f'(c) = -2.
  - E. There exists a value c on [-1, 2] such that  $f(c) \ge f(x)$  for all x on [-1, 2].
- 4. The graph shown depicts f', the derivative of f. The graph of f' has x-intercepts at -1 and 2 and a relative maximum where x = 0. At which value(s) of x does f have a point of inflection?



- A. -1 only
- B. -1 and 2
- C. 0 only
- D. 0 and 2
- E. -1, 0 and 2

5. A function h has the property that h > 0, h' < 0 and h'' > 0 for all real values of x. Which of these could be the graph of h?



6. Consider the graph of the function  $f(x) = \frac{1}{x}$  in the first quadrant, and a line *l* tangent to *f* at a point *P* where x = k.



- a. Find the slope of the line tangent to f at x = k in terms of k.
- b. Write an equation for the tangent line l in terms of k.
- c. Using the equation found in part (b), find the x and y intercepts of the line l.
- d. Find the area of the triangle formed by l and the coordinate axes.

7. Which of the following is equivalent to  $\int \frac{6x^2 - 4x}{x^4} dx$ ?

I. 
$$\frac{5(2x^3 - 2x^2)}{x^5} + C_1$$
  
II.  $\frac{-6x + 2}{x^2} + C_2$   
III.  $\frac{x^2 - 6x + 2}{x^2} + C_3$ 

- A. I only
- B. II only
- C. III only
- D. II and III
- E. I, II and III

# Solutions to Chapter 4 AP Problems

- 1. A basketball has a radius of 12 centimeters, measured on the outside, when properly inflated.  $(V = \frac{4}{3}\pi r^3)$ 
  - a. Write a formula for the linearization L(r) of the volume formula when the basketball is properly inflated.
    - $L(r) = 576\pi(r-12) + 2304\pi$  or  $576\pi r 4608\pi$
  - b. Using a linear approximation, estimate the change in the volume when the radius is increased by .1 centimeters from its proper inflation. Show the work that leads to your answer.

 $\Delta V \approx V'(12) \cdot \Delta r = 576\pi \cdot (.1) = 57.6\pi$  cubic centimeters

c. The basketball is considered to be in danger of bursting from inflation when its volume is more than 6% above the volume when its properly inflated. What is its radius at the point when it becomes in danger of bursting? Give your answer correct to three decimal places.

$$1.06V = 1.06\left(\frac{4}{3}\pi \cdot 12^3\right) = \frac{4}{3}\pi r^3$$

 $r^3 = 1.06(12^3)$ 

r = 12.235 centimeters [THIS QUESTION CORRESPONDS WITH SECTION 4.1]

- **2.** How many critical points does the function  $f(x) = |x^3 2x|$  have over its entire domain?
  - A. 2
  - B. 3
  - C. 4
  - D. 5

E. Infinitely many

# D [THIS QUESTION CORRESPONDS WITH SECTION 4.2]

- **3.** The function g is continuous on [-1, 2] and differentiable on (-1, 2). If g(-1) = 2 and g(2) = -4, which of the following statements is not necessarily true?
  - A. There exists a value c on (-1, 2) such that f(c) = 0.
  - B. There exists a value c on (-1, 2) such that f'(c) = 0.
  - C. There exists a value c on (-1, 2) such that f(c) = -3.
  - D. There exists a value c on (-1, 2) such that f'(c) = -2.

E. There exists a value c on [-1, 2] such that  $f(c) \ge f(x)$  for all x on [-1, 2]. B [THIS QUESTION CORRESPONDS WITH SECTION 4.3] 204

4. The graph shown depicts f', the derivative of f. The graph of f' has x-intercepts at -1 and 2 and a relative maximum where x = 0. At which value(s) of x does f have a point of inflection?



# A. -1 only

- B.  $-1 \mbox{ and } 2$
- C. 0 only
- D. 0 and 2
- E. -1, 0 and 2

# A [THIS QUESTION CORRESPONDS WITH SECTION 4.4]

5. A function h has the property that h > 0, h' < 0 and h'' > 0 for all real values of x. Which of these could be the graph of h?



A [THIS QUESTION CORRESPONDS WITH SECTION 4.6]

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6. Consider the graph of the function  $f(x) = \frac{1}{x}$  in the first quadrant, and a line *l* tangent to *f* at a point *P* where x = k.



a. Find the slope of the line tangent to f at x = k in terms of k.

$$\frac{-1}{k^2}$$

b. Write an equation for the tangent line l in terms of k.

$$y - \frac{1}{k} = -\frac{1}{k^2}(x - k)$$
 or  $y = -\frac{1}{k^2}x + \frac{2}{k}$ 

c. Using the equation found in part (b), find the x and y intercepts of the line l.

x-intercept: 2k; y-intercept:  $\frac{2}{k}$ 

d. Find the area of the triangle formed by l and the coordinate axes.

# <sup>2</sup> [THIS QUESTION CORRESPONDS WITH SECTION 4.7]

7. Which of the following is equivalent to  $\int \frac{6x^2 - 4x}{x^4} dx$ ?

I. 
$$\frac{5(2x^3 - 2x^2)}{x^5} + C_1$$
  
II.  $\frac{-6x + 2}{x^2} + C_2$   
III.  $\frac{x^2 - 6x + 2}{x^2} + C_3$ 

- A. I only
- B. II only
- C. III only
- D. II and III

E. I, II and III

D [THIS QUESTION CORRESPONDS WITH SECTION 4.9]

#### Ray Cannon's Chapter 5 Overview

Chapter 5 introduces a totally different topic from what has been covered up to now: the Definite Integral. Section 5.1 motivates the topic through the idea of area. Students need to be careful to understand that this motivation is for non-negative integrands. They are not responsible for the summation formulas in this section, but nonetheless need to know the definite integral is the limit of Riemann sums, which are formally defined in Section 5.2. The text returns to Numerical Approximations in Section 7.8, but here students should be familiar with Left-hand, right-hand, and midpoint Riemann sums. Section 5.2 also contains the basic properties of definite integrals such as linearity and additivity for adjacent intervals.

There are two parts to the Fundamental Theorem of Calculus, and Section 5.3 presents the part the shows how to evaluate definite integrals using antiderivatives. This is the form that most students remember; in fact, many forget this is a theorem, and so this point should be stressed. Section 5.4 gives the second part of the FTC, which is that every continuous function has an antiderivative. Many students have difficulty in understanding this part of the theorem, and questions testing it are always some of the more difficult on the AP exam. Students have difficulty understanding the topic of "functions defined by integrals" even leaving aside how to differentiate such functions. Be sure to cover examples using the chain rule.

Section 5.5 covers interpreting the definite integral as giving the net change in f given how f is changing, i.e. given f'. This is an important interpretation of the definite integral, and has appeared often on the AP exam in various settings. This is a section you may want to keep returning to when making assignments. Section 5.6 covers the technique of integration by substitution, and be sure students understand how to change the limits of integration and not always rely on back-substitution. The AP course description lists inverse trigonometric functions as "basic." So students need to know the formulas in Section 5.7 for antiderivatives, going beyond simply the power rule. The Chapter concludes with Section 5.8, discussing the importance of the differential equation y' = ky and showing how to solve it. This topic is profitably revisited in Chapter 9 when discussing "separation of variables."

#### 5.1. Limits: Approximating and Computing Area.

Class Time AB 2 periods; BC 2 periods. Essential.

#### **Key Points**

- Approximating area by rectangles.
  - (i) Riemann sums.
  - (ii) Right- and left-endpoint approximations.
- Midpoint approximation.

#### Lecture Material

First assume that f(x) is a continuous non-negative function over an interval [a, b]. Discuss what it means to partition [a, b] into n subintervals of equal width. Use the slide provided, a graphing program, or dynamic geometry software to approximate the area under the curve with n = 5 using the right endpoints, the left endpoints and the midpoints of the subinterval. Do Example 1 in the text, but also calculate the left-endpoint and midpoint approximations.

Review summation notation and its properties. Then derive the formulas for the rightendpoint, left-endpoint, and midpoint approximations. Do Example 2. This will be all you can cover in one hour.

Summation notation is not tested on the AB exam. On the BC exam it is used with infinite series and power series. Examples 3 and 4 show how infinite sums can be used to find areas without recourse to definite integrals or the Fundamental Theorem of Calculus (FTC). This type of problem does not need to be stressed for the AP exams.

Setting up and computing left-, right- and midpoint sums with a small number of partition points (Examples 1 and 2) is tested from function values given in tables (Exercises 3, 5, 6) or with an equation. These should be reviewed carefully and students should be given practice doing these.

Students should have a good graphical understanding of the sums, and the connection between them and the increasing/decreasing behaviors (Figures 7, 8, 9). They should know, for example, that a right-endpoint sum overestimates the area of an increasing function, etc.

#### Discussion Topics/Class Activities

Discuss how if an object moves in a straight line with constant velocity v, then the distance traveled over a time interval  $[t_1, t_2]$  is equal to the area of the rectangle with height v and width  $t_2 - t_1$ .

Suggested Problems (2 assignments)

 $Exercises \ 1, \ 3, \ 5, \ 7, \ 11, \ 13, \ 15, \ 17, \ 65, \ 67, \ 79, \ 86$ 

# Worksheet 5.1. Approximating and Computing Area

1. Calculate the area of the shaded rectangles in the figure. Which approximation to the area under the curve is this?



**2.** Estimate  $R_6$  and  $L_6$  for the function shown in the graph.



**3.** Evaluate the following sums (see Equations (3)-(5)). (optional)

a. 
$$\sum_{k=1}^{20} 2k + 1$$

b. 
$$\sum_{\ell=1}^{10} 2(\ell^3 - 2\ell^2)$$

4. Evaluate 
$$\lim_{N \to \infty} \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3}$$
. (optional)

5. Use Equations (3)–(5) to find a formula for  $R_N$  for  $f(x) = 3x^2 - x + 4$  over the interval [0, 1]. (optional)

6. Evaluate 
$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left(\frac{j}{N}\right)^2}$$
 by interpreting the limit as an area. (optional)

# Solutions to Worksheet 5.1

1. Calculate the area of the shaded rectangles in the figure below. Which approximation to the area under the curve is this?



Each rectangle has a width of 1 and the height is taken as the value of the function at the midpoint of the interval. Thus, the area of the shaded rectangles is

$$1\left(\frac{26}{29} + \frac{22}{13} + \frac{18}{5} + \frac{14}{5} + \frac{10}{13} + \frac{6}{29}\right) = \frac{18784}{1885} \approx 9.965$$

Because there are six rectangles and the height of each rectangle is taken as the value of the function at the midpoint of the interval, the shaded rectangles represent the approximation  $M_6$  to the area under the curve.

**2.** Estimate  $R_6$ , and  $L_6$  for the function shown below.



**3.** Evaluate the following sums (See formulas (3)-(5)).

a. 
$$\sum_{k=1}^{20} 2k + 1$$
$$\sum_{k=1}^{20} (2k+1) = 2\sum_{k=1}^{20} k + \sum_{k=1}^{20} 1 = 2\left(\frac{20^2}{2} + \frac{20}{2}\right) + 20 = 440.$$
b. 
$$\sum_{\ell=1}^{10} 2(\ell^3 - 2\ell^2)$$

$$\begin{split} \sum_{\ell=1}^{10} \left(\ell^3 - 2\ell^2\right) &= \sum_{\ell=1}^{10} \ell^3 - 2\sum_{\ell=1}^{10} \ell^2 \\ &= \left(\frac{10^4}{4} + \frac{10^3}{2} + \frac{10^2}{4}\right) - 2\left(\frac{10^3}{3} + \frac{10^2}{2} + \frac{10}{6}\right) \\ &= 2255. \end{split}$$

4. Evaluate  $\lim_{N \to \infty} \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3}$ Let

$$s_N = \sum_{i=1}^N \frac{i^2 - i + 1}{N^3}.$$

Then

$$s_{N} = \sum_{i=1}^{N} \frac{i^{2} - i + 1}{N^{3}} = \frac{1}{N^{3}} \left[ \left( \sum_{i=1}^{N} i^{2} \right) - \left( \sum_{i=1}^{N} i \right) + \left( \sum_{i=1}^{N} 1 \right) \right]$$
$$= \frac{1}{N^{3}} \left[ \left( \frac{N^{3}}{3} + \frac{N^{2}}{2} + \frac{N}{6} \right) - \left( \frac{N^{2}}{2} + \frac{N}{2} \right) + N \right] = \frac{1}{3} + \frac{2}{3N^{2}}.$$
$$\lim s_{N} = \frac{1}{2}.$$

Therefore,  $\lim_{N \to \infty} s_N = \frac{1}{3}$ .

5. Use formulas (3)–(5) to find a formula for  $R_N$  for  $f(x) = 3x^2 - x + 4$  over the interval [0, 1].

Let  $f(x) = 3x^2 - x + 4$  on the interval [0, 1]. Then  $\Delta x = \frac{1-0}{N} = \frac{1}{N}$  and a = 0. Hence,

$$R_{N} = \Delta x \sum_{j=1}^{N} f(0+j\Delta x) = \frac{1}{N} \sum_{j=1}^{N} \left( 3j^{2} \frac{1}{N^{2}} - j\frac{1}{N} + 4 \right)$$
$$= \frac{3}{N^{3}} \sum_{j=1}^{N} j^{2} - \frac{1}{N^{2}} \sum_{j=1}^{N} j + \frac{4}{N} \sum_{j=1}^{N} 1$$
$$= \frac{3}{N^{3}} \left( \frac{N^{3}}{3} + \frac{N^{2}}{2} + \frac{N}{6} \right) - \frac{1}{N^{2}} \left( \frac{N^{2}}{2} + \frac{N}{2} \right) + \frac{4}{N}N$$
$$= 1 + \frac{3}{2N} + \frac{1}{2N^{2}} - \frac{1}{2} - \frac{1}{2N} + 4$$

and

$$\lim_{N \to \infty} R_N = \lim_{N \to \infty} \left( 4.5 + \frac{1}{N} + \frac{1}{2N^2} \right) = 4.5.$$

6. Evaluate  $\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left(\frac{4j}{N}\right)^2}$  by interpreting the limit as an area. The limit

$$\lim_{N \to \infty} R_N = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^N \sqrt{1 - \left(\frac{j}{N}\right)^2}$$

represents the area between the graph of  $y = f(x) = \sqrt{1 - x^2}$  and the *x*-axis over the interval [0, 1]. This is the portion of the circular disk  $x^2 + y^2 \leq 1$  that lies in the first quadrant. Accordingly, its area is  $\frac{1}{4}\pi (1)^2 = \frac{\pi}{4}$ .

#### 5.2. The Definite Integral.

Class Time AB 2 periods; BC 1 period. Essential.

#### **Key Points**

- Definition of the definite integral.
- Properties of the definite integral (Theorems 2–5).
- Interpretation of "signed area."

#### Lecture Material

Explain that when we use the summation technique discussed in Section 5.1, we can use a partition with unequal subinterval lengths and the heights of the rectangles can be fevaluated at any point in the subinterval including the endpoints. The miracle is that if f is continuous, the limits of all the Riemann sums, as the norm of the partition Pgoes to 0, are all the same real number. This number is called the definite integral of ffrom a to b, denoted  $\int_{a}^{b} f(x) dx$ . Next discuss the graphical interpretation of the definite integral when f is nonnegative over [a, b] and when f is not always nonnegative over [a, b]. Finally, discuss the properties of the definite integral, including the integral of a constant over and interval, the linearity of the definite integral, reversing the limits of integration, additivity property for adjacent intervals, and the Comparison Theorem. If time permits,

use the right-endpoint approximations to prove that for b > 0,  $\int_0^b x^2 dx = \frac{b^3}{3}$ .

## Discussion Topics/Class Activities

Lead a discussion of the difference between the graphs of  $\int_{a}^{b} f(x) dx$  and  $\int_{a}^{b} |f(x)| dx$ . Use the same function, such as f(x) = 3 - x on intervals containing x = 3, and compare and contrast the answers.

#### Suggested Problems

Exercises 1, 3, 5, 7, 9, 23, 25, 31, 33, 43-46, 59, 61, 75, 83-84

# Worksheet 5.2. The Definite Integral



2. The graph of  $f(x) = \begin{cases} -\sqrt{1 - (x - 1)^2} & \text{if } 0 \le x \le 2\\ \sqrt{4 - (x - 4)^2} & \text{if } 2 < x \le 6 \end{cases}$  consists of two semicircles:

4

3

5

6

1

Evaluate the following integrals.

a. 
$$\int_{0}^{2} f(x) dx$$
  
b. 
$$\int_{0}^{6} f(x) dx$$
  
c. 
$$\int_{1}^{4} f(x) dx$$

**3.** Let  $g(x) = \begin{cases} x-1 & \text{if } 0 \le x \le 3\\ 8-2x & \text{if } 3 < x \le 5 \end{cases}$ . Use the graph of g(x) to evaluate the integrals.



a. 
$$\int_0^3 g(x) \, dx$$
  
b. 
$$\int_0^5 g(x) \, dx$$

4. Use the basic properties of the integral and Equations (4)–(6) to calculate the following integrals.

a. 
$$\int_{1}^{4} 6x^2 dx$$

b. 
$$\int_{-2}^{3} 3x + 4 \, dx$$

c. 
$$\int_{1}^{3} |2x - 4| \, dx$$

# Solutions to Worksheet 5.2



Evaluate the following integrals

a.  $\int_0^2 f(x) \, dx$ 

The definite integral  $\int_{2}^{2} f(x) dx$  is the signed area of a semicircle of radius 1 which lies below the x-axis. Therefore,

$$\int_{0}^{2} f(x) \, dx = -\frac{1}{2}\pi \left(1\right)^{2} = -\frac{\pi}{2}$$

b.  $\int_{0}^{6} f(x) dx$ 

The definite integral  $\int_0^6 f(x) dx$  is the signed area of a semicircle of radius 1 which lies below the x-axis and a semicircle of radius 2 which lies above the x-axis. Therefore,

$$\int_{0}^{6} f(x) \, dx = \frac{1}{2}\pi \left(2\right)^{2} - \frac{1}{2}\pi \left(1\right)^{2} = \frac{3\pi}{2}.$$

c.  $\int_{1}^{4} f(x) dx$ 

The definite integral  $\int_{1}^{4} f(x) dx$  is the signed area of one-quarter of a circle of radius 1 which lies below the x-axis and one-quarter of a circle of radius 2 which lies above the x-axis. Therefore,

$$\int_{1}^{4} f(x) \, dx = \frac{1}{4}\pi \left(2\right)^{2} - \frac{1}{4}\pi \left(1\right)^{2} = \frac{3}{4}\pi$$

**3.** Let  $g(x) = \begin{cases} x-1 & \text{if } 0 \le x \le 3\\ 8-2x & \text{if } 3 < x \le 5 \end{cases}$  Use the graph of g(x) given below to evaluate the following integrals.



a.  $\int_{0}^{3} g(x) dx$ The region bound by the curve y = g(x) and the x-axis over the interval [0,3] is comprised of two right triangles, one with area  $\frac{1}{2}$  below the axis, and one with area 2 above the axis. The definite integral is therefore equal to  $2 - \frac{1}{2} = \frac{3}{2}$ .

b.  $\int_0^5 g(x) \, dx$ 

The region bound by the curve y = g(x) and the x-axis over the interval [3,5] is comprised of another two right triangles, one with area 1 above the axis and one with area one below the axis. The definite integral is therefore equal to zero.

4. Use the basic properties of the integral and formulas (4)-(6) to calculate the following integrals.

a. 
$$\int_{1}^{4} 6x^{2} dx$$
$$\int_{1}^{4} 6x^{2} dx = 6 \int_{0}^{4} x^{2} dx - 6 \int_{0}^{1} x^{2} dx = 6 \left(\frac{1}{3}(4)^{3} - \frac{1}{3}(1)^{3}\right) = 126.$$
  
b. 
$$\int_{-2}^{3} 3x + 4 dx$$
$$\int_{-2}^{3} (3x + 4) dx = 3 \int_{-2}^{3} x dx + 4 \int_{-2}^{3} dx$$
$$= 3 \left(\int_{-2}^{0} x dx + \int_{0}^{3} x dx\right) + 4(3 - (-2))$$
$$= 3 \left(\int_{0}^{3} x dx - \int_{0}^{-2} x dx\right) + 20$$
$$= 3 \left(\frac{1}{2}3^{2} - \frac{1}{2}(-2)^{2}\right) + 20 = \frac{55}{2}.$$

c.  $\int_{1}^{3} |2x - 4| dx$ The area between |2x - 4| and the x axis consists of two triangles above the x-axis, each width 1 and height 2, and hence with area 1. The total area, and hence the definite integral, is 2.

Class Time AB 2 periods; BC 1 period. Essential.

#### **Key Points**

- Fundamental Theorem of Calculus, Part I: If f(x) is continuous on [a, b] and has an antiderivative on [a, b], then  $\int_{a}^{b} f(x) dx = F(b) F(a)$ .
- The Fundamental Theorem of Calculus, Part I enables us to evaluate the definite integral in the cases where the integrand has an antiderivative.

#### Lecture Material

Illustrate the Fundamental Theorem of Calculus, Part I, with the example  $\int_0^b x^2 dx = \frac{b^3}{3}$ .

State the Fundamental Theorem of Calculus, Part I, and prove it or at least sketch an outline of the proof. Do several examples where the integrand has a straightforward antiderivative, such as Exercises 9, 13, 16, 33, and 38.

#### **Discussion Topics/Class Activities**

With appropriate guidance, students could work Exercise 62 at their desks.

#### Suggested Problems

Exercises 1, 5, 11, 13, 17, 25, 27, 33, 35, 37, 43

# Worksheet 5.3. The Fundamental Theorem of Calculus, Part I

1. Sketch the graph of  $f(x) = \cos x$  over the interval  $[-\pi/2, \pi/2]$  and use Part I of the Fundamental Theorem of Calculus (FTC I) to calculate the enclosed area.



**2.** Evaluate using FTC I.

a. 
$$\int_{-2}^{2} (10x^9 + 3x^5) dx$$

b. 
$$\int_0^4 \sqrt{x} \, dx$$

c. 
$$\int_{4}^{9} \frac{16+t}{t^2} dt$$

d. 
$$\int_{\pi/4}^{3\pi/4} \sin\theta \, d\theta$$

e.  $\int_{0}^{5} |x^2 - 4x + 3| dx$  (Write as a sum if integrals without absolute values—then apply FTC I.)

# Solutions to Worksheet 5.3

1. Sketch the graph of  $f(x) = \cos x$  over the interval  $[-\pi/2, \pi/2]$  and use part I of the Fundamental Theorem of Calculus (FTC I) to calculate the enclosed area.



**2.** Evaluate using FTC I.

a. 
$$\int_{-2}^{2} (10x^{9} + 3x^{5}) dx$$
$$\int_{-2}^{2} (10x^{9} + 3x^{5}) dx = \left(x^{10} + \frac{1}{2}x^{6}\right)\Big|_{-2}^{2} = \left(2^{10} + \frac{1}{2}2^{6}\right) - \left(2^{10} + \frac{1}{2}2^{6}\right) = 0.$$
  
b. 
$$\int_{0}^{4} \sqrt{x} dx$$
$$\int_{0}^{4} \sqrt{x} dx = \int_{0}^{4} x^{1/2} dy = \frac{2}{3}x^{3/2}\Big|_{0}^{4} = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}.$$
  
c. 
$$\int_{4}^{9} \frac{16 + t}{t^{2}} dt$$
$$\int_{4}^{9} \frac{16 + t}{t^{2}} dt = \int_{4}^{9} 16t^{-2} + t^{-1} dt = -16t^{-1} + \ln t\Big|_{4}^{9} = \frac{20}{9} + \ln \frac{9}{4}$$
$$\int_{\pi/4}^{3\pi/4} \sin \theta d\theta$$
$$\int_{\pi/4}^{\frac{3\pi}{4}} \sin \theta d\theta = -\cos \theta \Big|_{\frac{3\pi}{4}}^{\frac{3\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}.$$

d.  $\int_{0}^{5} |x^2 - 4x + 3| dx$  (Write as a sum if integrals without absolute values—then apply FTC I.)

$$\begin{split} \int_{0}^{5} |x^{2} - 4x + 3| \, dx &= \int_{0}^{5} |(x - 3)(x - 1)| \, dx \\ &= \int_{0}^{1} (x^{2} - 4x + 3) \, dx + \int_{1}^{3} -(x^{2} - 4x + 3) \, dx + \\ &\int_{3}^{5} (x^{2} - 4x + 3) \, dx \\ &= \left(\frac{1}{3}x^{3} - 2x^{2} + 3x\right) \Big|_{0}^{1} - \left(\frac{1}{3}x^{3} - 2x^{2} + 3x\right) \Big|_{1}^{3} + \\ &\left(\frac{1}{3}x^{3} - 2x^{2} + 3x\right) \Big|_{3}^{5} \\ &= \left(\frac{1}{3} - 2 + 3\right) - 0 - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \\ &\left(\frac{125}{3} - 50 + 15\right) - (9 - 18 + 9) \\ &= \frac{28}{3}. \end{split}$$
Class Time AB and BC 2 periods. Essential.

#### **Key Points**

• The area function is

$$A(x) = \int_{a}^{x} f(t) \, dt$$

The Fundamental Theorem of Calculus, Part II: If f(x) is a continuous function on [a, b] and A(x) = ∫<sub>a</sub><sup>x</sup> f(t) dt, then A'(x) = f(x).
If A(x) = ∫<sub>a</sub><sup>g(x)</sup> f(t) dt, then A'(x) = f(g(x))g'(x).

#### Lecture Material

Define the area function and use the provided slide to illustrate it. State the Fundamental Theorem of Calculus, Part II, and prove it or at least sketch an outline of the proof. Consider the example  $\int_{1}^{x} t^2 dt$ . Integrate and then take the derivative. Then do the problem using the Fundamental Theorem of Calculus, Part II. Finally, do an example that combines the Fundamental Theorem of Calculus, Part II and the Chain Rule. Example 4 and Exercise 14 are good examples.

An alternative proof is this, which starts with the conclusion of the FTC part 1:

$$\int_{a}^{x} f(t) dt = F(x) - F(a), \text{ so}$$
$$\frac{d}{dx} \int_{a}^{x} f(t) dt = \frac{d}{dx} (F(x) - F(a))$$
$$= F'(x) - 0 = f(x)$$

Go over the "graphical insight" box and Figure 7. This way of determining the properties of a function (increasing/decreasing, local extreme values, etc.) from the definite integral is the same as was done in Chapter 4. Many students find this approach easier to understand.

#### **Discussion Topics/Class Activities**

First day: Exercise 39 or 43 or something similar could be discussed; Second day: Go over the "Graphical Insight" box on page 281. This approach to the relationship between f and f' is often tested on the AP Calculus exams.

Suggested Problems (2 assignments) Exercises 1, 3, 5, 7, 9, 15, 17, 19, 21, 25, 29, 31, 35–39, 43, 45

# Worksheet 5.4. The Fundamental Theorem of Calculus, Part II

**1.** If 
$$G(x) = \int_{1}^{x} \tan t \, dt$$
, find  
a.  $G(1)$ 

- b.  $G'(\pi/4)$
- 2. Find a formula for the function represented by the integral.

a. 
$$\int_2^x (t^2 - t) dt$$

b. 
$$\int_{\pi/4}^x \cos u \, du$$

**3.** Express the antiderivative F(x) of f(x) satisfying the given initial condition as an integral.

a. 
$$f(x) = \sqrt{x^4 + 1}, \quad F(3) = 0$$

b. 
$$f(x) = \sin(x^3)$$
,  $F(-\pi) = 0$ 

4. Calculate the derivatives.

a. 
$$\frac{d}{dx} \int_{1}^{x} \sin(t^2) dt$$

b. 
$$\frac{d}{dx} \int_{1}^{x^3} \tan t \, dt$$

5. Sketch the graph of  $A(x) = \int_0^x f(t) dt$  for the function f(t) over the interval [0, 4].



**6.** Let  $A(x) = \int_0^x f(t) dt$  for f(t) the piecewise linear function shown. Find a formula for A(x) (actually, two formulas-one for  $0 \le x \le 1$  and another for  $1 \le x \le 3$ ) and sketch the graph of A(x).



# Solutions to Worksheet 5.4

**1.** If 
$$G(x) = \int_{1}^{x} \tan t \, dt$$
, find  
a.  $G(1)$ 

b.  $G'(\pi/4)$ 

By definition,  $G(1) = \int_{\frac{1}{4}}^{1} \tan t \, dt = 0$ . By FTC,  $G'(x) = \tan x$ , whence  $G'(0) = \tan 0 = 0$  and  $G'(\frac{\pi}{4}) = \tan \frac{\pi}{4} = 1$ .

2. Find a formula for the function represented by the integral.

a. 
$$\int_{2}^{x} (t^{2} - t) dt$$
$$F(x) = \int_{2}^{x} (t^{2} - t) dt = \left(\frac{1}{3}t^{3} - \frac{1}{2}t^{2}\right)\Big|_{2}^{x} = \frac{1}{3}x^{3} - \frac{1}{2}x^{2} - \frac{2}{3}$$
b. 
$$\int_{\pi/4}^{x} \cos u \, du$$
$$F(x) = \int_{\pi/4}^{x} \cos u \, du = \sin u\Big|_{\pi/4}^{x} = \sin x - \frac{\sqrt{2}}{2}$$
$$F(x) = \int_{2}^{x} (t^{2} - t) \, dt = \left(\frac{1}{3}t^{3} - \frac{1}{2}t^{2}\right)\Big|_{2}^{x} = \frac{1}{3}x^{3} - \frac{1}{2}x^{2} - \frac{2}{3}$$

**3.** Express the antiderivative F(x) of f(x) satisfying the given initial condition as an integral.

a. 
$$f(x) = \sqrt{x^4 + 1}$$
,  $F(3) = 0$ .  
The antiderivative  $F(x)$  of  $f(x) = \sqrt{x^4 + 1}$  satisfying  $F(3) = 0$  is  
 $F(x) = \int_3^x \sqrt{t^4 + 1} dt$ 

b. 
$$f(x) = \sin(x^3)$$
,  $F(-\pi) = 0$ .  
The antiderivative  $F(x)$  of  $f(x) = \sin(x^3)$  satisfying  $F(-\pi) = 0$  is  
 $F(x) = \int_{-\pi}^x \sin(t^3) dt$ 

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4. Calculate the derivatives.

a. 
$$\frac{d}{dx} \int_{1}^{x} \sin(t^{2}) dt$$
  
By FTC II,  $\frac{d}{dx} \int_{1}^{x} \sin(t^{2}) dt = \sin x^{2}$ .  
b.  $\frac{d}{dx} \int_{1}^{x^{3}} \tan t dt$   
By combining the FTC and the Chain Rule,

$$G'(x) = \tan x^3 \cdot 3x^2 = 3x^2 \tan x^3$$

**5.** Sketch the graph of  $A(X) = \int_0^x f(t) dt$  for the function f(t) over the interval [0, 4].



$$A(x) = \begin{cases} 2x & \text{if } 0 \le x < 1\\ 1+x & \text{if } 1 \le x < 2\\ 5-x & \text{if } 2 \le x < 3\\ 2 & \text{if } 3 \le x < 4 \end{cases}$$



**6.** Let  $A(x) = \int_0^x f(t) dt$  for f(t) the piecewise linear function shown below. Find a formula for A(x) (actually, two formulas-one for  $0 \le x \le 1$  and another for  $1 \le x \le 3$ ) and sketch the graph of A(x).



$$A(x) = \begin{cases} x & \text{if } 0 \le x < 1\\ \frac{1}{2}(x^2 + 1) & \text{if } 1 \le x \le 3 \end{cases}$$



Class Time AB 2 periods; BC 1 period. Essential.

#### **Key Points**

• Net change as an integral

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} s'(t) dt$$

- If an object is traveling in a straight line at velocity v(t), then the net change (displacement) during  $[t_1, t_2] = \int_{t_1}^{t_2} v(t) dt$ .
- The total distance traveled during  $[t_1, t_2] = \int_{t_1}^{t_2} |v(t)| dt$ .

#### Lecture Material

Introduce the following problem. Water is flowing into an empty bucket at a constant rate of r(t) gallons per second. Ask how much water is in the bucket after 4 seconds. Show students that this is just the definite integral of r(t) over the interval [0, 4]. If the flow rate r(t) varies continuously over an interval  $[t_1, t_2]$ , then the quantity of water in the bucket is equal to the area under the graph of r(t) from  $t_1$  to  $t_2$ . This is because if s(t) is the amount of water in the bucket at time t, then s'(t) = r(t). In general, the quantity  $s(t_2) - s(t_1)$  is called the net change in s(t) over the interval  $[t_1, t_2]$  and is equal to  $\int_{t_1}^{t_2} s'(t) dt$ . Work Exercise 4.

We can apply the net change formula to find the displacement of an object traveling in a straight line at a velocity of v(t). Thus displacement during  $[t_1, t_2] = \int_{t_1}^{t_2} v(t)$ . To calculate the actual distance traveled rather than displacement, we must integrate the absolute value of velocity, which is the speed |v(t)|. Work Exercise 8.

The economic applications are not tested on the AP Calculus exams, but they are good applications to cover if you have time.

#### **Discussion Topics/Class Activities**

The analysis of the motion of a particle on a number line (AB and BC) and in the plane (BC only) is often tested on the AP exams. Do problems like Exercises 27 and 28 and supplement with actual AP exam questions.

#### Suggested Problems

Exercises 1, 2, 4, 8, 9, 10, 19, 25

# Worksheet 5.5. Net or Total Change as the Integral of a Rate

1. Find the displacement over the time interval [1, 6] of a helicopter whose vertical velocity at time t is

$$v(t) = .02t^2 + t$$
 ft/s

2. A particle is moving along a straight line with velocity  $v(t) = \cos t$  m/s. Find (a) the total displacement and (b) the total distance traveled over the time interval  $[0, 4\pi]$ .

**3.** The velocity (ft/s) of a car is recorded at half-second intervals:

t:	0	.5	1	1.5	2	2.5	3	3.5	4
v(t):	0	12	$\overline{20}$	$\overline{29}$	38	44	32	$\overline{35}$	30

Use the average of the left- and right-endpoint approximations as illustrated to estimate the total distance traveled over the time interval [0, 4].



- 4. The heat capacity C(T) of a substance is the amount of energy (joules) required to raise the temperature of one gram of the substance by one degree (°C) when its temperature is T.
  - a. Explain why the energy required to raise the temperature of one gram from  $T_1$  to  $T_2$  is  $\int_{T_1}^{T_2} C(T) dT$ .
  - b. If a substance has heat capacity  $C(T) = 6 + .2\sqrt{T}$ , calculate the energy required to raise the temperature of one gram of the substance from 50° to 100°C.

## Solutions to Worksheet 5.5

1. Find the displacement over the time interval [1,6] of a helicopter whose vertical velocity at time t is

$$v(t) = .02t^2 + t$$
 ft/s.

Given 
$$v(t) = \frac{1}{50}t^2 + t$$
 ft/s, the change in height over [1, 6] is  

$$\int_1^6 v(t) dt = \int_1^6 (\frac{1}{50}t^2 + t) dt = (\frac{1}{150}t^3 + \frac{1}{2}t^2) \Big|_1^6$$

$$= \left(\frac{1}{150}(6)^3 + \frac{1}{2}(6)^2\right) - \left(\frac{1}{150}(1)^3 + \frac{1}{2}(1)^2\right) = \frac{284}{15} \approx 18.93 \text{ ft.}$$

2. A particle is moving along a straight line with velocity  $v(t) = \cos t$  m/s. Find (a) the total displacement and (b) the total distance traveled over the time interval  $[0, 4\pi]$ .

Total displacement is given by  $\int_0^{4\pi} \cos t \, dt = \sin t |_0^{4\pi} = 0$  m, while total distance is given by

$$\int_{0}^{4\pi} |\cos t| dt = \int_{0}^{\pi/2} \cos t \, dt + \int_{\pi/2}^{3\pi/2} -\cos t \, dt + \int_{3\pi/2}^{5\pi/2} \cos t \, dt + \int_{5\pi/2}^{7\pi/2} -\cos t \, dt + \int_{7\pi/2}^{4\pi} \cos t \, dt$$
$$= \sin t |_{0}^{\pi/2} - \sin t |_{\pi/2}^{3\pi/2} + \sin t |_{3\pi/2}^{5\pi/2} - \sin t |_{5\pi/2}^{7\pi/2} + \sin t |_{7\pi/2}^{4\pi} = 8 \text{ m}$$

**3.** The velocity (ft/s) of a car is recorded at half-second intervals below.

Use the average of the left- and right-endpoint approximations as illustrated below to estimate the total distance traveled over the time interval [0, 4].



Let  $\Delta x = .5$ . Then

$$R_N = .5 \cdot (12 + 20 + 29 + 38 + 44 + 32 + 35 + 30) = 120 \text{ ft.}$$
  

$$L_N = .5 \cdot (0 + 12 + 20 + 29 + 38 + 44 + 32 + 35) = 105 \text{ ft.}$$

The average of  $R_N$  and  $L_N$  is 112.5 ft

- 4. The heat capacity C(T) of a substance is the amount of energy (joules) required to raise the temperature of one gram of the substance by one degree (°C) when its temperature is T.
  - a. Explain why the energy required to raise the temperature of one gram from  $T_1$  to  $T_2$  is  $\int_{T_1}^{T_2} C(T) dT$ .

Since C(T) is the energy required to raise the temperature of one gram of a substance by one degree when its temperature is T, the total energy required to raise the temperature from  $T_1$  to  $T_2$  is given by the definite integral  $\int_{T_1}^{T_2} C(T) dT$ . As C(T) > 0, the definite integral also represents the area under the graph of C(T).

b. If a substance has heat capacity  $C(T) = 6 + .2\sqrt{T}$ , calculate the energy required to raise the temperature of one gram of the substance from 50° to 100° C.

If  $C(T) = 6 + .2\sqrt{T} = 6 + \frac{1}{5}T^{1/2}$ , then the energy required to raise the temperature from 50°C to 100°C is  $\int_{50}^{100} C(T) dT$  or

$$\int_{50}^{100} \left(6 + \frac{1}{5}t^{1/2}\right) dt = \left(6t + \frac{2}{15}t^{3/2}\right)\Big|_{50}^{100}$$
$$= \left(6(100) + \frac{2}{15}(100)^{3/2}\right) - \left(6(50) + \frac{2}{15}(50)^{3/2}\right)$$
$$= \frac{1300 - 100\sqrt{2}}{3} \approx 386.19 \text{ Joules}$$

#### 5.6. Substitution Method.

Class Time AB and BC 2 periods. Essential.

#### **Key Points**

- Substitution Method of integrating.
- Change of Variables Formula for Definite Integrals.

#### Lecture Material

Explain the Substitution Method as the Chain Rule "in reverse." Illustrate with examples such as  $\int 2x \cos(x^2) dx$ ,  $\int \sqrt{1+2x} dx$ , and  $\int \frac{x}{\sqrt{x^2+9}} dx$ . Explain that the technique doesn't work for  $\int \cos(x^2) dx$  and  $\int \sqrt{1+2x^2} dx$ .

State the Change of Variables Formula for Definite Integrals and work two or three examples. Explain that you can change the limits to the new variable or keep the limits the same and change everything back to the old variable at the end.

## **Discussion Topics/Class Activities**

Have students work  $\int_{0}^{2} \frac{x+3}{x^{2}+6x+1} dx$  at their desks. Suggested Problems (2 assignments) Exercises 7, 11, 13, 15, 17, 21, 31, 33, 74, 75, 76, 79, 81, 85, 87, 89

# Worksheet 5.6. Substitution Method

1. Evaluate the indefinite integral.

a. 
$$\int x(x+1)^9 \, dx$$

b. 
$$\int \sin(2x-4) \, dx$$

c. 
$$\int \frac{x^3}{(x^4+1)^4} dx$$

d. 
$$\int \sqrt{4x-1} \, dx$$

e. 
$$\int x \cos(x^2) \, dx$$

f. 
$$\int \sin^5 x \cos x \, dx$$

g. 
$$\int \sec^2 x \tan^4 x \, dx$$

2. Use the Change of Variables Formula to evaluate the definite integral.

a. 
$$\int_0^1 \frac{x}{(x^2+1)^3} dx$$

b. 
$$\int_{10}^{17} (x-9)^{-2/3} dx$$

c. 
$$\int \frac{dx}{(2+\sqrt{x})^3}$$
 (Use  $u = 2 + \sqrt{x}$ .)

# Solutions to Worksheet 5.6

**1.** Evaluate the following indefinite integrals.

a. 
$$\int x(x+1)^9 dx$$
  
Let  $u = x + 1$ . Then  $x = u - 1$  and  $du = dx$ . Hence  

$$\int x(x+1)^9 dx = \int (u-1)u^9 du = \int (u^{10} - u^9) du$$
  

$$= \frac{1}{11}u^{11} - \frac{1}{10}u^{10} + C = \frac{1}{11}(x+1)^{11} - \frac{1}{10}(x+1)^{10} + C.$$
b. 
$$\int \sin(2x-4) dx$$
  
Let  $u = 2x - 4$ . Then  $du = 2 dx$  or  $\frac{1}{2} du = dx$ . Hence  

$$\int \sin(2x-4) dx = \frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos(2x-4) + C$$
c. 
$$\int \frac{x^3}{(x^4+1)^4} dx$$
  
Let  $u = x^4 + 1$ . Then  $du = 4x^3 dx$  or  $\frac{1}{4} du = x^3 dx$ . Hence  

$$\int \frac{x^3}{(x^4+1)^4} dx = \frac{1}{4} \int \frac{1}{u^4} du = -\frac{1}{12}u^{-3} + C = -\frac{1}{12}(x^4+1)^{-3} + C$$
d. 
$$\int \sqrt{4x-1} dx$$
  
Let  $u = 4x - 1$ . Then  $du = 4 dx$  or  $\frac{1}{4} du = dx$ . Hence  

$$\int \sqrt{4u-1} dx = \frac{1}{4} \int u^{1/2} du = \frac{1}{4} \cdot \frac{2}{3}u^{3/2} + C = \frac{1}{6}(4x-1)^{3/2} + C$$
e. 
$$\int x \cos(x^2) dx$$
  
Let  $u = x^2$ . Then  $du = 2x dx$  or  $\frac{1}{2} du = x dx$ . Hence,  

$$\int x \cos(x^2) dx = \frac{1}{2} \int \cos u \, du = \frac{1}{2} \sin u + C = \frac{1}{2}\sin(x^2) + C$$

f. 
$$\int \sin^5 x \cos x \, dx$$
  
Let  $u = \sin x$ . Then  $du = \cos x \, dx$ . Hence  
$$\int \sin^5 x \cos x \, dx = \int u^5 \, du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6 x + C$$
$$\int \sec^2 x \tan^4 x \, dx \text{ Let } u = \tan x. \text{ Then } du = \sec^2 x \, dx. \text{ Hence}$$
$$\int \sec^2 x \tan^4 x \, dx = \int u^4 \, du = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5 x + C$$

2. Use the change of variables formula to evaluate the following definite integrals.

a. 
$$\int_{0}^{1} \frac{x}{(x^{2}+1)^{3}} dx$$
  
Let  $u = x^{2} + 1$ . Then  $du = 2x \, dx$  or  $\frac{1}{2} \, du = x \, dx$ . Hence
$$\int_{0}^{1} \frac{x}{(x^{2}+1)^{3}} dx = \frac{1}{2} \int_{1}^{2} \frac{1}{u^{3}} du = \frac{1}{2} \cdot -\frac{1}{2} u^{-2} \Big|_{1}^{2}$$
$$= -\frac{1}{16} + \frac{1}{4} = \frac{3}{16} = 0.1875.$$

b. 
$$\int_{10}^{17} (x-9)^{-2/3} dx$$
  
Let  $u = x - 9$ . Then  $du = dx$ . Hence
$$\int_{10}^{17} (x-9)^{-2/3} dx = \int_{1}^{8} u^{-2/3} dx$$
$$= 3u^{1/3} \Big|_{1}^{8}$$
$$= 3(2-1) = 3$$

c. 
$$\int \frac{dx}{(2+\sqrt{x})^3} \quad (\text{Use } u = 2 + \sqrt{x}.)$$
  
Let  $u = 2 + \sqrt{x}$ . Then  $du = \frac{1}{2\sqrt{x}}dx$ , so that  
 $2\sqrt{x} \, du = dx$   
 $2(u-2) \, du = dx$ 

From this, we get:

$$\int \frac{dx}{(2+\sqrt{x})^3} = \int 2\frac{u-2}{u^3} du$$
  
=  $2 \int (u^{-2} - 2u^{-3}) du$   
=  $2 (-u^{-1} + u^{-2}) + C$   
=  $2 \left(-\frac{1}{2+\sqrt{x}} + \frac{1}{(2+\sqrt{x})^2}\right) + C$   
=  $2 \left(\frac{-2-\sqrt{x}+1}{(2+\sqrt{x})^2}\right) + C$   
=  $-2 \frac{1+\sqrt{x}}{(2+\sqrt{x})^2} + C$ 

**Class Time** AB and BC 1 period. Essential. **Key Points** 

- $\ln x = \int_1^x \frac{dt}{t}, \quad x > 0.$
- Integral formulas for inverse trigonometric functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$$
$$\int \frac{dx}{x^2+1} = \tan^{-1}x + C$$
$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}x + C$$

• Integrals of exponential functions  $(b > 0, b \neq 1)$ :

$$\int e^x dx = e^x + C, \qquad \int b^x dx = \frac{b^x}{\ln b} + C$$

#### Lecture Material

First point out that the integral formula  $\int_{a}^{b} \frac{dx}{x} = \ln \left| \frac{a}{b} \right|$  can be used to obtain  $\ln x = \int_{1}^{x} \frac{dt}{t}$  for x > 0 by setting a = 1 and b = x. Work Exercise 4 to illustrate the use of this formula. Then point out that in a similar fashion, the derivative formulas for the inverse trigonometric functions can be used obtain corresponding integration formulas. Work Exercises 16 and 20 to illustrate the use of these formulas. Similarly, the differentiation formula for  $e^x$  and  $b^x$  can be used to obtain the integration formulas  $\int e^x dx = e^x + C$ 

and  $\int b^x dx = \frac{b^x}{\ln h} + C$ . Work Exercises 28 and 34 to illustrate the use of this formula.

### **Discussion Topics/Class Activities**

Work Exercise 73 to give an antiderivative for  $\sin^{-1} t$ .

#### Suggested Problems

Exercises 1-9 odd (basic, definite), 13-31 every other odd (indefinite), Review of integration: 33–69 every other odd (indefinite)

# Worksheet 5.7. Further Transcendental Functions

Evaluate the integral.

1. 
$$\int_{2}^{4} \frac{dt}{3t+4}$$

$$2. \int \frac{dt}{\sqrt{1-16t^2}}$$

$$3. \int \frac{dx}{4+x^2}$$

**4.** 
$$\int_0^1 3^{-x} dx$$

5. 
$$\int e^{4x} dx$$

# Solutions to Worksheet 5.7

Evaluate the integral.

1. 
$$\int_{2}^{4} \frac{dt}{3t+4} = \frac{1}{3} \ln \frac{16}{10} = \frac{1}{3} \ln \frac{8}{5}$$
2. 
$$\int \frac{dt}{\sqrt{1-16t^{2}}} \int \frac{dt}{\sqrt{1-16t^{2}}} = \frac{1}{4} \sin^{-1}(4t)$$
3. 
$$\int \frac{dx}{4+x^{2}} \int \frac{dx}{4+x^{2}} = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)$$
4. 
$$\int_{0}^{1} 3^{-x} dx = \frac{2}{3 \ln 3} = \frac{2}{\ln 27}$$
5. 
$$\int e^{4x} dx = \frac{e^{4x}}{4}$$

#### 5.8. Exponential Growth and Decay.

**Class Time** AB and BC 2 periods. Very important.

### **Key Points**

- Exponential growth,  $P(t) = P_0 e^{kt}$ , and decay,  $P(t) = P_0 e^{-kt}$ , k > 0.
- Differential equation y' = ky.
- Doubling times and half-lives.
- Compound interest.

• 
$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$
.

• Continuously compounded interest.

#### Lecture Material

Unconstrained population growth and radioactive decay are important examples of exponential growth and decay. Example 1 illustrates exponential growth and Example 6 illustrates exponential decay. Applications include doubling time (Example 4) and half-life (Example 5). If the rate of change is proportional to the amount present at time t, then we have the differential equation y'(t) = ky(t) for some k > 0. The unique solution of this differential equation is  $y(t) = y(0)e^{kt}$ . Work Examples 2 and 3.

If  $P_0$  is invested at an annual rate of r compounded M times per year, then the value of the investment after t years is

$$P(t) = P_0 \left(1 + \frac{r}{M}\right)^{Mt}$$

Compare the value of an initial investment of \$1000 after 10 years at an annual rate of 5% compounded

- (1) 1 time/year (simple interest):  $1000(1 + .05/1)^{10} \approx 1628.89$
- (2) 12 times/year (compounded monthly):  $1000(1 + .05/12)^{120} \approx 1647.01$
- (3) 26 times/year (biweekly):  $1000(1 + .05/26)^{260} \approx 1647.93$
- (4) 365 times/year (daily):  $1000(1 + .05/365)^{3650} \approx 1648.66$
- (5) every minute (365 \* 24 \* 60 = 525600 times/year): $1000(1 + .05/525600)^{5256000} \approx 1648.72$

It appears that as  $M \to \infty$ , the values  $1000 \left(1 + \frac{.05}{M}\right)^{M*10}$  approach a limit  $L \approx 1648.72$ . This section provides a very nice argument  $\left(1 + \frac{1}{n}\right)^n \to e$  as  $n \to \infty$  based on the Squeeze Theorem and geometric bounds for  $\ln\left(1 + \frac{x}{n}\right) = \int_{1}^{1 + \frac{x}{n}} \frac{1}{t} dt$  (see Figure

11). Exercise 25 asks for a modification to prove  $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$ . Thus  $P_0$  invested at an annual rate r compounded continuously has value after t years

$$P(t) = P_0 e^{rt} = \lim_{M \to \infty} P_0 \left( 1 + \frac{rt}{Mt} \right)^{Mt}$$

Our computation of the value of an initial investment of \$1000 after 10 years at an annual rate of 5% compounded every minute shows that  $P_0 e^{.05*10} \approx 1648.7212707$  is a reasonable approximation to the value of  $1000(1 + .05/M)^{10*M}$  for values  $M \ge 12$ .

Present value of an income stream is not tested on the AB or BC exams and may be omitted.

#### **Discussion Topics/Class Activities**

Exercise 25 outlines a discussion of Moore's Law.

#### Suggested Problems

Exercises 1, 5, 7, 11, 12, 15, 31, 33

# Worksheet 5.8. Exponential Growth and Decay

- 1. A quantity P obeys the exponential growth law  $P = 2000e^{1.3t}$  (t in years). a. What is the doubling time for P?
  - b. At what time is P = 10000?
- 2. Radium-226 has a half-life of 1,622 years. If a sample initially contains 200 grams of Radium-226, how much will it contain 500 years later?

- **3.** Find the solution of the differential equation y' = .7y satisfying y(0) = 15.
- 4.  $C^{14}$  decays exponentially with decay constant k = -.000121A years<sup>-1</sup>. Suppose an analysis of cave paintings indicated a  $C^{14}$  to  $C^{12}$  ratio of 3% of that found in the atmosphere. Approximate the age of the paintings.

- Suppose that \$1000 is deposited in an account paying 4% annual interest. Compute the balance after 12 years if the interest is compounded a. monthly.
  - b. daily.
  - c. continuously.

6. If an investment increases in value at a continuous compounded annual rate of 6%, how long will it take to double in value?

## Solutions to Worksheet 5.8

- **1.** A quantity P obeys the exponential growth law  $P = 2000e^{1.3t}$  (t in years).
  - a. What is the doubling time for P? t = .53319.
  - b. At what time is P = 10000? t = 1.23803
- Radium-226 has a half-life of 1,622 years. If a sample initially contains 200 grams of Radium-226, how much will it contain 500 years later? 161.523 grams
- **3.** Find the solution of the differential equation y' = .7y satisfying y(0) = 15.  $y = 15e^{.7t}$
- 4.  $C^{14}$  decays exponentially with decay constant k = -.000121 years<sup>-1</sup>. Suppose an analysis of cave paintings indicated a  $C^{14}$  to  $C^{12}$  ratio of 3% of that found in the atmosphere. Approximate the age of the paintings.

The age of the paintings is approximately  $\frac{\ln(.03)}{k} \approx 28980$  years.

5. Suppose that \$1000 is deposited in an account paying 4% annual interest. Compute the balance after 12 years if the interest is compounded

a. monthly.  

$$A = 1000(1 + .04/12)^{12*12} \approx 1614.78$$

- b. daily.  $A = 1000(1 + .04/365)^{12*365} \approx 1616.03$
- c. continuously.  $A = 1000 e^{.04*12} \approx 1616.07$
- 6. If an investment increases in value at a continuous compounded annual rate of 6%, how long will it take to double in value?

$$t = \frac{\ln 2}{.06} \approx 11.5525 \text{ years}$$

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# Chapter 5 AP Problems

For 1–3 and 6–9, a calculator may be used. For 4 and 5, no calculator allowed.

- 1. Use the values of f(x), a continuous function, given in the table below to find a trapezoidal approximation of  $\int_{0}^{9} f(x) dx$  with 3 equal subdivisions.  $\frac{x \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9}{f(x) \quad 4 \quad 6 \quad 7 \quad 8 \quad 7 \quad 5 \quad 4 \quad 6 \quad 7 \quad 10}$ A. 39 B. 48 C. 57
  - D. 66
  - E. 78
- **2.** In the figure below, the area of region A is 2 and the area of region B is 11. Find  $\int_0^3 (f(x) 2) \, dx.$



- A. -3
- B. 3
- C. 7
- D. 11
- E. 15

**3.** If a particle is moving in a straight line with a velocity of v(t) = 2t - 3 ft/sec and its position at t = 2 sec is -10 ft, find its position at t = 5 sec.

A. -22 ft B. 2 ft  $\,$ C. 10 ft D. 12 ft E. 22 ft 4. If  $f(x) = \int_0^{x^2} \sqrt{1+t^3} \, dt$ , then  $f'(\sqrt{2}) = ?$ A.  $\sqrt{3}$ B. 3 C.  $2\sqrt{14}$ D.  $6\sqrt{2}$ E. 12 5.  $\int 2x^2(\sec(x^3))^2 dx = ?$ A.  $-\frac{2}{3}\tan(x^3) + C$ B.  $2\tan(x^3) + C$ C.  $-2\tan(x^3) + C$ D.  $\frac{2}{3}\tan(x^3) + C$ E.  $6\tan(x^3) + C$ 

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6. A car is traveling on a straight road. The velocity of the car is recorded in the table below in 20 second intervals for  $0 \le t \le 120$  seconds.

$t \; (sec)$	0	20	40	60	80	100	120
v(t) (ft/sec)	45	58	65	72	68	63	48

a. Using correct units, explain the meaning of  $\int_0^{120} v(t) dt$  in terms of this car.

- b. Use a midpoint Riemann Sum with 3 equal subintervals to approximate  $\int_0^{120} v(t) dt$ .
- c. The car's velocity can be estimated by  $v(t) = \frac{-7}{1000}t^2 + \frac{9}{10}t + 42$  for  $0 \le t \le 120$  sec. If the car's initial position is 20 ft, use the formula to find the car's position at t = 60 sec.
- d. At what time does the equation for v(t) predict the car's displacement will be 3000 ft?

7. The following chart shows a runner's speed at various times during a race. Using a Riemann sum approximation, determine the length of the race.

Speed $(m/s)$	0	3.896	4.016	4.406	4.140
Time (s)	0	5.13	9.96	13.62	19.32

8. A six-lane swimming pool is 75 feet long by 30 feet wide. Its depth is measured at varying distances shown in the table below.

Distance (ft)	0	6	7	9	10	11	13	23	25
Depth (ft)	3.5	3.5	4	6	7	9	10	14	12

- Using a right Riemann sum approximation, what is the volume of the pool?
- **9.** Green Lake has an average depth of 25 feet. The measurements were taken 20 feet apart. The values represent the width of the lake at that point. Use a Trapezoidal approximation to estimate the total volume of Green Lake.



# Solutions to Chapter 5 AP Problems

1. Use the values of f(x), a continuous function, given in the table below to find a trapezoidal approximation of  $\int_{0}^{9} f(x) dx$  with 3 equal subdivisions.  $\frac{x \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9}{f(x) \mid 4 \mid 6 \mid 7 \mid 8 \mid 7 \mid 5 \mid 4 \mid 6 \mid 7 \mid 10}$ A. 39 B. 48 C. 57 D. 66 E. 78

## C [THIS QUESTION CORRESPONDS WITH SECTION 5.5]

2. In the figure below, the area of region A is 2 and the area of region B is 11. Find  $\int_0^3 (f(x) - 2) \, dx.$ 



A. -3

B. 3

- C. 7
- D. 11
- E. 15

### B [THIS QUESTION CORRESPONDS WITH SECTION 5.2]

**3.** If a particle is moving in a straight line with a velocity of v(t) = 2t - 3 ft/sec and its position at t = 2 sec is -10 ft, find its position at t = 5 sec.

A.  $-22~{\rm ft}$ 

B. 2 ft  $\,$ 

C. 10 ft  $\,$ 

D. 12 ft  $\,$ 

E. 22 ft  $\,$ 

### B [THIS QUESTION CORRESPONDS WITH SECTION 5.5]

4. If 
$$f(x) = \int_{0}^{x^{2}} \sqrt{1 + t^{3}} dt$$
, then  $f'(\sqrt{2}) = ?$   
A.  $\sqrt{3}$   
B. 3  
C.  $2\sqrt{14}$   
D.  $6\sqrt{2}$   
E. 12  
D [THIS QUESTION CORRESPONDS WITH SECTION 5.4]  
5.  $\int 2x^{2}(\sec(x^{3}))^{2} dx = ?$   
A.  $-\frac{2}{3}\tan(x^{3}) + C$   
B.  $2\tan(x^{3}) + C$   
C.  $-2\tan(x^{3}) + C$   
D.  $\frac{2}{3}\tan(x^{3}) + C$ 

E.  $6 \tan(x^3) + C$ 

D [THIS QUESTION CORRESPONDS WITH SECTION 5.6]

6. A car is traveling on a straight road. The velocity of the car is recorded in the table below in 20 second intervals for  $0 \le t \le 120$  seconds.

a. Using correct units, explain the meaning of  $\int_0^{120} v(t) dt$  in terms of this car.

$$\int_0^{120} v(t) dt$$
 gives the car's displacement in feet from  $t = 0$  to  $t = 120$  seconds.

b. Use a midpoint Riemann Sum with 3 equal subintervals to approximate  $\int_0^{120} v(t) dt$ .

 $7720~{\rm ft}$ 

c. The car's velocity can be estimated by  $v(t) = \frac{-7}{1000}t^2 + \frac{9}{10}t + 42$  for  $0 \le t \le 120$  sec. If the car's initial position is 20 ft, use the formula to find the car's position at t = 60 sec.

 $7508~{\rm ft}$ 

d. At what time does the equation for v(t) predict the car's displacement will be 3000 ft?

## t = 50.958 sec [THIS QUESTION CORRESPONDS WITH SECTION 5.1]
7. The following chart shows a runner's speed at various times during a race. Using a Riemann sum approximation, determine the length of the race.

Speed $(m/s)$	0	3.896	4.016	4.406	4.140
Time (s)	0	5.13	9.96	13.62	19.32

### 199.980m (should be a 200m race) [THIS QUESTION CORRESPONDS WITH SECTION 5.1]

8. A six-lane swimming pool is 75 feet long by 30 feet wide. Its depth is measured at varying distances shown in the table below.

Distance (ft)	0	6	7	9	10	11	13	23	25
Depth (ft)	3.5	3.5	4	6	7	9	10	14	12

Using a right Riemann sum approximation, what is the volume of the pool?

## 7110 cubic feet [THIS QUESTION CORRESPONDS WITH SECTION 5.1]

**9.** Green Lake has an average depth of 25 feet. The measurements were taken 20 feet apart. The values represent the width of the lake at that point. Use a Trapezoidal approximation to estimate the total volume of Green Lake.



### Ray Cannon's Chapter 6 Overview

Just as Chapter 4 showed some applications of the derivative, Chapter 6 gives some applications of the definite integral. The AP course description acknowledges that not all possible applications can be covered in any one course, but this text does cover all the applications listed as providing a common foundation.

Section 6.1 revisits the motivating idea in Section 5.1 to talk about the area between two curves, and shows how to integrate along either axis. Section 6.2 first shows how to find the volume of a solid with known cross-sections, mean (average) value of a function and density. (Flow rate and center of mass are not tested on either the AB or BC exams.) As examples of other applications, this section presents total mass as the integral of a density function and volume of a flow as an integral of the velocity of the flow. Both of these applications are developed as suggested by the AP course description, i.e. "setting up an approximating Riemann sum and representing its limit as a definite integral." Section 6.3 deals with volumes of revolution, which is just a special case of known crosssections.

Section 6.4 continues in the same manner, developing the "shell method." The "shell method" is not tested on either the AB or BC exam. Section 6.5 uses the application of work, which used to be a required application but is no longer. Again, the application is developed by using approximating Riemann sums.

# 6. Applications of the Integral

#### 6.1. Area Between Two Curves.

Class Time AB 2 periods; BC 1 period. Essential.

### Key Point

• Area between two curves: If  $f(x) \ge g(x)$  on [a, b], then the area between the graphs over [a, b] is  $\int_{a}^{b} (f(x) - g(x)) dx$ .

#### Lecture Material

Suppose  $f(x) \ge g(x)$  for all  $x \in [a, b]$ . Explain that the area of the region between y = f(x) and y = g(x) from x = a to x = b is represented by  $\int_{a}^{b} f(x) dx - \int_{a}^{b} g(x) dx = \int_{a}^{b} (f(x) - g(x)) dx$ . Do Example 1 and Exercise 4 in class.

If  $g_2(y) \ge g_1(y)$ , then the area between the graphs of these two functions is  $\int_a^b (g_2(y) - g_1(y)) \, dy$ . Work Exercise 33.

## Discussion Topics/Class Activities

Have students work Exercise 3 at their desks.

#### Suggested Problems

Exercises 5, 6, 13, 17, 19, 31, 35, 39, 43, 45, 47, 53, 55

# Worksheet 6.1. Area Between Two Curves

- 1. Let f(x) = 8x 10 and  $g(x) = x^2 4x + 10$ .
  - a. Find the points of intersection of the graphs and draw the region bounded by these two functions.



b. Compute the area of this region.

**2.** Find the area of the region bounded by  $x = y^2 + 4y - 22$  and x = 3y + 8.



**3.** Sketch the region enclosed by the curves  $x = \sin y$  and  $x = \frac{2}{\pi}y$ , and find its area.



4. Sketch the region enclosed by  $y = \sin x$ ,  $y = \csc x$ ,  $x = \frac{\pi}{4}$ , and  $x = \frac{3\pi}{4}$  and find its area.



5. Sketch the region whose area is represented by

$$\int_{\frac{-1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (\sqrt{1-x^2} - |x|)$$

and compute its area using the formula for the area of a sector subtended by angle  $\theta$  with radius r;  $A = \frac{r^2 \theta}{2}$ .



## Solutions to Worksheet 6.1

- 1. Let f(x) = 8x 10 and  $g(x) = x^2 4x + 10$ .
  - a. Find the points of intersection of the graphs and draw the region bounded by these two functions.



b. Compute the area of this region.

Area = 
$$\int_{2}^{10} ((8x - 10) - (x^2 - 4x + 10)) dx = \frac{256}{3}$$

**2.** Find the area of the region bounded by  $x = y^2 + 4y - 22$  and x = 3y + 8.



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**3.** Sketch the region enclosed by the curves  $x = \sin y$  and  $x = \frac{2}{\pi}y$  and find its area.



Area = 
$$\int_{-\pi/2}^{\pi/2} |\sin y - \frac{2}{\pi}y| \, dy = 2 \int_{0}^{\pi/2} (\sin y - \frac{2}{\pi}y) \, dy = 2 - \frac{\pi}{2}$$

4. Sketch the region enclosed by  $y = \sin x$ ,  $y = \csc x$ ,  $x = \frac{\pi}{4}$ , and  $x = \frac{3\pi}{4}$  and find its area.



5. Sketch the region whose area is represented by

$$\int_{\frac{-1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (\sqrt{1-x^2} - |x|) \, dx$$

and compute its area using the formula for the area of a sector subtended by angle  $\theta$ with radius r;  $A = \frac{r^2\theta}{2}$ . Since the region is the portion of the disk  $x^2 + y^2 \le 1$  subtended by a central angle of

measure  $\pi/2$ , Area =  $\pi/4$ .



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Class Time AB 3 periods; BC 2 periods.

Fluid flow is not tested on the AP Calculus exams and may be omitted.

#### Key Points

- The volume V of a solid body is equal to the integral of the area of the horizontal cross sections A(y), that is,  $V = \int_{a}^{b} A(y) dy$ .
- Average or mean value of a function on an interval:

$$M = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.$$

• Mean Value Theorem for Integrals: If f(x) is continuous on [a, b] with average value M, then f(c) = M for some  $c \in [a, b]$ .

#### Lecture Material

Using the slide provided, explain the basic idea behind finding the volume of a solid object by partitioning it into N slices. Work Example 1 in the text, which uses horizontal cross sections. We can also use use vertical cross sections. Work Example 3 in the text.

Suppose f(x) is a continuous function on [a, b]. If we partition [a, b] into  $\{x_i\}, i = 0, \ldots, N$  then we can interpret  $R_N = \frac{b-a}{N}(f(x_1) + f(x_2) + \cdots + f(x_N))$ , as an average value. Dividing by b-a gives

$$\frac{1}{b-a}R_N = \frac{f(x_1) + f(x_2)\dots + f(x_N)}{N}$$

If we take N to infinity, we can define this as the average of f(x) on [a, b]. Thus the average value of f on [a, b] is  $\frac{1}{b-a} \int_{a}^{b} f(x) dx$ . It is also helpful to draw pictures like Figures 13 and 16 in the text. Work Example 8 in the text. Since f is continuous, it takes on its average at some value  $c \in [a, b]$ . Thus we have the Mean Value Theorem for Integrals: If f(x) is continuous on [a, b], then there exists a value  $c \in [a, b]$  such that

$$f(c) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$$

Given a density function (mass per unit length, population per square mile, mass per cubic volume and the like), the amount is found by integrating the density function multiplied by the length, area or volume. Work Examples 4 and 5.

## **Discussion Topics/Class Activities**

Students could work Exercise 46 at their desks or in groups using their graphing calculators.

## Suggested Problems (2–3 assignments)

Exercises 1, 5, 8, 9, 11, 13, 24 (Volume); 25, 27, 29, 31, 33 (Density); 39, 45, 49, 51, 57, 58, 59 (Average Value)

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## Worksheet 6.2. Setting up Integrals: Volume, Density, Average Value

- 1. Calculate the volume of the ramp in the figure by integrating the area of the cross sections
  - a. perpendicular to the x-axis (rectangles),
  - b. perpendicular to the *y*-axis (triangles),
  - c. perpendicular to the z-axis.



2. Find the total mass of a 4-m rod whose linear density function is  $\rho(x) = 1 + \cos(\frac{\pi}{2}x)$  kg/m.

3. Odzala National Park in the Republic of Congo has a high density of gorillas. Suppose that the radial density function is  $\rho(r) = 10(1+r)^{-2}$  gorillas per square kilometer, where r is the distance from a large grassy clearing with a source of food and water. Calculate the number of gorillas within a 5-km radius of the clearing.

4. What is the average area of the circles whose radii vary from 0 to 1?

5. Let M be the average value of  $f(x) = 2x^2$  on [0,2]. Find a value c such that f(c) = M.

## Solutions to Worksheet 6.2

- 1. Calculate the volume of the ramp in the figure in three was by integrating the area of the cross sections
  - a. Perpendicular to the x-axis (rectangles),

Cross sections perpendicular to the x-axis are rectangles of width 4 and height  $2 - \frac{1}{3}x$ . The volume of the ramp is then

$$\int_{0}^{6} 4\left(-\frac{1}{3}x+2\right) \, dx = \left(-\frac{2}{3}x^{2}+8x\right)\Big|_{0}^{6} = 24$$

b. Perpendicular to the y-axis (triangles),

Cross sections perpendicular to the y-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then

$$\int_0^4 \left(\frac{1}{2} \cdot 2 \cdot 6\right) \, dy = \left. (6y) \right|_0^4 = 24$$

c. Perpendicular to the z-axis.

Cross sections perpendicular to the z-axis are rectangles of length 6-3z and width 4. The volume of the ramp is then

$$\int_{0}^{2} 4\left(-3(z-2)\right) \, dz = \left(-6z^{2}+24z\right)\Big|_{0}^{2} = 24$$



2. Find the total mass of a 4-m rod whose linear density function is  $\rho(x) = 1 + \cos(\frac{\pi}{2}x)$  kg/m.

The total mass of the rod is

$$\int_{0}^{4} \rho(x) \, dx = \int_{0}^{4} \left( 1 + \cos \frac{\pi x}{2} \right) \, dx = \left( x + \frac{2 \sin \frac{\pi x}{2}}{\pi} \right) \Big|_{0}^{4} = 4 \, \mathrm{kg}$$

3. Odzala National Park in the Republic of Congo has a high density of gorillas. Suppose that the radial density function is  $\rho(r) = 10(1+r)^{-2}$  gorillas per square kilometer, where r is the distance from a large grassy clearing with a source of food and water. Calculate the number of gorillas within a 5-km radius of the clearing.

The number of gorillas within a 5-km radius of the clearing is

$$2\pi \int_0^5 r \cdot \rho(r) dr = \int_0^5 \frac{104\pi r}{(1+r^2)^2}$$
$$= -\frac{52\pi}{1+r^2} \Big|_0^5 = 50\pi \approx 157$$

4. What is the average area of the circles whose radii vary from 0 to 1? The average area is

$$\frac{1}{1-0} \int_0^1 \pi r^2 \, dr = \frac{\pi}{3} r^3 \Big|_0^1 = \frac{\pi}{3}$$

5. Let M be the average value of  $f(x) = 2x^2$  on [0,2]. Find a value c such that f(c) = M.

$$M = \frac{1}{2 - 0} \int_0^2 2x^2 \, dx = \frac{2}{2} \int_0^2 x^2 \, dx = \frac{1}{3} x^3 \Big|_0^2 = \frac{8}{3}$$
$$= 2c^2 = \frac{8}{2} \text{ implies } c = \pm \frac{4}{\sqrt{2}}.$$

Then  $M = f(c) = 2c^2 = \frac{8}{3}$  implies  $c = \pm \frac{4}{\sqrt{3}}$ .

#### 6.3. Volumes of Revolution.

Class Time AB 3 periods; BC 2–3 periods. Essential.

#### **Key Points**

- Volumes of revolution
  - (i) Disk method: If we rotate the region under the graph of f(x) about the x-axis for  $a \le x \le b$ , we obtain a solid whose volume V is

$$V = \pi \int_{a}^{b} f(x)^{2} \, dx$$

(ii) Washer method: If  $f(x) \ge g(x)$  for  $a \le x \le b$ , then the volume V of the solid obtained by rotating the region between f and g about the x-axis is

$$V = \pi \int_{a}^{b} (f(x)^{2} - g(x)^{2}) \, dx$$

- These formulas may be adapted for regions revolved around the *y*-axis
  - (i) Disk method: If we rotate the region under the graph of x = f(y) about the y-axis for  $c \le y \le d$ , we obtain a solid whose volume V is

$$V = \pi \int_c^d (f(y))^2 \, dy$$

(ii) Washer method: If  $f(y) \ge g(y)$  for  $c \le y \le d$ , then the volume V of the solid obtained by rotating the region between f and g about the y-axis is

$$V = \pi \int_{c}^{d} (f(y)^{2} - g(y)^{2}) \, dy$$

- When the region is rotated around a horizontal or vertical line that is not an axis, these formulas need to be modified
  - (i) Disk method: If we rotate the region under the graph of y = f(x) about the line y = k for  $a \le x \le b$ , we obtain a solid whose volume V is

$$V = \pi \int_{a}^{b} (f(x) - k)^2 \, dx$$

(ii) Washer method: If  $f(x) \ge g(x) \ge k$  for  $a \le x \le b$ , then the volume V of the solid obtained by rotating the region between f and g about the line y = k is

$$V = \pi \int_{a}^{b} ((f(x) - k)^{2} - (g(x) - k)^{2}) dx$$

Notice that the general form of each of the formulas above is  $\int_a^b \pi (\text{outer radius})^2 - \pi (\text{inner radius})^2 dx$  where the inner radius may be zero.

#### Lecture Material

Using the slide provided, explain the basic idea behind solids formed by revolving a continuous function about the x-axis over [a, b]. The cross section of this solid will be a circle with a radius of f(x) and area  $\pi f(x)^2$ . Thus the volume of the solid will be  $\int_a^b \pi f(x)^2 dx$ . This is called the disk method. Illustrate by working Exercises 6 and 12. Suppose f and g are two nonnegative continuous functions such that  $f(x) \ge g(x)$  on [a, b]. If we rotate the region between f and g on [a, b], then our solid of revolution has a "hole" in it. So the cross section obtained will have area  $\pi f(x)^2 - \pi g(x)^2$ . Thus the volume of this solid will be  $\pi \int_a^b (f(x)^2 - g(x)^2) dx$ . This is called the washer method. Illustrate by working Exercises 16 and 18.

There are variations of these two methods. You can rotate about a horizontal line other than the x-axis, and you can rotate about vertical lines. Work Exercises 49 and 50.

#### **Discussion Topics/Class Activities**

Students could work Exercise 59 at their desks or in groups.

#### Suggested Problems

Exercises 3, 5, 11, 21, 23 (disk); 15, 19, 23, 27, 29, 31, 43, 51 (washer); 53, 54, 56, 59 (harder)

# Worksheet 6.3. Volumes of Revolution

1. Find the volume of the solid obtained by rotating the region under the graph of the function about the x-axis over the given interval.

a. 
$$f(x) = \frac{1}{x^2}$$
, [1, 4]





a.  $y = x^2$  and y = 2x + 3

2. Sketch the region enclosed by the two curves and find the volume of the solid obtained by rotating the region about the *x*-axis.

b. 
$$y = \sec x, y = \csc x, y = 0, x = 0, \text{ and } x = \frac{\pi}{2}$$



**3.** Find the volume of the solid obtained by rotating the region enclosed by the graph about the given line.



b. y = 16 - x, y = 3x + 12, x = 0 about x = 2



# Solutions to Worksheet 6.3

1. Find the volume of the solid obtained by rotating the region under the graph of the function about the x-axis over the given interval.



The volume of the solid of revolution is

$$\pi \int_{1}^{4} (x^{-2})^{2} dx = \pi \int_{1}^{4} x^{-4} dx = \pi \left( -\frac{1}{3} x^{-3} \right) \Big|_{1}^{4} = \frac{21\pi}{64}$$

b. 
$$f(x) = \sqrt{\cos x \sin x}$$



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The volume of the solid of revolution is

$$\pi \int_0^{\pi/2} (\sqrt{\cos x \sin x})^2 dx = \pi \int_0^{\pi/2} (\cos x \sin x) dx$$
$$= \pi \int_0^1 u \, du = \pi \left(\frac{1}{2}u^2\right) \Big|_0^1 = \frac{\pi}{2}$$

2. Sketch the region enclosed by the two curves and find the volume of the solid obtained by rotating the region about the *x*-axis.

$$0 = x^{2} - 2x - 3 = (x - 3)(x + 1).$$

The two curves therefore intersect at x = -1 and x = 3. The region enclosed by the two curves is shown in the figure.

- (2) When the region is rotated about the x-axis, each cross section is a washer with outer radius R = 2x + 3 and inner radius  $r = x^2$ .
- (3) The volume of the solid of revolution is

$$\pi \int_{-1}^{3} \left( (2x+3)^2 - (x^2)^2 \right) dx = \pi \int_{-1}^{3} (4x^2 + 12x + 9 - x^4) dx$$
$$= \pi \left( \frac{4}{3}x^3 + 6x^2 + 9x - \frac{1}{5}x^5 \right) \Big|_{-1}^{3} = \frac{1088\pi}{15}$$





- (1) The region in question is shown in the figure.
- (2) When the region is rotated about the x-axis, cross sections for  $x \in [0, \pi/4]$  are circular disks with radius  $R = \sec x$ , whereas cross sections for  $x \in [\pi/4, \pi/2]$  are circular disks with radius  $R = \csc x$ .
- (3) The volume of the solid of revolution is

$$\int_{0}^{\pi/4} \pi \sec^2 x \, dx + \int_{\pi/4}^{\pi/2} \pi \csc^2 x \, dx = \pi (\tan x) \left|_{0}^{\pi/4} + \pi (-\cot x) \right|_{\pi/4}^{\pi/2}$$
$$= \pi (1) + \pi (1)$$
$$= 2\pi$$

**3.** Find the volume of the solid obtained by rotating the region enclosed by the graphs about the given line.

a. 
$$y = \frac{1}{x}, y = \frac{5}{2} - x$$
, about  $y = -1$ 

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The region enclosed by the two curves is shown in the figure. Rotating the region about the line y = -1 produces a solid whose cross sections have outer radius  $R = \frac{1}{x} - (-1) = \frac{1}{x} + 1$  and inner radius  $r = \frac{5}{2} - x - (-1) = \frac{7}{2} - x$ . Setting  $\frac{1}{x} = \frac{5}{2} - x$  and solving for x, the curves  $y = \frac{1}{x}$  and  $y = \frac{5}{2} - x$  intersect at  $x = \frac{1}{2}$  and x = 2. The volume of the solid of revolution is

$$\pi \int_{\frac{1}{2}}^{2} \left( \left(\frac{1}{x} + 1\right)^{2} - \left(\frac{7}{2} - x\right)^{2} \right) dx = \pi \int_{\frac{1}{2}}^{2} \left(\frac{1}{x^{2}} + \frac{2}{x} - \frac{41}{4} + 7x - x^{2}\right) dx$$
$$= 2\pi \left( -\frac{1}{x} + 2\ln x - \frac{41}{4}x + \frac{7}{2}x^{2} - \frac{1}{3}x^{3} \right) \Big|_{\frac{1}{2}}^{2} = \pi (\ln 16 - \frac{27}{8})^{2}$$

b. y = 16 - x, y = 3x + 12, x = 0 about x = 2.



Rotating the region enclosed by y = 16 - x, y = 3x + 12 and the y-axis (see the figure in Exercise 3 (a)) about x = 2 produces a solid with two different cross sections. For each  $y \in [12, 15]$ , the cross section is a washer with outer radius R = 2 and inner radius  $r = 2 - \frac{1}{3}(y - 12) = 6 - \frac{1}{3}y$ ; for each  $y \in [15, 16]$ , the cross section is a washer with outer radius R = 2 and inner radius r = 2 - (16 - y) = y - 14. The volume of the solid of revolution is

$$\pi \int_{12}^{15} \left( (2)^2 - \left(6 - \frac{1}{3}y\right)^2 \right) dy + \pi \int_{15}^{16} \left( (2)^2 - (y - 14)^2 \right) dy$$
  
=  $\pi \int_{12}^{15} \left( -\frac{1}{9}y^2 + 4y - 32 \right) dy + \pi \int_{15}^{16} (-y^2 + 28y - 192) dy$   
=  $\pi \left( -\frac{1}{27}y^3 + 2y^2 - 32y \right) \Big|_{12}^{15} + \pi \left( -\frac{1}{3}y^3 + 14y^2 - 192y \right) \Big|_{15}^{16}$   
=  $\frac{20}{3}\pi$ 

## Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS. **Key Points**

• If  $f(x) \ge 0$ , then the volume V of the solid obtained by rotating the region underneath the graph of y = f(x) over [a, b] about the y-axis is

$$V = 2\pi \int_{a}^{b} x f(x) \, dx$$

If  $g(y) \ge 0$ , then the volume V of the solid obtained by rotating the region to the left of the graph of x = q(y) over [a, b] about the x-axis is

$$V = 2\pi \int_{a}^{b} yg(y) \, dy$$

• If we revolve f(x) from [a, b] about x = c where  $c \neq 0$ , then the volume of the resulting solid is

$$V = 2\pi \int_{a}^{b} (x - c)f(x) \, dx$$

#### Lecture Material

Using the slide provided, explain the basic idea behind the Method of Cylindrical Shells. Cylindrical shells are used to find the volume of a solid of revolution. We want the volume of a cylindrical shell with height h, outer radius R, and inner radius r. This is  $\pi R^2 h - \pi r^2 h = \pi h (R+r) (\Delta r)$ , where  $\Delta r = R - r$ . If the shell is very thin, then we can replace R + r with 2r. So the volume of the cylindrical shell is approximately  $2\pi hr\Delta r$ .

If we rotate a continuous function y = f(x) from x = a to x = b about the y-axis. Then we can partition the resulting solid into N cylindrical shells each of whose volume is  $2\pi x_i f(x_i) \Delta x$ , where  $x_i$  is the right or left endpoint of the subintervals. Adding and then taking the limit as N goes to infinity gives  $V = 2\pi \int_{a}^{b} xf(x) dx$ . Work Exercises 2

and 5.

You can use the Method of Cylindrical Shells to find the volume of a solid obtained by rotating the region between two continuous functions of x about the y-axis or rotating about another vertical line besides the y-axis, or rotating continuous functions of y about the x-axis or other horizontal lines. Work Exercises 12, 18, 24, and 34, if time permits.

#### **Discussion Topics/Class Activities**

Students could work Exercise 52 at their desks or in groups.

#### Suggested Problems

Exercises 1, 4, 6 (graphical), 8, 9, 15, 18, 21, 26 (basic), 47, 48, 49, 50 (harder)

# Worksheet 6.4. The Method of Cylindrical Shells

- 1. Sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the y- axis and find its volume.
  - y 2 -4 -3 -2 -1 1 2 3 4 x

b. 
$$f(x) = \sqrt{x^2 + 9}, [0,3]$$

a.  $f(x) = \sqrt{x}, [0, 4]$ 



- 2. Use the Shell Method to compute the volume of the solid obtained by rotating the region enclosed by  $y = 8 x^3$  and y = 8 4x about the y-axis.
- **3.** Sketch the solid obtained by rotating the region underneath the graph of  $f(x) = x^3$  over [0, 1] about the vertical line x = -2.



4. Use the Shell Method to calculate the volume of the region found by rotating  $y = 4 - x^2$  on [0, 2] about the *y*-axis.

5. Use the Shell Method to find the volume of the solid found by rotating the region below y = 6 and above  $y = x^2 + 2$  over [0, 2] about the vertical line x = -3.



## Solutions to Worksheet 6.4

Sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the y axis and find its volume.
a. f(x) = √x, [0,4]



Each shell has radius x and height  $\sqrt{x}$ , so the volume of the solid is

$$2\pi \int_0^4 x \sqrt{x} \, dx = 2\pi \int_0^4 x^{3/2} \, dx = 2\pi \left(\frac{2}{5}x^{5/2}\right) \Big|_0^4 = \frac{128}{5}\pi$$
  
b.  $f(x) = \sqrt{x^2 + 9}, \ [0, 3]$ 



Each shell has radius x and height  $1 + x^2$ , so the volume of the solid is

$$2\pi \int_{1}^{3} x(1+x^{2}) dx = 2\pi \int_{1}^{3} (x+x^{3}) dx$$
$$= 2\pi \left(\frac{1}{2}x^{2} + \frac{1}{4}x^{4}\right)\Big|_{1}^{3} = 48\pi$$

- 2. Use the Shell Method to compute the volume of the solid obtained by rotating the region enclosed by  $y = 8 - x^3$  and y = 8 - 4x about the y-axis.
  - Following is a sketch of the solid.



Notice that the solid is generated by rotating two vertical strips  $8 - 4x \le y \le 8 - x^3$ and  $x^3 + 8 \le y \le 8 + 4x$  for every  $0 \le x \le 2$ . Thus each of the two shells has radius x and height  $4x - x^3$ . The volume of the resulting solid is

$$4\pi \int_0^2 x(4x - x^3) \, dx = 4\pi \int_0^2 (4x^2 - x^4) \, dx$$
$$= 4\pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{256\pi}{15}$$

**3.** Sketch the solid obtained by rotating the region underneath the graph of  $f(x) = x^3$  over [0,1] about the vertical line x = -2.



Each shell has radius x - (-2) = x + 2 and height  $x^3$ , so the volume of the solid is

$$2\pi \int_0^1 (2+x) (x^3) dx = 2\pi \int_0^1 (2x^3 + x^4) dx$$
$$= 2\pi \left(\frac{x^4}{2} + \frac{x^5}{5}\right) \Big|_0^1 = \frac{7\pi}{5}$$

4. Use the Shell Method to calculate the volume of the region found by rotating  $y = 4 - x^2$  on [0, 2] about the *y*-axis.



When the region in the figure is rotated about the x-axis, each shell has radius y and height  $\sqrt{4-y}$ . The volume of the resulting solid is

$$2\pi \int_0^4 y\sqrt{4-y}\,dy$$

Let u = 4 - y. Then du = -dy, y = 4 - u, and

$$2\pi \int_{0}^{4} y\sqrt{4-y} \, dy = -2\pi \int_{4}^{0} (4-u)\sqrt{u} \, du$$
$$= 2\pi \int_{0}^{4} \left(4\sqrt{u} - u^{3/2}\right) \, du$$
$$= 2\pi \left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_{0}^{4} = \frac{256\pi}{15}$$

5. Use the Shell Method to find the volume of the solid found by rotating the region below y = 6 and above  $y = x^2 + 2$  over [0, 2] about the vertical line x = -3.



#### 6.5. Work and Energy.

#### Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS.

## **Key Points**

- The work performed when a constant force F is applied to an object through a distance d is  $F \times d$ .
- If the force is a continuous function of x, denoted F(x), the the work done in moving an object along the x-axis from a to b when F is applied is  $\int_{a}^{b} F(x) dx$ .
- Hooke's Law: A spring exerts a restoring force in the opposite direction of magnitude. That is, F(x) = -kx, where k is the spring constant measured in kilograms per second squared.
- In some cases, the work is computed by dividing the object into N thin layers each with a thickness of  $\Delta y = \frac{b-a}{N}$ , where the object extends from a to b. We approximate the work  $W_i$  performed on the *i*th layer as  $W(y_i)\Delta y$ . Then total work is  $W = \int_a^n W(y) \, dy$ .

#### Lecture Material

Using the slide provided, illustrate that work is force times distance. Work Exercise 2.

If the force can be represented as a continuous function, F(x), applied to an object along the x-axis from a to b, then we must partition [a, b] into N subintervals of length  $\Delta x = \frac{b-a}{N}$  and approximate the work done on each subinterval as  $F(x_i)\Delta x$ , where  $x_i$ is the right endpoint of the subintervals. Then we add up subintervals and take the limit as N goes to infinity. Hence the work W performed by F in moving an object along the x-axis from a to b is  $W = \int_a^b F(x) dx$ .

An application is finding the work required to stretch or compress a spring. Hooke's Law states that a spring exerts a restoring force in the opposite direction of magnitude F(x) = -kx, where k is the spring constant measured in units of kilograms per second squared. Work Example 1 in the text.

Sometimes each layer of an object is moved through a different distance, such as building a cement column of height h and a square base. We need to divide [0, h] into N subintervals and approximate the work done on each layer using the formula of force times distance. Then we sum the approximate work done on each layer up and take the limit as N goes to infinity. Thus the work W performed is  $\int_{0}^{h} W(y) \, dy$ , where W(y)

is the work performed to raise a layer to height y. Work Example 2, Exercise 12, and Exercise 20, if time permits.

## **Discussion Topics/Class Activities**

Students could work Exercise 17 at their desks or in groups.

## Suggested Problems

Exercises 1, 2, 3, 5 (basic), 13, 15, 19, 21, 23, 27, 29 (harder)
## Worksheet 6.5. Work and Energy

1. Find the work in joules to compress a spring 4 cm past equilibrium, assuming that the spring constant is  $k = 150 \text{ kg/s}^2$ 

.

- 2. Calculate the work against gravity to build a cylindrical tower of height 20 ft and radius 10 ft, assuming that the brick has density 80 lb/ft<sup>3</sup>.
- **3.** Calculate the work in joules required to pump all the water out of the conical tank pictured.



4. How much work is done in lifting a 3-m chain over the side of a building, if the chain has mass density 4 kg/m?

## Solutions to Worksheet 6.5

1. Find the work in joules to compress a spring 4 cm past equilibrium, assuming that the spring constant is  $k = 150 \text{ kg/s}^2$ .

The work required to compress the spring 4 cm past equilibrium is

$$\int_0^{-.04} 150x \, dx = 75x^2 \big|_0^{-.04} = 0.12 \text{ J}$$

2. Calculate the work against gravity to build a cylindrical tower of height 20 ft and radius 10 ft, assuming that the brick has density 80  $lb/ft^3$ .

The area of the base is  $100\pi$  ft<sup>2</sup>, so the volume of each small layer is  $100\pi\Delta y$  ft<sup>3</sup>. The weight of one layer is then  $8000\pi\Delta y$  lb. Finally, the total work done against gravity to build the tower is

$$\int_0^{20} 8000\pi y \, dy = 1.6 \times 10^6 \pi \text{ ft-lb.}$$

**3.** Calculate the work in joules required to pump all the water out of the conical tank pictured.



Place the origin at the vertex of the inverted cone, and let the positive y-axis point upward. Consider a layer of water at a height of y meters. From similar triangles, the area of the layer is

 $\pi \left(\frac{y}{2}\right)^2 m^2$  $\pi \left(\frac{y}{2}\right)^2 \Delta y m^3$ 

so the volume is

Thus the weight of one layer is

$$9800\pi \left(\frac{y}{2}\right)^2 \Delta y$$
 N

The layer must be lifted 12 - y meters, so the total work needed to empty the tank is

$$\int_{0}^{10} 9800\pi \left(\frac{y}{2}\right)^{2} (12 - y) \, dy = \pi (3.675 \times 10^{6}) \text{ J}$$
  
$$\approx 1.155 \times 10^{7} \text{ J}$$

4. How much work is done in lifting a 3-m chain over the side of a building, if the chain has mass density 4 kg/m?

Consider a segment of the chain of length  $\Delta y$  located a distance  $y_j$  meters from the top of the building. The work needed to lift this segment of the chain to the top of the building is approximately

$$W_j \approx (4\Delta y)(9.8)y_j$$
 J

Summing all segments of the chain and passing to the limit as  $\Delta y \to 0$ , it follows that the total work is

$$\int_0^3 4 \cdot 9.8y \, dy = 19.6y^2 \big|_0^3 = 176.4 \text{ J}$$

# Chapter 6 AP Problems

For 2, 4, 5, 6 and 7, a calculator may be used. For 1 and 3, no calculator allowed.

1. Which of the following equations would give you the area shaded in the graph below?



A. 
$$A = \int_0^{\pi} [\sin(2x) - \sin x] dx$$
  
B.  $A = \int_0^{\pi} [\sin x - \sin(2x)] dx$   
C.  $A = \int_0^{\pi/3} \sin x \, dx + \int_{\pi/3}^{\pi} \sin(2x) \, dx$   
D.  $A = \int_0^{\pi/3} [\sin(2x) - \sin x] \, dx + \int_{\pi/3}^{\pi} [\sin x - \sin(2x)] \, dx$   
E.  $A = \int_0^{\pi/3} [\sin x - \sin(2x)] \, dx + \int_{\pi/3}^{\pi} [\sin(2x) - \sin x] \, dx$ 

- 2. The velocity of an object in m/s is given by  $v(t) = 2t \ln t$ . What is the average velocity from t = 1 to t = 5 seconds?
  - A. 3.219 m/s
    B. 4.024 m/s
    C. 7.059 m/s
    D. 16.094 m/s
  - E. 28.236 m/s

**3.** The region between the curves  $y = x^2$  and  $y = \sqrt{x}$  in the first quadrant is the base of a solid. For this solid, each cross section perpendicular to the y-axis is a rectangle whose height (above the xy-plane) is half of its length (in the xy-plane). Which of the following integrals gives the volume of this solid?

A. 
$$\pi \int_0^1 (\sqrt{y} - y^2)^2 - \left(\frac{1}{2}(\sqrt{y} - y^2)\right)^2 dy$$
  
B.  $\frac{\pi}{2} \int_0^1 (\sqrt{y} - y^2)^2 dy$   
C.  $\int_0^1 (\sqrt{y} - y^2)^2 - \left(\frac{1}{2}(\sqrt{y} - y^2)\right)^2 dy$   
D.  $\int_0^1 (y - y^4) dy$   
E.  $\frac{1}{2} \int_0^1 (\sqrt{y} - y^2)^2 dy$ 

- 4. The area bounded by the curve  $y = \frac{1}{2}x^2$ , the x-axis, and the lines x = k and x = 2, where 0 < k < 2, is rotated about the x-axis. The volume of the solid formed is 5. Find the value of k.
  - A. 0.002B. 0.701
  - C. 0.967
  - D. 1.565
  - E. 3.248

- 5. Find the volume of the solid formed when the area in the first quadrant between  $f(x) = \sin(2x)$  and g(x) = x is rotated about the line y = 2.
  - A. 0.057B. 0.179
  - C. 0.576
  - D. 1.809
  - E. 3.479
- 6. Let R be the region in the first quadrant shown in the figure below. Region R is bounded by  $f(x) = \tan(x/2)$  and  $g(x) = \sin x$ . (calculator required)



- a. Find the area of R.
- b. Find the volume of the solid generated when R is revolved about the x-axis.
- c. Find the volume of the solid generated when R is revolved about the line x = -2.
- d. The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a semicircle. Find the volume of this solid.

7. The function S(t) given below can be used to estimate the number of hours of daylight in Liberty Hill, TX for a given day of the year for  $0 \le t \le 365$  (t is in days and t = 0 is midnight on January 1). (calculator required)

$$S(t) = -1.95 \cos\left(\frac{\pi}{189}(x+24)\right) + 12.15 \text{ hours/day}$$

Use the given formula to find the following:

- a. What is the total number of daylight hours from the  $100^{\text{th}}$  day to the  $200^{\text{th}}$  day?
- b. Is the length of the day increasing or decreasing on the 100<sup>th</sup> day? Explain.
- c. What is the average number of hours of daylight from day 30 to day 250?
- d. On what day is the Summer Solstice (longest day of the year)? Explain.

## Solutions to Chapter 6 AP Problems

1. Which of the following equations would give you the area shaded in the graph below?



- D [THIS PROBLEM CORRESPONDS WITH SECTION 6.1]
- 2. The velocity of an object in m/s is given by  $v(t) = 2t \ln t$ . What is the average velocity from t = 1 to t = 5 seconds?
  - A. 3.219 m/s
  - B. 4.024 m/s
  - C. 7.059 m/s
  - D. 16.094 m/s
  - E. 28.236 m/s

### C [THIS PROBLEM CORRESPONDS WITH SECTION 6.2]

**3.** The region between the curves  $y = x^2$  and  $y = \sqrt{x}$  in the first quadrant is the base of a solid. For this solid, each cross section perpendicular to the y-axis is a rectangle whose height (above the xy-plane) is half of its length (in the xy-plane). Which of the following integrals gives the volume of this solid?

A. 
$$\pi \int_{0}^{1} (\sqrt{y} - y^{2})^{2} - (\frac{1}{2}(\sqrt{y} - y^{2}))^{2} dy$$
  
B.  $\frac{\pi}{2} \int_{0}^{1} (\sqrt{y} - y^{2})^{2} dy$   
C.  $\int_{0}^{1} (\sqrt{y} - y^{2})^{2} - (\frac{1}{2}(\sqrt{y} - y^{2}))^{2} dy$   
D.  $\int_{0}^{1} (y - y^{4}) dy$   
E.  $\frac{1}{2} \int_{0}^{1} (\sqrt{y} - y^{2})^{2} dy$   
E [THIS PROBLEM CORRESPONDS WITH SECTION 6.2]

4. The area bounded by the curve  $y = \frac{1}{2}x^2$ , the *x*-axis, and the lines x = k and x = 2, where 0 < k < 2, is rotated about the *x*-axis. The volume of the solid formed is 5. Find the value of k.

A. 0.002B. 0.701

C. 0.967

D. 1.565

E. 3.248

## B [THIS PROBLEM CORRESPONDS WITH SECTION 6.3]

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- 5. Find the volume of the solid formed when the area in the first quadrant between  $f(x) = \sin(2x)$  and g(x) = x is rotated about the line y = 2.
  - A. 0.057
  - B. 0.179
  - C. 0.576
  - D. 1.809
  - E. 3.479

## D [THIS PROBLEM CORRESPONDS WITH SECTION 6.3]

6. Let R be the region in the first quadrant shown in the figure below. Region R is bounded by  $f(x) = \tan(x/2)$  and  $g(x) = \sin x$ . (calculator required)



a. Find the area of R.

## 0.307 [THIS PROBLEM CORRESPONDS WITH SECTION 6.1]

b. Find the volume of the solid generated when R is revolved about the x-axis.

## 1.119 [THIS PROBLEM CORRESPONDS WITH SECTION 6.3]

c. Find the volume of the solid generated when R is revolved about the line x = -2.

## 5.470 [THIS PROBLEM CORRESPONDS WITH SECTION 6.3]

d. The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a semicircle. Find the volume of this solid.

## 0.029 [THIS PROBLEM CORRESPONDS WITH SECTION 6.2]

7. The function S(t) given below can be used to estimate the number of hours of daylight in Liberty Hill, TX for a given day of the year for  $0 \le t \le 365$  (t is in days and t = 0 is midnight on January 1). (calculator required)

$$S(t) = -1.95 \cos\left(\frac{\pi}{189}(x+24)\right) + 12.15 \text{ hours/day}$$

Use the given formula to find the following:

a. What is the total number of daylight hours from the 100<sup>th</sup> day to the 200<sup>th</sup> day?

### 1382.954 hours [THIS PROBLEM CORRESPONDS WITH SECTION 5.5]

b. Is the length of the day increasing or decreasing on the 100<sup>th</sup> day? Explain.

S'(100) = .029. The length of the day is increasing on the 100<sup>th</sup> day because S'(100) is positive.

## [THIS PROBLEM CORRESPONDS WITH SECTION 4.3]

c. What is the average number of hours of daylight from day 30 to day 250?

### 13.094 hours/day [THIS PROBLEM CORRESPONDS WITH SECTION 6.2]

d. On what day is the Summer Solstice (longest day of the year)? Explain.

Local maximums are at t = 165 (because S'(t) = 0 and changing from positive to negative) and at t = 365 (increasing to right endpoint). We have

$$S(165) = 14.1$$
 hours/day  
 $S(365) = 10.233$  hours/day

The Summer Solstice is at the absolute maximum which is the 165<sup>th</sup> day of the year. [THIS PROBLEM CORRESPONDS WITH SECTION 4.2]

### Ray Cannon's Chapter 7 Overview

The preponderance of material in Chapter 7 pertains only to the BC course description. For example, Section 7.1, integration by parts, is not part of the AB course description, but many AB teachers like to cover this material anyway; it is a very important topic in the BC course description. AP students are not responsible for the reduction formulas in Section 7.2, but should know the antiderivatives of  $\tan(x)$  and  $\sec(x)$ , and how to use substitution for integrals like  $\int \sin^3(x) \cos(x) \, dx$ . Neither section 7.3 which treats a special form of substitution, known as "trigonometric substitution" nor the hyperbolic functions treated in Section 7.4 are part of either AP course description. Section 7.5 deals with the method of partial fractions in depth; BC students are only required to be able to deal with non-repeated linear factors in the denominator.

The topic of Section 7.6 is improper integrals, which are also included in the BC course description. BC students must be able to handle both unbounded domains, and unbounded functions. They should also realize improper integrals are the limit of definite integrals, and be careful in their treatment of the Fundamental Theorem of Calculus during evaluations. Sec 7.7 is another section that shows applicability of the definite integral, but the specific topic is optional. Finally, Section 7.8 discusses numerical approximations to a definite integral. Along with the left- and right-side approximations, students should be familiar with using trapezoids and midpoint rectangles to approximate the value of a definite integral. The formulas for the Trapezoidal rule and the Midpoint rule need not be memorized. Calculus students can always benefit from a discussion of error bounds, but these formulas are not tested.

# 7. Techniques of Integration

### 7.1. Integration by Parts.

Class Time AB 0 periods; BC 1 period. Essential.

BC only. This material is not tested on the AB exam.

### Key Point

• Integration by Parts Formula:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

### Lecture Material

Remind students of the Product Rule,

$$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x)$$

Taking the antiderivative of both sides gives

$$u(x)v(x) = \int u(x)v'(x) \, dx + \int u'(x)v(x) \, dx$$

Rearranging gives the Integration by Parts Formula:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

This formula also applies to definite integrals.

$$\int_{a}^{b} u(x)v'(x) \, dx = u(x)v(x)|_{a}^{b} - \int_{a}^{b} u'(x)v(x) \, dx$$

Work examples  $\int x \sin x \, dx$ ,  $\int x e^x \, dx$ , and  $\int_1^3 \ln x \, dx$ .

Sometimes we need to integrate more than once. Work  $\int x^2 \sin x \, dx$  and  $\int e^x \sin x$ . If time permits, derive the reduction formula  $\int x^n e^x = x^n e^x - n \int x^{n-1} e^x \, dx$ . Work

 $\int_0^1 x^3 e^x dx$ . Reduction formulas are not tested on the AP exams.

**Discussion Topics/Class Activities** Discuss Exercise 79.

Suggested Problems

Exercises 7, 9, 11, 13, 15 (basic), 23, 25, 29 (harder), 49, 53 (basic), 59–64 (review), 71 (application)

# Worksheet 7.1. Integration by Parts

1. Use Integration by Parts to evaluate  $\int x e^{-x} dx$ .

**2.** Use the substitution  $u = x^2$  and then Integration by Parts to evaluate  $\int x^3 e^{x^2} dx$ .

**3.** Compute the definite integral 
$$\int_{1}^{3} \ln x \, dx$$
.

4. Find the volume of the solid obtained by revolving  $y = \cos x$  for  $0 \le x \le \frac{\pi}{2}$  about the *y*-axis.



## Solutions to Worksheet 7.1

- 1. Use integration by parts to evaluate  $\int xe^{-x} dx$ . Let u = x and  $v' = e^{-x}$ . Then we have  $u = x, v = -e^{-x}$ 
  - $u' = 1 \text{ and } v' = e^{-x}$

Using Integration by Parts, we get

$$\int x e^{-x} dx = x(-e^{-x}) - \int (1)(-e^{-x}) dx$$
$$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C$$
$$= -e^{-x}(x+1) + C$$

**2.** Use the substitution  $u = x^2$  then integration by parts to evaluate  $\int x^3 e^{x^2} dx$ . Let  $w = x^2$ . Then dw = 2x dx, and

$$\int x^3 e^{x^2} \, dx = \frac{1}{2} \int w e^w \, dw.$$

Using Integration by Parts, we let u = w and  $v' = e^w$ . Then we have

$$\int we^w \, dw = we^w - \int (1)e^w \, dw = we^w - e^w$$

Substituting back in terms of x, we get

$$\int x^3 e^{x^2} dx = \frac{1}{2} \left( x^2 e^{x^2} - e^{x^2} \right) + C$$

- 3. Compute the definite integral  $\int_{1}^{3} \ln x \, dx$ . Let  $u = \ln x$  and v' = 1. Then u' = 1/x and v = x. Using Integration by Parts,  $\int_{1}^{3} \ln x \, dx = x \ln x - x |_{1}^{3} = 3 \ln 3 - 3 - ((1) \ln 1 - 1)$  $= 3 \ln 3 - 2$
- 4. Find the volume of the solid obtained by revolving  $y = \cos x$  for  $0 \le x \le \frac{\pi}{2}$  about the y-axis.



Use the Cylindrical Shells method, where the volume V is given by

$$V = \int_{a}^{b} (2\pi r)h \, dx = 2\pi \int_{0}^{\frac{\pi}{2}} x \cos x \, dx$$

and the radius r = x and varies from 0 to  $\frac{\pi}{2}$ , and the height  $h = y = \cos x$ . Using Integration by Parts, with u = x and  $v' = \cos x$ , we get

$$V = 2\pi \left[ x \sin x + \cos x \right]_0^{\frac{\pi}{2}} = 2\pi \left[ \left( \frac{\pi}{2} + 0 \right) - (0+1) \right]$$
$$= \pi(\pi - 2)$$

# Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS. Key Points

• Odd power of sine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then use the substitution  $u = \cos x, du = -\sin x dx$ .

• Odd power of cosine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

Then use the substitution  $u = \sin x, du = \cos x dx$ .

• Sine and cosine both occur to an even power:

$$\int \sin^m x \cos^n s \, dx = \int (1 - \cos^2 x)^{m/2} \cos^n x \, dx \text{ if } m > n$$
$$\int \sin^m x \cos^n s \, dx = \int \sin^m x (1 - \sin^2 x)^{n/2} \, dx \text{ if } m < n$$

Then expand the right-hand side to obtain the integral of a sum of powers of  $\cos x$  or powers of  $\sin x$ . These integrals can now be evaluated using the reduction formulas:

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$
$$\int \cos^n x \, dx = -\frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

#### Lecture Material

With the availability of computer algebra systems and powerful calculators, the topic of trigonometric integrals is not as important as it once was. However the basic principle behind the techniques is important. When the integrand involves an odd power of sine or an odd power of cosine, you should use the Pythagorean Theorem  $\sin^2 x + \cos^2 x = 1$  to rewrite the integrand and then make the appropriate substitution. Work Examples 1 and 2.

Remind students of the trigonometric identities  $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$  and  $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ , both of which are derived from the identity  $\cos(2x) = \cos^2 x - \sin^2 x$ . Using the trigonometric identities, derive formulas for  $\int \sin^2 x \, dx$  and  $\int \cos^2 x \, dx$ . If time permits, prove one of the reduction formulas found in Box 1 (in the text) using integration by parts. Now work Example 4.

Derive the formula  $\int \tan x \, dx = \ln |\sec x| + C$  by using the substitution  $u = \cos x$ . Derive  $\int \sec c \, dx = \ln |\sec x + \tan x| + C$  by using the substitution  $u = \sec x + \tan x$ . If time permits, work Examples 6 and 7 and discuss integrals of the type  $\int \sin(mx) \cos(nx)$ .

### Discussion Topics/Class Activities

Discuss the Preliminary Questions with the students.

#### **Suggested Problems**

Exercises 1, 3, 5, 9, 11 (basic), 13, 31, 47, 57 (harder)

# Worksheet 7.2. Trigonometric Integrals

1. Evaluate  $\int \cos x \sin^5 x \, dx$ .

**2.** Evaluate  $\int \cos^2 \theta \sin^2 \theta \, d\theta$ .

**3.** Compute the definite integral  $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$ .

4. Evaluate the definite integral  $\int_{-\frac{\pi}{4}}^{-\frac{\pi}{4}} \sec^4 x \, dx$ .

# Solutions to Worksheet 7.2

1. Evaluate 
$$\int \cos x \sin^5 x \, dx$$
.  
Write  $\sin^5 x = \sin^4 x \sin x = (1 - \cos^2 x)^2 \sin x$ . Then

$$\int \cos x \sin^5 x \, dx = \int \cos x \left(1 - \cos^2 x\right)^2 \sin x \, dx$$

Now use the substitution  $u = \cos x$ ,  $du = -\sin x \, dx$ :

$$\int \cos x \sin^5 x \, dx = -\int u \left(1 - u^2\right)^2 du = -\int u \left(1 - 2u^2 + u^4\right) du$$
$$= \int \left(-u + 2u^3 - u^5\right) du = -\frac{1}{2}u^2 + \frac{1}{2}u^4 - \frac{1}{6}u^6 + C$$
$$= -\frac{1}{2}\cos^2 x + \frac{1}{2}\cos^4 x - \frac{1}{6}\cos^6 x + C$$

2. Evaluate  $\int \cos^2 \theta \sin^2 \theta \, d\theta$ . First use the identity  $\cos^2 \theta = 1 - \sin^2 \theta$  to write

$$\int \cos^2 \theta \sin^2 \theta \, d\theta = \int \left(1 - \sin^2 \theta\right) \sin^2 \theta \, d\theta = \int \sin^2 \theta \, d\theta - \int \sin^4 \theta \, d\theta$$

Using the reduction formula for  $\sin^m x$ ,

$$\int \cos^2 \theta \sin^2 \theta \, d\theta = \int \sin^2 \theta \, d\theta - \left[ -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx \right]$$
$$= \frac{1}{4} \sin^3 x \cos x + \frac{1}{4} \int \sin^2 \theta \, d\theta$$
$$= \frac{1}{4} \sin^3 x \cos x + \frac{1}{4} \left( -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx \right)$$
$$= \frac{1}{4} \sin^3 x \cos x - \frac{1}{8} \sin x \cos x + \frac{1}{8} x + C$$

**3.** Compute the definite integral 
$$\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$$
.

Use the reduction formula for  $\cos^m x$ :

$$\int_{0}^{\pi/2} \cos^{3} x \, dx = \frac{1}{3} \cos^{2} x \sin x \Big|_{0}^{\pi/2} + \frac{2}{3} \int_{0}^{\pi/2} \cos x \, dx$$
$$= \left[ \frac{1}{3} (0)(1) - \frac{1}{3} (1)(0) \right] + \frac{2}{3} \sin x \Big|_{0}^{\pi/2}$$
$$= 0 + \frac{2}{3} (1 - 0) = \frac{2}{3}$$

4. Evaluate the definite integral  $\int_{-\frac{\pi}{4}}^{-\frac{\pi}{4}} \sec^4 x \, dx$ . First use the reduction formula for  $\sec^m x$  to evaluate the indefinite integral:

$$\int \sec^4 x \, dx = \frac{1}{3} \tan x \sec^2 x + \frac{2}{3} \int \sec^2 x \, dx$$
$$= \frac{1}{3} \tan x \sec^2 x + \frac{2}{3} \tan x + C$$

Now compute the definite integral:

$$\int_{-\pi/4}^{\pi/4} \sec^4 x \, dx = \frac{1}{3} \tan x \sec^2 x + \frac{2}{3} \tan x \Big|_{-\pi/4}^{\pi/4}$$
$$= \left[ \frac{1}{3} (1) \left( \sqrt{2} \right)^2 + \frac{2}{3} (1) \right] - \left[ \frac{1}{3} (-1) \left( \sqrt{2} \right)^2 + \frac{2}{3} (-1) \right]$$
$$= \frac{4}{3} - \left( -\frac{4}{3} \right) = \frac{8}{3}$$

### 7.3. Trigonometric Substitution.

## Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS.

### **Key Points**

- If the integrand involves  $\sqrt{a^2 x^2}$ , use the trigonometric substitution  $x = a \sin \theta$ . Then  $\sqrt{a^2 - x^2} = a \cos \theta$ .
- If the integrand involves  $\sqrt{a^2 + x^2}$ , use the trigonometric substitution  $x = a \tan \theta$ . Then  $\sqrt{a^2 + x^2} = a \sec \theta$ .
- If the integrand involves  $\sqrt{x^2 a^2}$ , use the trigonometric substitution  $x = a \sec \theta$ . Then  $\sqrt{x^2 - a^2} = a \tan \theta$ .

### Lecture Material

Using the slide provided, discuss the basic principle behind trigonometric substitution. Emphasize that students should draw a right triangle and label the sides before tackling the integration. There are three important cases, outlined in the Key Points. Work Examples 1 and 2 and then Exercises 22 and 24.

Remind students of how to complete the square, then work either Example 5 or Exercise 36.

### **Discussion Topics/Class Activities**

Discuss the preliminary questions with the students.

#### Suggested Problems

Exercises 1, 3 (basic), 13, 15, 21, 29 (harder), 35, 37, 39 (challenging), 43–51 odd, 53 (intuitive), 55, 56 (application)

# Worksheet 7.3. Trigonometric Substitution

1. Use trigonometric substitution to evaluate  $\int \frac{dx}{\sqrt{x^2 - 9}} dx$ .

**2.** Evaluate the definite integral 
$$\int_0^1 \frac{dx}{(16+x^2)^2}$$
.

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3. Evaluate  $\int \frac{dx}{\sqrt{12x - x^2}}$  by first completing the square and then using trigonometric substitution.

4. Indicate a method for evaluating the given integral, but don't actually integrate. a.  $\int \sin^3 x \cos^3 dx$ 

b. 
$$\int \sqrt{4x^2 - 1} \, dx$$

c. 
$$\int \ln x \, dx$$

### Solutions to Worksheet 7.3

1. Use trigonometric substitution to evaluate  $\int \frac{dx}{\sqrt{x^2-9}} dx$ . Let  $x = 3 \sec \theta$ . Then  $dx = 3 \sec \theta \tan \theta \, d\theta$ , and  $x^2 - 9 = 9 \sec^2 \theta - 9 = 9(\sec^2 \theta - 1) = 9(\sec^2 \theta - 1)$  $9\tan^2\theta$ , gives

$$I = \int \frac{dx}{\sqrt{x^2 - 9}} = \int \frac{3 \sec \theta \tan \theta \, d\theta}{3 \tan \theta} = \int \sec \theta \, d\theta$$
$$= \ln|\sec \theta + \tan \theta| + C$$

Since  $x = 3 \sec \theta$ , we construct a right triangle with  $\sec \theta = \frac{x}{3}$ . From this we see that  $\tan \theta = \sqrt{x^2 - 9}/3$ , so

$$I = \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| + C_1 = \ln \left| x + \sqrt{x^2 - 9} \right| + \ln \left( \frac{1}{3} \right) + C_1$$
  
=  $\ln \left| x + \sqrt{x^2 - 9} \right| + C$ 

where  $C = \ln(\frac{1}{3}) + C_1$ .

2. Evaluate the definite integral  $\int_0^1 \frac{dx}{(16+x^2)^2}$ . Let  $x = 4 \tan \theta$ . Then  $dx = 4 \sec^2 \theta \, d\theta$ , and  $16 + x^2 = 16 + 16 \tan^2 \theta = 16(1 + \tan^2 \theta) =$  $16 \sec^2 \theta$ . Then

$$I = \int_0^1 \frac{dx}{(16+x^2)^2} = \int_0^1 \frac{4\sec^2\theta \,d\theta}{(16\sec^2\theta)^2} = \frac{4}{256} \int_0^1 \frac{\sec^2\theta \,d\theta}{\sec^4\theta}$$
$$= \frac{1}{64} \int_0^1 \cos^2\theta \,d\theta = \frac{1}{64} \left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right]_{x=0}^{x=1}$$

Since  $x = 4 \tan \theta$ , we see that  $\sin \theta = x/\sqrt{x^2 + 16}$  and  $\cos \theta = 4/\sqrt{x^2 + 16}$ . Therefore,

$$I = \frac{1}{128} \left[ \tan^{-1} \left( \frac{x}{4} \right) + \left( \frac{x}{\sqrt{x^2 + 16}} \right) \left( \frac{4}{\sqrt{x^2 + 16}} \right) \right]_0^1$$
  
=  $\frac{1}{128} \left[ \tan^{-1} \left( \frac{x}{4} \right) + \frac{4x}{\sqrt{x^2 + 16}} \right]_0^1 = \frac{1}{128} \left[ \left( \tan^{-1} \left( \frac{1}{4} \right) + \frac{4}{17} \right) - (0+0) \right]$   
=  $\frac{1}{128} \left[ \tan^{-1} \left( \frac{1}{4} \right) + \frac{4}{17} \right] \approx 0.003752$ 

3. Evaluate  $\int \frac{dx}{\sqrt{12x - x^2}}$  by first completing the square and then using trigonometric substitution.

First complete the square:

$$12x - x^{2} = -(x^{2} - 12x + 36 - 36) = -(x^{2} - 12x + 36) + 36$$
$$= 36 - (x - 6)^{2}$$

Now let u = x - 6, and du = dx. This gives

$$I = \int \frac{dx}{\sqrt{12x - x^2}} = \int \frac{dx}{\sqrt{36 - (x - 6)^2}} = \int \frac{du}{\sqrt{36 - u^2}}$$

Now let  $u = 6 \sin \theta$ . Then  $du = 6 \cos \theta \, d\theta$ , and  $36 - u^2 = 36 - 36 \sin^2 \theta = 36(1 - \sin^2 \theta) = 36 \cos^2 \theta$ . So then

$$I = \int \frac{6\cos\theta \,d\theta}{6\cos\theta} = \int \,d\theta = \theta + C$$

Substituting back, we get

$$I = \sin^{-1}\left(\frac{u}{6}\right) + C = \sin^{-1}\left(\frac{x-6}{6}\right) + C$$

- 4. Indicate a method for evaluating the given integral, but don't actually integrate.
  - a.  $\int \sin^3 x \cos^3 dx$

Use the following trig method: Substitute  $\sin^2 x = 1 - \cos^2 x$  and let  $u = \cos x$ .

b.  $\int \sqrt{4x^2 - 1} \, dx$ 

Use trigonometric substitution, with  $x = \frac{1}{2} \sec \theta$ .

c.  $\int \ln x \, dx$ 

Use Integration by Parts, with  $u = \ln x$  and v' = 1.

### 7.4. Integrals of Hyperbolic and Inverse Hyperbolic Functions.

## Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS. Key Points

• Integrals of hyperbolic functions:

$$\int \sinh x dx = \cosh x + C \qquad \int \cosh x dx = \sinh x + C$$
$$\int \operatorname{sech}^2 x dx = \tanh x + C \qquad \int \operatorname{csch}^2 dx = -\coth x + C$$
$$\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C \qquad \int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + C$$

• Integrals involving inverse hyperbolic functions:

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x + C$$

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x + C \qquad (\text{for } x > 1)$$

$$\int \frac{dx}{1 - x^2} = \tanh^{-1} x + C \qquad (\text{for } |x| < 1)$$

$$\int \frac{dx}{1 - x^2} = \coth^{-1} x + C \qquad (\text{for } |x| < 1)$$

$$\int \frac{dx}{x\sqrt{1 - x^2}} = -\operatorname{sech}^{-1} x + C \qquad (\text{for } 0 < x < 1)$$

$$\int \frac{dx}{|x|\sqrt{1 - x^2}} = -\operatorname{csch}^{-1} x + C \qquad (\text{for } x \neq 0)$$

### Lecture Material

First point out that the differentiation formulas for the hyperbolic functions give integration formulas, and state them. Also point out the hyperbolic identities listed in this section and that similar techniques to those involving the integrals of trigonometric functions can be used for the hyperbolic trigonometric functions. Work Exercises 4 and 14 to demonstrate use of these formulas. Now work Exercise 18 to show that hyperbolic substitutions can be made analogously to trigonometric substitution, and then state the integral formulas for the inverse hyperbolic functions.

## Discussion Topics/Class Activities

With students, work Exercise 32 which shows that the answers obtained from either trigonometric substitution or hyperbolic substitution are the same in a particular case.

### Suggested Problems

Exercises 1–15 odd (hyperbolic integrals), 17–29 odd (hyperbolic substitution integrals)

# Worksheet 7.4. Integrals of Hyperbolic and Inverse Hyperbolic Functions

Calculate the integrals.

1. 
$$\int \sinh^2 x \cosh x dx$$

**2.** 
$$\int \tanh(3t) \operatorname{sech}(3t) dt$$

$$3. \int \frac{dx}{\sqrt{x^2 - 4}}$$

# Solutions to Worksheet 7.4

Calculate the integrals.

1. 
$$\int \sinh^2 x \cosh x \, dx$$
$$\int \sinh^2 x \cosh x \, dx = \frac{1}{3} \sinh^3 x$$
  
2. 
$$\int \tanh(3t) \operatorname{sech}(3t) \, dt$$
$$\int \tanh(3t) \operatorname{sech}(3t) \, dt = -\frac{1}{3} \operatorname{sech}(3x) + C$$
  
3. 
$$\int \frac{dx}{\sqrt{x^2 - 4}}$$
$$\int \frac{dx}{\sqrt{x^2 - 4}} = \int \frac{dx}{2\sqrt{(\frac{x}{2})^2 - 1}} = \cosh^{-1}(\frac{x}{2})$$

### 7.5. The Method of Partial Fractions.

Class Time AB 0 periods; BC 1–2 periods. Essential.

BC only. This material is not tested on the AB exam.

### **Key Points**

- Suppose  $f(x) = \frac{P(x)}{Q(x)}$  such that the degree of P is less than the degree of P. Then we use the Method of Partial Fractions.
- If  $Q(x) = (x a_1)(x a_2) \cdots (x a_n)$ , where the roots  $a_j$  are distinct, then there exist constants  $A_1, \ldots, A_n$  such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} \dots + \frac{A_n}{x - a_n}$$

• BC students may be tested on partial fractions involving non-repeated linear factors only. These expressions may appear in the context of the logistic differential equation (Section 9.4). The rest of the material in this chapter is optional.

### Lecture Material

Suppose P and Q are polynomials and the degree of P is less than the degree of Q. To integrate  $\int \frac{P(x)}{Q(x)} dx$ , we factor Q. We are concerned only with the case where Q is a product of distinct linear factors. First, set

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2}\cdots + \frac{A_n}{x-a_n}$$

Then solve for the constants  $A_1, \ldots, A_n$ . You can show students the trick of plugging in the roots to simplify the algebra. Work Example 1, 2 (omit 4–6) or Exercise 12.

If P has a degree larger than or equal to Q, then divide Q into P using long division (Example 3).

**Discussion Topics/Class Activities** 

Discuss Exercise 49.

## Suggested Problems

Exercises 2, 5, 7, 9, 11, 13, 15, 28

# Worksheet 7.5. The Method of Partial Fractions

**1.** Use the Method of Partial Fractions to evaluate  $\int \frac{(3x+5)dx}{x^2-4x-5}$ .

**2.** Use the Method of Partial Fractions to evaluate  $\int \frac{3dx}{(x+1)(x^2+x)}$ .
**3.** Use long division to write  $\frac{x^3-1}{x^2-x}$  as the sum of a polynomial and a proper rational fraction. Then calculate the integral.

## Solutions to Worksheet 7.5

1. Use the Method of Partial Fractions to evaluate  $\int \frac{(3x+5)dx}{x^2-4x-5}$ . The denominator factors as  $x^2 - 4x - 5 = (x-5)(x+1)$ , so the partial fraction decomposition has the form

$$\frac{3x+5}{x^2-4x-5} = \frac{3x+5}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$

Clearing denominators gives

$$3x + 5 = A(x + 1) + B(x - 5)$$

Setting x = 5,

$$20 = A(6) + 0 \Rightarrow A = \frac{20}{6} = \frac{10}{3}$$

Setting x = -1,

$$2 = 0 + B(-6) \Rightarrow B = -\frac{1}{3}$$

The result is

$$\frac{3x+5}{x^2-4x-5} = \frac{\frac{10}{3}}{x-5} + \frac{-\frac{1}{3}}{x+1}$$
$$\int \frac{(3x+5)\,dx}{x^2-4x-5} = \frac{10}{3}\int \frac{dx}{x-5} - \frac{1}{3}\int \frac{dx}{x+1}$$
$$= \frac{10}{3}\ln|x-5| - \frac{1}{3}\ln|x+1| + C$$

2. Use the Method of Partial Fractions to evaluate  $\int \frac{3dx}{(x+1)(x^2+x)}$ . The partial fraction decomposition has the form

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{(x+1)(x)(x+1)} = \frac{3}{x(x+1)^2}$$
$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Clearing denominators gives

$$3 = A(x+1)^2 + Bx(x+1) + Cx$$

Setting x = 0 gives

$$3 = A(1) + 0 + 0 \Rightarrow A = 3$$
$$3 = 0 + 0 + C(-1) \Rightarrow C = -3$$

Setting x = -1 gives

$$3 = 0 + 0 + C(-1) \Rightarrow C = -3$$

Now plug in A = 3 and C = -3:

$$3 = 3(x+1)^2 + Bx(x+1) - 3x$$

The constant B can be determined by plugging in for x any value other than 0 or -1. Plugging in x = 1 gives

$$3 = 3(4) + B(1)(2) - 3 \Rightarrow B = -3$$

The result is

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-3}{(x+1)^2}$$
$$\int \frac{3 \, dx}{(x+1)(x^2+x)} = 3 \int \frac{dx}{x} - 3 \int \frac{dx}{x+1} - 3 \int \frac{dx}{(x+1)^2}$$
$$= 3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + C$$

**3.** Use long division to write  $\frac{x^3-1}{x^2-x}$  as the sum of a polynomial and a proper rational fraction. Then calculate the integral.

Long division gives

$$\frac{x^3 - 1}{x^2 - x} = x + 1 + \frac{x - 1}{x^2 - x} = x + 1 + \frac{x - 1}{x(x - 1)} = x + 1 + \frac{1}{x}$$

Therefore the integral is

$$\int \frac{x^3 - 1}{x^2 - x} dx = \int (x + 1) dx + \int \frac{dx}{x}$$
$$= \frac{1}{2}x^2 + x + \ln|x| + C$$

#### 7.6. Improper Integrals.

Class Time AB 0 periods; BC 2 periods. Essential.

BC only. This material is not tested on the AB exam.

**Key Points** 

•  $\int_{a}^{\infty} f(x) dx$ ,  $\int_{-\infty}^{a} f(x) dx$ ,  $\int_{-\infty}^{\infty} f(x) dx$  are called improper integrals. •  $\int_{a}^{\infty} f(x) dx = \lim_{R \to \infty} \int_{a}^{R} f(x) dx$ •  $\int_{-\infty}^{a} f(x) dx = \lim_{R \to -\infty} \int_{R}^{a} f(x) dx$ •  $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \to -\infty} \int_{R}^{0} f(x) dx + \lim_{R \to \infty} \int_{0}^{R} f(x) dx$ If the limits exist, then the integrals are said to converge. Otherwise, they diverge.

• 
$$\int_{1}^{\infty} \frac{dx}{x^{p}} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1\\ \infty, & \text{if } p \le 1 \end{cases}$$

• If f(x) is continuous on [a, b) but discontinuous at x = b, then

$$\int_{a}^{b} f(x) dx = \lim_{R \to b^{-}} \int_{a}^{R} f(x) dx$$

• If f(x) is continuous on (a, b] but discontinuous at x = a, then

$$\int_{a}^{b} f(x) dx = \lim_{R \to a^{+}} \int_{R}^{b} f(x) dx$$

If the limits exists, then the integral is said to converge. Otherwise it diverges.

- $\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p}, & \text{if } p < 1\\ \infty, & \text{if } p \ge 1 \end{cases}$
- The Comparison Test for Improper Integrals (Theorem 3)

#### Lecture Material

It is possible to integrate a continuous function over an infinite integral and get a finite answer. An important example of this is the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2} \, dx$$

which is related to the standard normal distribution  $y = e^{-x^2/2}$ . We evaluate such integrals using limits. The three cases are outlined in the Key Points. Work Examples 1 and 2 and Exercise 54.

State and prove Theorems 1 and 2. Work Exercises 6 and 8.

Another type of improper integral occurs when the integrand f(x) becomes infinite at one or both endpoints of the interval of integration. For example,  $\int_{1}^{2} \frac{dx}{(x-1)^{2}}$  is improper because  $\lim_{x\to 1^+} \frac{1}{(x-1)^2} = \infty$ . Again, we use limits to evaluate these integrals. The two cases are listed in the Key Points. Work Exercises 22 and 23. State and prove

Sometimes it is not possible to evaluate an improper integral, but we can determine whether it converges or not by comparing it to an integral that we can evaluate. We can use the Comparison Test for Improper Integrals. Suppose  $f(x) \ge g(x) > 0$  for  $x \ge a$ . If  $\int_{a}^{\infty} f(x) dx$  converges, then  $\int_{a}^{\infty} g(x) dx$  also converges and if  $\int_{a}^{\infty} g(x) dx$  diverges, then  $\int_{a}^{\infty} g(x) dx$  also diverges.

Theorem 2.

Vork Exercises 58 and 70 and Example 9.

#### **Discussion Topics/Class Activities**

Discuss Example 3 (using L'Hôpital's Rule). This is the only situation where AP students may have to use L'Hôpital's Rule. Of course, they may use it with regular limits.

#### Suggested Problems

Exercises 5, 9, 11, 13, 15, 21, 35 (basic), 39, 43 (harder), 45, 47 (doubly improper), 61, 63, 67 (comparison test)

# Worksheet 7.6. Improper Integrals

Determine whether the improper integral converges, and if it does, evaluate it.

**1.** 
$$\int_{1}^{\infty} \frac{dx}{x^{20/19}}$$

$$2. \quad \int_2 0^\infty \frac{dt}{t}$$

3. 
$$\int_0^5 \frac{dx}{x^{19/20}}$$

$$4. \quad \int_1^3 \frac{dx}{\sqrt{3-x}}$$

5. 
$$\int_{-2}^{4} \frac{dx}{(x+2)^{1/3}}$$

## Solutions to Worksheet 7.6

Determine whether the improper integral converges and if it does, evaluate it.

1.  $\int_{1}^{\infty} \frac{dx}{x^{20/19}}$ First evaluate the integral over the finite interval [1, R] for R > 1:

$$\begin{split} \int_{1}^{R} \frac{dx}{x^{20/19}} &= -19x^{-1/19} \Big|_{1}^{R} = \frac{-19}{R^{1/19}} - (-19) = 19 - \frac{19}{R^{1/19}} \\ \int_{1}^{\infty} \frac{dx}{x^{20/19}} &= \lim_{R \to \infty} \int_{1}^{R} \frac{dx}{x^{20/19}} = \lim_{R \to \infty} \left( 19 - \frac{19}{R^{1/19}} \right) \\ &= 19 - 0 = 19 \end{split}$$

2.  $\int_{20}^{\infty} \frac{dt}{t}$ 

First evaluate the integral over the finite interval [20, R] for 20 < R:

$$\int_{20}^{R} \frac{dt}{t} = \ln |t||_{20}^{R} = \ln R - \ln 20$$
$$\int_{20}^{\infty} \frac{dt}{t} = \lim_{R \to \infty} \int_{20}^{R} \frac{dt}{t} = \lim_{R \to \infty} (\ln R - \ln 20) = \infty$$

The integral does not converge.

3. 
$$\int_0^5 \frac{dx}{x^{19/20}}$$

The function  $x^{-19/20}$  is infinite at the endpoint 0, so we first evaluate the integral on the finite interval [R, 5] for 0 < R < 5:

$$\int_{R}^{5} \frac{dx}{x^{19/20}} = 20x^{1/20} \Big|_{R}^{5} = 20 \left( 5^{1/20} - R^{1/20} \right)$$
$$\int_{0}^{5} \frac{dx}{x^{19/20}} = \lim_{R \to 0^{+}} \int_{R}^{5} \frac{dx}{x^{19/20}} = \lim_{R \to 0^{+}} 20 \left( 5^{1/20} - R^{1/20} \right)$$
$$= 20 \left( 5^{1/20} - 0 \right) = 20 \cdot 5^{1/20}$$

$$4. \quad \int_1^3 \frac{dx}{\sqrt{3-x}}$$

The function  $f(x) = 1/\sqrt{3-x}$  is infinite at x = 3 and is left continuous at x = 3, so we first evaluate the integral on the interval [1, R] for 1 < R < 3:

$$\int_{1}^{R} \frac{dx}{\sqrt{3-x}} = 2\sqrt{3-x} \mid_{0}^{R}$$
$$= -2\sqrt{3-R} + 2\sqrt{2}$$
$$\lim_{R \to 3} \int_{1}^{R} \frac{dx}{\sqrt{3-x}} = 0 + 2\sqrt{2}$$

Therefore the integral is equal to  $2\sqrt{2}$ .

5. 
$$\int_{-2}^{4} \frac{dx}{(x+2)^{1/3}}$$

The function  $(x+2)^{-1/3}$  is infinite at x = -2 and right-continuous at x = -2, so we first evaluate the integral on the interval [R, 4] for -2 < R < 4:

$$\int_{R}^{4} \frac{dx}{(x+2)^{1/3}} = \frac{3}{2}(x+2)^{2/3} \Big|_{R}^{4} = \frac{3}{2} \left( 6^{2/3} - (R+2)^{2/3} \right)$$
$$\int_{-2}^{4} \frac{dx}{(x+2)^{1/3}} = \lim_{R \to -2^{+}} \int_{R}^{4} \frac{dx}{(x+2)^{1/3}} = \lim_{R \to 2^{+}} \frac{3}{2} \left( 6^{2/3} - (R+2)^{2/3} \right)$$
$$= \frac{3}{2} \left( 6^{2/3} - 0 \right) = \frac{3 \cdot 6^{2/3}}{2}$$

## 7.7. Probability and Integration.

 Key Points None
 Lecture Material None
 Discussion Topics/Class Activities None
 Suggested Problems None

**Class Time** 0 periods; omit for AB and BC.

#### 7.8. Numerical Integration.

Class Time AB 2 periods; BC 1 period. Essential.

#### **Key Points**

• We consider two numerical approximations to  $\int_{a}^{b} f(x) dx$ : the Trapezoidal Rule  $T_{N}$  and the Midpoint Rule  $M_{N}$ . For a positive integer N let  $\Delta x = \frac{b-a}{N}$  and  $y_{j} = f(a+j\Delta x), j = 0, 1, ..., N$ . Then

$$T_N = \frac{1}{2}\Delta x (y_0 + 2y_1 + 2y_2 + \dots + 2y_{N-1} + y_N)$$

and

$$M_N = \Delta x (f(c_1) + f(c_2) + \dots + f(c_N)), \text{ where } c_j = a + (j - \frac{1}{2})\Delta x.$$

- For both the AB and BC exams students are expected to be able to approximate a definite integral using a left sum, a right sum, a midpoint sum (see also Section 5.1) and the Trapezoidal Rule using a small number of intervals.
- The function values needed to use these formulas are often given in a table and sometimes in a graph rather than an equation. Uneven subintervals are not uncommon. For this reason, the formulas need not be memorized if the students have a good graphical understanding of the process. For example, drawing the few trapezoids required and finding the sum of their areas results in the Trapezoidal Rule approximation.
- Simpson's Rule and the Error Bound formulas for numerical integration are not tested on either exam. Both of these may be omitted.

#### Lecture Material

We cannot always find a formula for the antiderivative of the integrand. In these instances we need to integrate numerically. An important example is the standard normal distribution  $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$  used in statistical applications. The three numerical techniques discussed in the text are the Trapezoidal Rule, the Midpoint Rule, and Simpson's Rule.

For the Trapezoidal Rule, we partition [a, b] into N intervals of width  $\Delta x = \frac{b-a}{N}$  and compute the area of each trapezoid with lengths  $f(x_{j-1})$  and  $f(x_j)$  and height  $\Delta x$  for all

 $j = 1, \ldots N$ . Then we add them up. This is equivalent to computing the average of the left- and right-endpoint approximations. So the formula for a given N is

$$T_N = \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + \dots + 2y_{N-1} + y_N)$$

where  $y_j = f(a + j\Delta x), j = 0, 1, \dots, N$ . Work Example 1.

The Midpoint Rule is obtained by summing the areas of the rectangles with widths  $\Delta x$  and heights  $f(c_j)$ , where  $c_j$  is the midpoint of the interval, for  $j = 1 \dots N$ . This can also be interpreted as the area of the "tangential" trapezoids, illustrated in Figure 5 in the text. You can use the slide provided to discuss the Midpoint Rule. The formula is

$$M_N = \Delta x (f(c_1) + f(c_2) + \dots + f(c_N))$$

where  $c_j = a + (j - \frac{1}{2})\Delta x$ . You may omit the discussion of the error bounds.

#### **Discussion Topics/Class Activities**

For both the Trapezoidal and Midpoint approximations, as well as left- and right-sums, stress the graphical aspects. That is, show the rectangles or trapezoids for a small value of n on the graph. Find the approximation without direct use of the formulas.

Demonstrate that the difference between the exact value of the integral and the approximation depends on whether the function is increasing or decreasing, or concave up or down. For example, if the graph is concave down, the trapezoidal approximation underestimates the value while the midpoint approximation overestimates the value.

#### Suggested Problems

Exercises 1, 7, 11, (numerical), 23, 27 (use trapezoids), 28 (use midpoints), also p. 381 #56 and p. 477 #1, 10

# Worksheet 7.8. Numerical Integration

**1.** Find 
$$T_4$$
 for  $\int_0^4 \sqrt{x} \, dx$ .

**2.** Find 
$$M_8$$
 for  $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx$ .

**3.** State whether  $M_{10}$  underestimates or overestimates  $\int_{1}^{4} \ln x \, dx$ .

## Solutions to Worksheet 7.8

1. Find  $T_4$  for  $\int_0^4 \sqrt{x} \, dx$ . Let  $f(x) = \sqrt{x}$ . We divide [0, 4] into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1$$

with endpoints 0, 1, 2, 3, 4, and midpoints 0.5, 1.5, 2.5, 3.5. With this data, we get:

$$T_4 = \frac{1}{2} \Delta x \left( \sqrt{0} + 2\sqrt{1} + 2\sqrt{2} + 2\sqrt{3} + \sqrt{4} \right)$$
  

$$\approx 5.14626$$
  

$$M_4 = \Delta x \left( \sqrt{0.5} + \sqrt{1.5} + \sqrt{2.5} + \sqrt{3.5} \right)$$
  

$$\approx 5.38382$$

2. Find  $M_8$  for  $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx$ . Let  $f(x) = \sqrt{\sin x}$ . We divide  $[0, \pi/2]$  into 8 subintervals of width

$$\Delta x = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

with endpoints

$$0, \frac{\pi}{16}, \frac{2\pi}{16}, \cdots, \frac{8\pi}{16} = \frac{\pi}{2},$$

and midpoints

$$\frac{\pi}{32}, \frac{3\pi}{32}, \cdots, \frac{15\pi}{32}.$$

With this data, we get:

$$T_8 = \frac{1}{2} \left(\frac{\pi}{16}\right) \left(\sqrt{\sin(0)} + 2\sqrt{\sin(\pi/16)} + \dots + \sqrt{\sin(8\pi/16)}\right)$$
  

$$\approx 1.18005$$
  

$$M_8 = \frac{\pi}{16} \left(\sqrt{\sin(\pi/32)} + \sqrt{\sin(3\pi/32)} + \dots + \sqrt{\sin(15\pi/32)}\right)$$
  

$$\approx 1.20344$$

**3.** State whether  $M_{10}$  underestimates or overestimates  $\int_{1}^{4} \ln x \, dx$ .

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Let  $f(x) = \ln x$ . Then f'(x) = 1/x and

$$f''(x) = -\frac{1}{x^2} < 0$$

on [1, 4], so f(x) is concave down, and  $M_{10}$  overestimates the integral.

## Chapter 7 AP Problems

For 1, 2, 4 and 5, a calculator may be used. For 3 and 6, no calculator allowed.

1. A continuous function f has the values shown in the table below for selected x-values in the interval [0, 4].

If  $T_2$  is the trapezoidal approximation of  $\int_0^4 f(x) dx$  with two equal subintervals and  $T_4$  is the trapezoidal approximation of  $\int_0^4 f(x) dx$  with four equal subintervals, then  $T_2 - T_4$  is equal to:

A. 0

- B. a b
- C. a + b
- D. a + b + c
- E. b + 2c
- **2.** For k > 0 and n an integer such that  $n \ge 2$ ,  $\int x^{n-1} \ln(x) dx =$

A. 
$$\frac{n}{x}x^{n} + C$$
  
B. 
$$nx^{n} - \ln(x) + C$$
  
C. 
$$\frac{x^{n}}{2}(\ln(x))^{2} + C$$
  
D. 
$$x^{n-2} - x\ln(x) + C$$
  
E. 
$$\frac{x^{n}}{n}\ln(x) - \frac{1}{n^{2}}x^{n} + C$$

**3.** If the substitution  $u = \sec x$  is used to transform  $\int_0^{\pi/4} \tan x \, dx$ , which of these is an equivalent integral expression?

A. 
$$\int_{0}^{\pi/4} u \, du$$
  
B. 
$$\int_{0}^{\pi/4} \frac{1}{u} \, du$$
  
C. 
$$\int_{1}^{\sqrt{2}} u \sqrt{u} \, du$$
  
D. 
$$\int_{1}^{\sqrt{2}} \frac{1}{u} \, du$$
  
E. 
$$\int_{0}^{\pi/4} u \sqrt{1+u^{2}} \, du$$

4. If the substitution  $x = 3\sin\theta$  is used, then  $\int \frac{x^2}{(9-x^2)^{3/2}} dx$  is equivalent to

A. 
$$\int \frac{1}{3} \sec^{3}(\theta) d\theta$$
  
B. 
$$\int \frac{1}{3} \tan^{3}(\theta) d\theta$$
  
C. 
$$\int \tan^{2}(\theta) d\theta$$
  
D. 
$$\int \csc(\theta) d\theta$$
  
E. 
$$\int \frac{1}{3} \tan^{2}(\theta) \sec(\theta) d\theta$$

- 5. A region R in the first quadrant is bounded by the curve  $f(x) = \frac{1}{x(x-k)}$ , the x-axis, and the lines x = a and x = b, where k > 0 and a < b.
  - a. Set up, but do not integrate, and integral expression that could be used to find the area of R.
  - b. Using integration by partial fractions, find the general antiderivative of the expression in part (a). Show the work that leads to your answer.
  - c. Compute the area of R when a = 2k and b = 4k.
- 6. What are ALL values of p for which  $\int_{1}^{\infty} \frac{1}{x^{2p-1}} dx$  diverges?
  - A. p < 1/2B. p > 1/2C. p > 1D. p < 1
  - E. The integral diverges for all values of p

## Solutions to Chapter 7 AP Problems

1. A continuous function f has the values shown in the table below for selected x-values in the interval [0, 4].

If  $T_2$  is the trapezoidal approximation of  $\int_0^4 f(x) dx$  with two equal subintervals and  $T_4$  is the trapezoidal approximation of  $\int_0^4 f(x) dx$  with four equal subintervals, then  $T_2 - T_4$  is equal to:

A. 0

- B. a b
- C. a + b
- D. a + b + c

E. b + 2c

# **B** [THIS PROBLEM CORRESPONDS WITH SECTION 7.8]

**2.** For k > 0 and n an integer such that  $n \ge 2$ ,  $\int x^{n-1} \ln(x) dx =$ 

A.  $\frac{n}{x}x^n + C$ B.  $nx^n - \ln(x) + C$ C.  $\frac{x^n}{2}(\ln(x))^2 + C$ D.  $x^{n-2} - x\ln(x) + C$ E.  $\frac{x^n}{n}\ln(x) - \frac{1}{n^2}x^n + C$  **3.** If the substitution  $u = \sec x$  is used to transform  $\int_0^{\pi/4} \tan x \, dx$ , which of these is an equivalent integral expression?

A. 
$$\int_{0}^{\pi/4} u \, du$$
  
B. 
$$\int_{0}^{\pi/4} \frac{1}{u} \, du$$
  
C. 
$$\int_{1}^{\sqrt{2}} u \sqrt{u} \, du$$
  
D. 
$$\int_{1}^{\sqrt{2}} \frac{1}{u} \, du$$
  
E. 
$$\int_{0}^{\pi/4} u \sqrt{1 + u^{2}} \, du$$

# D [THIS PROBLEM CORRESPONDS WITH SECTION 7.2]

4. If the substitution  $x = 3\sin\theta$  is used, then  $\int \frac{x^2}{(9-x^2)^{3/2}} dx$  is equivalent to

A. 
$$\int \frac{1}{3} \sec^{3}(\theta) d\theta$$
  
B. 
$$\int \frac{1}{3} \tan^{3}(\theta) d\theta$$
  
C. 
$$\int \tan^{2}(\theta) d\theta$$
  
D. 
$$\int \csc(\theta) d\theta$$
  
E. 
$$\int \frac{1}{3} \tan^{2}(\theta) \sec(\theta) d\theta$$
  
C [THIS PROBLEM CORRESPONDS WITH SECTION 7.3]

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- 5. A region R in the first quadrant is bounded by the curve  $f(x) = \frac{1}{x(x-k)}$ , the x-axis, and the lines x = a and x = b, where k > 0 and a < b.
  - a. Set up, but do not integrate, and integral expression that could be used to find the area of R.

$$\int_{a}^{b} \frac{1}{x(x-k)} \, dx$$

b. Using integration by partial fractions, find the general antiderivative of the expression in part (a). Show the work that leads to your answer.

$$\frac{1}{x(x-k)} = \frac{A}{x} + \frac{B}{(x-k)}$$
  

$$1 = A(x-k) + Bx$$
  

$$A = -\frac{1}{k}; B = \frac{1}{k}$$
  

$$\int \frac{1}{x(x-k)} dx = \frac{1}{k} \int -\frac{1}{x} + \frac{1}{(x-k)} dx = \frac{1}{k} \ln\left(\frac{x-k}{x}\right) + C$$

c. Compute the area of R when a = 2k and b = 4k.

$$\int_{2k}^{3k} \frac{1}{x(x-k)} dx = \frac{1}{k} \ln\left(\frac{x-k}{x}\right) \Big|_{2k}^{4k} = \frac{1}{k} [\ln(3/4) - \ln(1/2)] = \frac{1}{k} \ln(3/2)$$
  
[THIS PROBLEM CORRESPONDS WITH SECTION 7.5]

- 6. What are ALL values of p for which  $\int_{1}^{\infty} \frac{1}{x^{2p-1}} dx$  diverges?
  - A. p < 1/2
  - B. p > 1/2
  - C. p > 1
  - D. *p* < 1

E. The integral diverges for all values of p

## D [THIS PROBLEM CORRESPONDS WITH SECTION 7.6]

#### Ray Cannon's Chapter 8 Overview

Chapter 8 is a relatively short chapter, but covers two important topics in the BC course description. Section 8.1 deals with arc length of a graph (in the BC course description) and area of a surface of revolution (optional). The text continues to carefully develop applications of definite integrals via setting up approximating Riemann sums as recommended in the BC course description. This method continues in the next two sections, which cover the optional topics of fluid pressure and its related force in Section 8.2, and Moments and the center of mass in Section 8.3.

Section 8.4 deals with the very important topic of Taylor polynomials, the special case of Maclaurin polynomials, and the establishment of the error bound on the difference between the actual value of f(x) and its approximation by P(x). Students should be familiar with all the terminology in this section, and with the techniques for manipulating Taylor polynomials found in the exercises. The bound established in the text (Theorem 2) is the same as the Lagrange error bound called for in the course description, but is arrived at in a way different from using the Lagrange form for the remainder in what is called Taylors Theorem.

# 8. Further Applications

#### 8.1. Arc Length and Surface Area.

#### Class Time AB 0 periods; BC 1 period. Essential.

This is a BC only topic. Since so little of this chapter needs to be covered in AP courses, you may wish to hold this section until you get to Section 11.2 where arc length in parametric form is discussed and include it there. Since the problems are so similar, this should not present difficulties.

#### **Key Points**

• The formula for the arc length of a smooth function f(x) on [a, b] is

$$\int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx$$

• Surface area is not tested on the AB or BC exams.

#### Lecture Material

Discuss how the arc length of a continuous function f(x) over [a, b] can be approximated by partitioning the interval [a, b] and adding up the lengths of the line segments that connect the points on the f that correspond to the partition points. Then the actual arc length is obtained by taking the limit as the width of the partition goes to 0. Use the Pythagorean Theorem to find the length of each line segment and then apply the Mean Value Theorem to get the arc length formula. Work Examples 2 and 3 or Exercises 10 and 12.

#### **Discussion Topics/Class Activities**

Students could work Exercise 15 in groups or independently.

#### Suggested Problems

Exercises 7, 9 (basic), 11, 12 (numerical), 18, 25 (harder)

# Worksheet 8.1. Arc Length and Surface Area

**1.** Calculate the arc length of the curve  $y = \ln(\cos x)$  over the interval  $[0, \frac{\pi}{4}]$ .

2. Approximate the arc length of the curve  $y = \sin x$  over the interval  $[0, \frac{\pi}{2}]$  using the Midpoint Rule  $M_8$ .

#### Solutions to Worksheet 8.1

1. Calculate the arc length of the curve  $y = \ln(\cos x)$  over the interval  $[0, \frac{\pi}{4}]$ .  $1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$  and  $e^{\pi/4}$ 

$$\int_0^{\pi/4} \sqrt{1 + (y')^2} \, dx = \int \sec x \, dx = \ln |\sec x + \tan x| \Big|_0^{\pi/4} = \ln |\sqrt{2} + 1|$$

2. Approximate the arc length of the curve  $y = \sin x$  over the interval  $[0, \frac{\pi}{2}]$  using the Midpoint Rule  $M_8$ .

Since  $y = \sin x$ ,

$$\begin{aligned} 1+y'^2 &= 1+(\cos x)^2\\ \text{Therefore } \sqrt{1+y'^2} &= \sqrt{1+(\cos x)^2}, \text{ and the arc length over } [0,\pi/2] \text{ is }\\ \int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \end{aligned}$$

Let  $f(x) = \sqrt{1 + \cos^2 x}$ .  $M_8$  is the midpoint approximation with eight subdivisions. Since, n = 8,

$$\Delta x = \frac{\pi/2}{8} = \pi/16$$

$$x_i = 0 + \left(i + \frac{1}{2}\right) \Delta x \ (i = 0 \dots 8)$$

$$y_i = f\left(\left(i + \frac{1}{2}\right) \Delta x\right)$$

$$M_8 = \sum_{i=1}^8 (y_i) \Delta x$$

Since  $f(x_1) = 1.41081, f(x_1) = 1.3841, f(x_3) = 1.3334, f(x_4) = 1.26394, f(x_5) = 1.18425, f(x_6) = 1.10554, f(x_7) = 1.04128$ , and  $f(x_8) = 1.00479$ , the arc length is approximately 1.9101.

# Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS.

## Key Points

- The fluid pressure at depth h is wh, where w is the density of the fluid.
- If an object is submerged vertically extending from a depth y = a to a depth
  - y = b, then the total force on a side of the object is  $F = w \int_{a}^{b} yf(y) \, dy$ , where

f(y) is the horizontal width of the side at depth y.

#### Lecture Material

Remind students that water pressure is proportional to depth and that the fluid pressure at a depth h in a fluid of density w (weight per unit volume) is wh. The pressure acts at each point on the object in the direction perpendicular to the object's surface at that point. If the pressure is constant, then the total force acting on a surface is pressure times area. Students need to know that the density of water is  $w = 62.5 \text{ lb/ft}^2$ . Work Example 1.

Next discuss how an integral can be used to calculate the force on the side of a box submerged in water. This can be generalized to the side of an object with a width that can be expressed as a continuous function of y. Work Examples 2 and 3.

If time permits, you can discuss the fluid force on an inclined surface and work Example 4.

#### **Discussion Topics/Class Activities**

The students could work Exercise 15 in groups or independently.

#### Suggested Problems

Exercises 1, (basic), 2, 4, 5 (algebraic), 18, 20 (harder)

# Worksheet 8.2. Fluid Pressure and Force

1. A thin plate in the shape of an isosceles triangle with base 1 and height 2 ft is submerged so that the top of the triangle is 3 feet below the surface of the water, see the figure. Given that the density of water is  $62.5 \text{ lb/ft}^2$ , write an approximation for the total force F on the side of the plate as a Riemann sum and indicate the integral to which it converges.



2. Calculate the force on one side of a circular plate of radius 2 ft, submerged vertically in a tank of water so that the highest point is tangent to the surface of the water.

**3.** Calculate the fluid force on a square plate of side 3 ft submerged at an angle of 30 degrees with its top edge at the surface of the water.

4. The trough in the figure is filled with corn syrup, whose density is 90  $lb/ft^3$ , Calculate the force on the front of the trough.



### Solutions to Worksheet 8.2

1. A thin plate in the shape of an isosceles triangle with base 1 and height 2 ft is submerged so that the top of the triangle is 3 feet below the surface of the water, see the figure. Given that the density of water is  $62.5 \text{ lb/ft}^2$ , write an approximation for the total force F on the side of the plate as a Riemann sum and indicate the integral to which it converges.



2. Calculate the force on one side of a circular plate of radius 2 ft, submerged vertically in a tank of water so that the highest point is tangent to the surface of the water.

The width at any depth y is  $2\sqrt{4-(2-y)^2} = 2\sqrt{4y-y^2}$ . Thus,

$$F = 2\omega \int_0^4 y\sqrt{4y - y^2} \, dy = 2\omega 4\pi = 500\pi \text{ lb}$$

**3.** Calculate the fluid force on a square plate of side 3 ft, submerged at an angle of 30 degrees with its top edge at the surface of the water.

$$F = \frac{\omega}{\sin 30^{\circ}} \int_0^3 3y \, dy = 1687.5$$

4. The trough in the figure is filled with corn syrup, whose density is  $90 \text{ lb/ft}^3$ . Calculate the force on the front of the trough.



$$F = \omega \int_0^h h\left(\frac{y(b-a)}{h}\right) dy = \omega \int_0^h y(b-a) dy = 45(b-a)h^2$$

## Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS. **Key Points**

• If there are n particles with coordinates  $(x_j, y_j)$  and mass  $m_j$ , j = 1, ..., n, then the moments of the system are

 $M_x = m_1 y_1 + \dots + m_n y_n$  and  $M_y = m_1 x_1 + \dots + m_n x_n$ 

The center of mass of the system is the point  $(x_{cm}, y_{cm})$ , where  $x_{cm} = \frac{M_y}{M}$  and  $y_{cm} = \frac{M_x}{M}$  with  $M = m_1 + \dots + m_n$ .

• Suppose a lamina of constant mass density  $\rho$  occupies the region between two continuous functions,  $f_1(x)$  and  $f_2(x)$  from x = a to x = b, where  $f_1(x) \ge f_2(x)$ .

The y-moment is  $M_y = \rho \int_{a}^{b} x(f_1(x) - f_2(x)) dx$  and the x-moment is  $M_x =$  $\frac{\rho}{2}\int_{a}^{b}(f_1(x)^2-f_2(x)^2)\,dx.$  The total mass of the lamina is  $M=\rho\int_{a}^{b}(f_1(x)-f_2(x)^2)\,dx.$  $f_2(x)$  dx. The center of mass called the centroid is the point  $(x_{cm}, y_{cm})$ , where  $x_{cm} = \frac{M_y}{M}$  and  $y_{cm} = \frac{M_x}{M}$ . • Symmetry Principle: If a lamina of constant mass density is symmetric with

- respect to a given line, then the centroid lies on that line.
- Additivity Principle: If a region S consists of two or more smaller regions, then each moment of S is the sum of the corresponding moments of the smaller regions.

#### Lecture Material

Every object has a balance point called the center of mass. The object does not rotate when a force is applied along a line passing through its center of mass. The center of mass is expressed in terms of quantities called moments. The moment with respect to x measures the tendency of the object to rotate about the x-axis and the moment with respect to y measures the tendency of the object to rotate about the y-axis.

First discuss the moments of an individual particle with a given mass and then a system of n particles with their masses. Do Example 1.

Next consider a plate so thin that it can be thought of as two dimensional. Laminas used in scientific experiments are considered to be examples. Suppose we have a lamina of constant mass density  $\rho$  occupying a region between two continuous functions,  $f_1(x)$ and  $f_2(x)$  from x = a to x = b, where  $f_1(x) \ge f_2(x)$ . The total mass, M, of the lamina is density times area; thus  $M = \rho \int_{a}^{b} (f_1(x) - f_2(x)) dx$ . Use Riemann sums to derive the formulas for the moments with respect to y and x. The disadvantage of Equation 2 is that the lamina has to occupy a region that is both x-simple and y-simple. You might want to just use Equation 3. Work Exercises 13 and 18.

Discuss the Symmetry Principle and work Exercise 27. End the lecture with the Additivity Principle and work Example 5.

#### **Discussion Topics/Class Activities** Discuss Exercise 42.

Discuss Exercise 42.

#### Suggested Problems

Exercises 1, (basic), 4, 5, 9, 14, 15, 19, 23 (algebraic), 26, 32, 33 (harder)

# Worksheet 8.3. Center of Mass

1. Find the center of mass of the system of particles of mass 4, 2, 5, 1 located at (1, 2), (-3, 3), (2, -1) and (4, 0) respectively.

2. Find the centroid of the region lying underneath the graph of  $y = 9 - x^2$  over the interval [0, 3].

**3.** Find the centroid of the region enclosed by x = 0, y = x - 1, and  $y = (1 - x)^3$ .



4. Use additivity of moments to find the center of mass of three quarters of the unit circle. (Remove the part in the fourth quadrant.)



# Solutions to Worksheet 8.3

1. Find the center of mass of the system of particles of mass 4, 2, 5, 1 located at (1, 2), (-3, 3), (2, -1) and (4, 0) respectively.

$$M_x = 4(2) + 2(2) + 5(-1) + 1(0) = 7$$
  

$$M_y = 4(1) + 2(-3) + 5(2) + 1(4) = 12$$
  

$$M = 12, \text{ COM} = \left(1, \frac{7}{12}\right)$$

2. Find the centroid of the region lying underneath the graph of  $y = 9 - x^2$  over the interval [0, 3].

$$M_x = \frac{1}{2} \int_0^3 (9 - x^2)^2 \, dx = 324/5$$
$$M_y = \int_0^3 x(9 - x^2) \, dx = 81/4$$
$$M = \int_0^3 (9 - 2x) \, dx = 18$$
$$COM = \left(\frac{9}{8}, \frac{18}{5}\right)$$
**3.** Find the centroid of the region enclosed by x = 0, y = x - 1, and  $y = (1 - x)^3$ .



4. Use additivity of moments to find the center of mass of three quarters of the unit circle. (Remove the part in the fourth quadrant.)



By the Symmetry Principle, COM must lie on line y = -x. Let region 1 be the semicircle above the x-axis and region 2 be the quarter circle in Quadrant III. Then

$$M_y^1 = 0 \text{ by the Symmetry Principle}$$
$$M_y^2 = \int_{-1}^0 x \sqrt{1 - x^2} \, dx = -1/3$$
$$M_y = -1/3$$
$$M = 3\pi/4$$
$$\text{COM} = (-.14, .14)$$

#### 8.4. Taylor Polynomials.

Class Time AB 0 periods; BC 2–3 periods. Essential.

### **Key Points**

• The *n*th Taylor polynomial centered at x = a for the function f(x) is

$$T_n(x) = f(x) + \frac{f'(x)}{1!}(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(x)}{n!}(x-a)^n$$

When a = 0, this is called the *n*th Maclaurin poynomial.

• The expression  $|f^{(n+1)}(u)| \frac{|x-a|^{n+1}}{(n+1)!}$ , where u is a number between a and x, is called the Lagrange Form of the Remainder or the Lagrange Error Bound. This is a name that BC students should know. They should also know how to use the Lagrange Error Bound formula to estimate the absolute value of the error of a Taylor polynomial approximation.

#### Lecture Material

Discuss the wish to approximate a nonpolynomial function with a polynomial. Suppose a function has k derivatives. Define the nth Taylor polynomial for f centered at a for n = 1, ..., k as

$$T_n(x) = f(x) + \frac{f'(x)}{1!}(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(x)}{n!}(x-a)^n$$

When a = 0, it is called the *n*th Maclaurin polynomial. Derive the *n*th Maclaurin polynomials for  $f(x) = e^x$ ,  $\sin x$ ,  $\ln x$ , and  $\frac{1}{1-x}$ . These four (only) should be memorized. Do an example of finding a Taylor series centered at some point other than the origin. Exercises 20 and 22 would be good illustrations.

For a given function f(x) and its Taylor polynomial, define the *n*th remainder for f(x) at x = a by  $R_n(x) = f(x) - T_n(x)$ . State and prove Taylor's Theorem that for a function f whose n + 1 derivatives exist,

$$R_n(x) = \frac{1}{n!} \int_a^x (x-u)^n f^{(n+1)}(u) \, du$$

The proof uses Integration by Parts and the Fundamental Theorem of Calculus. State and prove the error bound for the Taylor polynomial. Now use Taylor's Theorem to estimate the size of the error. That is, if K is a number such that  $|f^{(n+1)}(u)| \leq K$  for all u between a and x, then the error bound is  $|T_n(x) - f(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$ . Do Examples 6 and 7. Use a graphing calculator or computer graphing program to reproduce Figure 5 and continue with  $n = 12, 14, 16 \dots$ 

# Discussion Topics/Class Activities

Have students use their graphing calculators to work Exercise 28 in groups.

# Suggested Problems (2–3 assignments)

Exercises 1, 3, 21, 23, 25 (basic), 27 (graphical), 30, 31, 32 (numerical), 33 (graphical), 37 (algebraic)

# Worksheet 8.4. Taylor Polynomials

1. Find the Taylor polynomial  $T_3(x)$  for  $f(x) = \frac{1}{1+x}$  centered at a = 1.

**2.** Find  $T_n(x)$  for  $f(x) = \cos x$  centered at  $x = \frac{\pi}{4}$ .

**3.** Find *n* such that  $|T_n(1) - \sqrt{1.3}| \le 10^{-6}$ , where  $T_n$  is the Taylor polynomial for  $\sqrt{x}$  at a = 1.

- 4. Consider the function  $f(x) = \frac{1}{1-x}$ .
  - a. Find the Maclaurin series for f.
  - b. Find the Maclaurin series for  $g(x) = \frac{1}{1+x^2}$  by substituting  $-x^2$  for x in the Maclaurin series for f found in part (a).

# Solutions to Worksheet 8.4

**1.** Find the Taylor polynomial  $T_3(x)$  for  $f(x) = \frac{1}{1+x}$  centered at a = 1.

$$f(x) = \frac{1}{1+x} \qquad f(a) = 1/2$$
  

$$f'(x) = \frac{-1}{(1+x)^2} \qquad f'(a) = -1/4$$
  

$$f''(x) = \frac{2}{(1+x)^3} \qquad f''(a) = 1/4$$
  

$$f'''(x) = \frac{-6}{(1+x)^4} \qquad f'''(a) = -3/8$$

$$T_2(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8}$$
$$T_3(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{16}$$

**2.** Find  $T_n(x)$  for  $f(x) = \cos x$  centered at  $x = \frac{\pi}{4}$ .

**3.** Find *n* such that  $|T_n(1.3) - \sqrt{1.3}| \le 10^{-6}$ , where  $T_n$  is the Taylor polynomial for  $\sqrt{x}$  at a = 1.

$$f(x) = \frac{1}{x+1} \qquad f(1) = \frac{1}{2}$$

$$f'(x) = \frac{-1}{(x+1)^2} \qquad f'(1) = -\frac{1}{4}$$

$$f''(x) = \frac{2}{(x+1)^3} \qquad f''(1) = \frac{1}{4}$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \qquad f^{(n)}(1) = \frac{(-1)^n n!}{2^{n+1}}$$

$$T_n(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{(x-1)^2}{2 \cdot 4} + \dots + (-1)^n \frac{(x-1)^n}{2^{n+1}}$$

$$|T_n(1.3) - \sqrt{1.3}| \le K \frac{|.3|^{n+1}}{(n+1)!}. \text{ For } f(x) = \sqrt{x}, \ f^{(n)}(x) \text{ is either negative or decreasing}$$

for x > 1, hence the maximum value of  $f^{(n)}(x)$  will happen at x = 1.

$$|f^{(n)}(1)| = \frac{n!}{2^{n+1}} \text{ (for } n \ge 2) \le n! \frac{(.3)^{n+1}}{(n+1)!} < n \frac{3^{n+1}}{10^{n+1}} < 10^{-6} \Rightarrow n \ge 9$$

4. Consider the function f(x) = 1/(1-x).
a. Find the Maclaurin series for f.

$$f(x) = \frac{1}{1-x} \qquad f(0) = 1$$
  

$$f'(x) = \frac{1}{(1-x)^2} \qquad f'(0) = 1$$
  

$$f''(x) = \frac{2}{(1-x)^3} \qquad f''(0) = 2$$
  

$$f'''(x) = \frac{6}{(1-x)^4} \qquad f'''(0) = 6$$
  

$$\vdots \qquad \vdots$$
  

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \qquad f^{(n)}(0) = n!$$

 $T_n(x) = 1 + x + 2x^2/2 + 6x^3/6 + \dots + n!x^n/n! = 1 + x + x^2 + x^3 + \dots + x^n$ 

b. Find the Maclaurin series for  $g(x) = \frac{1}{1+x^2}$  by substituting  $-x^2$  for x in the Maclaurin series for f found in part (a).  $f(x) = \frac{1}{1+x^2} \Rightarrow T(x) = 1 + x + x^2 + \dots + x^n \Rightarrow T(-x^2) = 1 - x^2 + x^4 - x^6 + \dots + (-x^2)^n$ 

$$f(u) = \frac{1}{1-u} \Rightarrow T_n(u) = 1 + u + u^2 + \dots + u^n \Rightarrow T_n(-x^2) = 1 - x^2 + x^4 - x^6 + \dots + (-x^2)^n$$

# Chapter 8 AP Problems

For 4, a calculator may be used. For 1, 2, 3 and 5, no calculator allowed.

- 1. The length of a curve from x = 2 to x = 3 is given by  $\int_{2}^{3} \sqrt{1 + (12x^2 4)^2} dx$ . Which of the following is the equation of the curve if the curve goes through the point (2, 22)?
  - A.  $y = 4x^{3} 4x$ B.  $y = 4x^{3} - 4x + 2$ C.  $y = 4x^{3} - 4x - 2$ D.  $y = 12x^{2} - 4$ E.  $y = 12x^{2} - 94$
- 2. Given the third degree Taylor polynomial  $5 2(x 1) + 3(x 1)^2 6(x 1)^3$  of f(x), what is the value of f''(1)?
  - A. -36
    B. -6
    C. 0
    D. 6
    E. 36

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- **3.** Given f(0) = 2, f'(0) = 3, and f''(0) = 4, which of the following is the second degree Maclaurin polynomial for f(x)?
  - A.  $2 3x + 2x^{2}$ B.  $2 - 3x + 4x^{2}$ C.  $-2 + 3x - 2x^{2}$ D.  $2 + 3x + 2x^{2}$ E.  $2 + 3x + 4x^{2}$
- 4. Given the third degree Taylor polynomial  $5+2(x-3)+\frac{3}{2}(x-3)^2+\frac{5}{4}(x-3)^3$ , what is the Lagrange Error Bound when this polynomial is used to estimate f(3.2) if  $|f^{(4)}(x)| \le 4$  for all x on the closed interval [3, 3.2]?
  - A. 0.0000107
    B. 0.000267
    C. 0.005333
    D. 0.013333
  - E. 0.053333
- **5.** The Maclaurin series for f(x) is given by:

$$\frac{1}{2} - \frac{x^2}{2^3 \cdot 3!} + \frac{x^4}{2^5 \cdot 5!} - \frac{x^6}{2^7 \cdot 7!} + \dots + \frac{(-1)^n \cdot x^{2n}}{2^{2n+1} \cdot (2n+1)!}$$

a. Find f'(0) and f''(0) and explain why f(x) has a local maximum, local minimum, or neither at x = 0.

b. Let g(x) = 2xf(2x). Give the first three nonzero terms and the general term for g(x).

c. Write g(x) as a familiar function and write f(x) in terms of the same function.

# Solutions to Chapter 8 AP Problems

- 1. The length of a curve from x = 2 to x = 3 is given by  $\int_{2}^{3} \sqrt{1 + (12x^2 4)^2} dx$ . Which of the following is the equation of the curve if the curve goes through the point (2, 22)?
  - A.  $y = 4x^3 4x$ B.  $y = 4x^3 - 4x + 2$ C.  $y = 4x^3 - 4x - 2$ D.  $y = 12x^2 - 4$ E.  $y = 12x^2 - 94$ C [THIS PROBLEM CORRESPONDS WITH SECTION 8.1]
- 2. Given the third degree Taylor polynomial  $5 2(x 1) + 3(x 1)^2 6(x 1)^3$  of f(x), what is the value of f''(1)?
  - A. -36 B. -6 C. 0
  - D. 6
  - E. 36

# A [THIS PROBLEM CORRESPONDS WITH SECTION 8.4]

**3.** Given f(0) = 2, f'(0) = 3, and f''(0) = 4, which of the following is the second degree Maclaurin polynomial for f(x)?

A.  $2 - 3x + 2x^2$ 

B.  $2 - 3x + 4x^{2}$ C.  $-2 + 3x - 2x^{2}$ D.  $2 + 3x + 2x^{2}$ E.  $2 + 3x + 4x^{2}$ 

# D [THIS PROBLEM CORRESPONDS WITH SECTION 8.4]

- 4. Given the third degree Taylor polynomial  $5+2(x-3)+\frac{3}{2}(x-3)^2+\frac{5}{4}(x-3)^3$ , what is the Lagrange Error Bound when this polynomial is used to estimate f(3.2) if  $|f^{(4)}(x)| \le 4$  for all x on the closed interval [3, 3.2]?
  - A. 0.0000107
  - B. 0.000267
  - C. 0.005333
  - D. 0.013333

E. 0.053333

# B [THIS PROBLEM CORRESPONDS WITH SECTION 8.4]

**5.** The Maclaurin series for f(x) is given by:

$$\frac{1}{2} - \frac{x^2}{2^3 \cdot 3!} + \frac{x^4}{2^5 \cdot 5!} - \frac{x^6}{2^7 \cdot 7!} + \dots + \frac{(-1)^n \cdot x^{2n}}{2^{2n+1} \cdot (2n+1)!}$$

a. Find f'(0) and f''(0) and explain why f(x) has a local maximum, local minimum, or neither at x = 0.

$$f'(0) = 0; f''(0) = -\frac{1}{24}$$

There is a critical point at x = 0 since f'(0) = 0, and it is a local maximum since f''(0) < 0 (Second Derivative Test).

b. Let g(x) = 2xf(2x). Give the first three nonzero terms and the general term for g(x).

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$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

c. Write g(x) as a familiar function and write f(x) in terms of the same function.

 $g(x) = \sin x$ ;  $f(x) = \frac{\sin(x/2)}{x}$  if  $x \neq 0$ , and f(x) = 1/2 if x = 0[THIS PROBLEM CORRESPONDS WITH SECTION 8.4]

# Ray Cannon's Chapter 9 Overview

Section 9.1 is an overview of what a differential equation is, and emphasizes that a solution is a function, which has a <u>domain</u>. Several questions on the AP examination have demanded exactly this knowledge from students. Section 9.2 presents several applications that lead to the equation y' = k(y-b) and equations of this type have frequently appeared on both AB and BC exams. Refer back to the growth-decay differential equation in section 5.8 at this point. Section 9.3 introduces slope fields and solution curves, which are common topics. This section also treats Eulers method of numerical approximations, which is a BC-only topic. The logistic differential equation is also a BC-only topic, and is covered in Section 9.4 The last section deals with first-order linear equations and integrating factor which are not required topics in either AP course description.

# 9. Introduction to Differential Equations

## 9.1. Solving Differential Equations.

Class Time AB 3 periods; BC 2 periods. Essential.

## **Key Points**

- General and particular solutions of a differential equation.
- Classification of differential equations.
  - (i) The order of a differential equation.
  - (ii) Linear and nonlinear differential equations.
  - (iii) Separable equations
- Initial value problems.

### Lecture Material

Introduce differential equations with examples such as

(1) 
$$y'(x) = 3x^2$$

(2) 
$$(y')^3 + y'' = \sin x$$

(3) 
$$\frac{d^2y}{dx^2} + y = e^x$$

(4) 
$$y''' - y'' + y' - y = t$$

(5) 
$$\frac{dy}{dt} = t^{-2}(y-1)$$

Examples (2), (3) and (4) are types of differential equations that are not tested on either the AB or BC exams. Examples (1) and (5) are separable differential equations, which are tested.

A solution to a differential equation is a function (not a number) that when substituted into the given differential equation with its derivative(s) satisfies the differential equation.

A particular solution of a differential equation has a domain that is the largest open interval containing the initial condition and on which the solution satisfies the differential equation. In other words, the solution cannot cross an asymptote or contain a point where the differential equation or its solution is not defined.

The only method that AB and BC students are expected to know is separation of variables. The general solution includes an arbitrary constant. Evaluating the constant using a given initial condition gives the particular solution — this is called an *initial value problem*.

Separable equations are equations that can be expressed in the form y'(x) = f(x) g(y). Solve Equation (5) over the interval  $0 < t < \infty$  by separating variables and integrating. See Examples 1, 2, and 3, and Exercises 29, 33, 39, and 41.

# Discussion Topics/Class Activities

Exercises 52, 54, 56 and 58 provide nice examples of how separable equations arise in applications.

Suggested Problems (2–3 assignments)

Exercises 5, 9, 10, 11, 15, 16, 18, 19, 21, 23, 27, 31, 33, 41

# Worksheet 9.1. Solving Differential Equations

- 1. Find the general solutions of the following using separation of variables.
  - a.  $\frac{dy}{dt} 2y = 1$

b. 
$$(1+x^2)y' = x^3y$$

2. Solve the initial value problem a.  $\begin{cases} y' + 2y = 0 \\ 0 \end{cases}$ 

$$\int y(\ln 2) = 3$$

b. 
$$\begin{cases} \frac{dy}{dx} = (x-1)(y-2) \\ y(0) = 3 \end{cases}$$





# Solutions to Worksheet 9.1

1. Find the general solutions of the following using separation of variables.

a. 
$$\frac{dy}{dt} - 2y = 1$$
:  $y = -\frac{1}{2} + Ce^{2t}$ .  
b.  $(1 + x^2)y' = x^3y$ :  $y = \frac{Ce^{x^2/2}}{\sqrt{1 + x^2}}$ .

2. Solve the initial value problem

a. 
$$\begin{cases} y' + 2y = 0\\ y(\ln 2) = 3 \end{cases} \quad y = 12e^{-2x}.$$
  
b. 
$$\begin{cases} \frac{dy}{dx} = (x - 1)(y - 2)\\ y(0) = 3 \end{cases} \quad y = 2 + e^{x^2/2 - x}.$$

3. Find the family of curves satisfying y' = x/y and sketch several members of the family.  $y^2 - x^2 = C$ 



## 9.2. Models involving y' = k(y - b).

# Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS.

### **Key Points**

- The general solution of y' = k(y b) is y = b + Ce<sup>kt</sup> where C is a constant.
  Long term behavior of y = b + Ce<sup>kt</sup> and y = b + Ce<sup>-kt</sup> (k > 0).
- Applications.
  - (1) Newton's Law of Cooling.
  - (2) Modeling a free-falling body with air resistance.

### Lecture Material

Derive the general solution  $y = b + Ce^{kt}$  of y'(t) = k(y(t)-b) and sketch graphs of solutions for a few values C (positive and negative) and k (positive and negative). Example 1 applies the differential equation to model how an object cools. Example 2 models the velocity of a free-falling body with air resistance, and Example 3 discusses annuities. Exercise 23 gives an application to electrical circuits using Ohm's Law.

## **Discussion Topics/Class Activities**

Some people believe that, everything else being equal, warmer water placed in an ice cube tray in the freezer will freeze faster than cooler water. Is this fact or fiction?

Suppose that the ambient temperature of the freezer is  $10^{\circ}$  Fahrenheit, and a tray of  $60^{\circ}$  water takes 3 hours to reach  $32^{\circ}$ . (Assume that the freezer is large enough so that the water does not appreciably raise the ambient temperature of the freezer.) According to Newton's law of cooling, the temperature function has form  $y_{60}(t) = 10 + 50e^{-kt}$ . Since  $y_{60}(3) = 32$ , it follows that

$$k = -\frac{1}{3}\ln\left(\frac{22}{50}\right).$$

The temperature of a tray of water with initial temperature  $x > 32^{\circ}$  after t hours in the freezer is therefore  $y_x(t) = 10 + (x - 10)e^{\frac{t}{3}\ln .44}$ , and the time it takes to cool to 32° is

$$T(x) = \frac{3}{\ln .44} \ln \left(\frac{22}{x - 10}\right).$$

Thus  $T'(x) = \frac{-3}{\ln .44}(x-10)^{-2}$ . Since T'(x) > 0 for all x > 32, the time for water at an initial temperature x > 32 to freeze is an increasing function of x. The assertion is false. It is true, however, that warmer water is always cooling faster than cooler water.

#### Suggested Problems

Exercises 1, 2 (basic), 4, 5, 6 (cooling), 11, 13, 14 (free-fall), 15, 17 (annuities), 23 (electrical current)

# Worksheet 9.2. Models Involving y' = k(y - b)

1. Find the general solution of y' = -3(y-2) and graph two solutions satisfying (a) y(0) = 0 (b) y(0) = 4.



- 2. A hot metal bar is submerged in a large reservoir of  $70^{\circ}$  water. After 30 seconds, the temperature of the bar is  $200^{\circ}$ , and its temperature is  $150^{\circ}$  30 seconds later.
  - a. Determine the cooling constant k
  - b. Find the formula for the temperature of the bar t seconds after it is immersed.
  - c. What was the temperature of the bar at the moment it was submerged?

3. Let v(t) denote the velocity (m/s) of a 1 kg object dropped from the top of a tall building t seconds after its release. If air resistance at time t is 4v(t)kg m/s<sup>2</sup>,
a. find the formula for v(t).

- b. What is the terminal velocity of the object?
- 4. What is the minimum initial deposit necessary to fund an annuity for 20 years if with-drawals are made at a rate of 20,000 dollars per year at an annual interest rate of 5%?

## Solutions to Worksheet 9.2

1. Find the general solution of y' = -3(y-2) and graph two solutions satisfying



- 2. A hot metal bar is submerged in a large reservoir of  $70^{\circ}$  water. After 30 seconds, the temperature of the bar is  $200^{\circ}$ , and its temperature is  $150^{\circ}$  30 seconds later.
  - a. Determine the cooling constant k:  $k = \frac{1}{30} \ln \left(\frac{8}{13}\right)$
  - b. Find the formula for the temperature of the bar t seconds after it is immersed.  $T(t) = 70 + \frac{845}{4}e^{kt}$ .
  - c. What was the temperature of the bar at the moment it was submerged?  $T(0) = 281.25^{\circ}$
- **3.** Let v(t) denote the velocity (m/s) of a 1 kg object dropped from the top of a tall building t seconds after its release. If air resistance at time t is  $4v(t)kg m/s^2$ ,
  - a. find the formula for v(t).  $v(t) = \frac{gm}{k}(e^{-4t} 1) = 2.45(e^{-4t} 1) m/s.$
  - b. What is the terminal velocity of the object?  $\lim_{t \to \infty} v(t) = 2.45m/2$ .
- 4. What is the minimum initial deposit necessary to fund an annuity for 20 years if withdrawals are made at a rate of \$20,000 dollars per year at an annual interest rate of 5%?  $0 = P(20) = \frac{20000}{.05} + Ce^{.05*20} \Rightarrow C = -\frac{400000}{e} \approx -147151.78.$  Thus the minimum initial deposit is  $400,000 \frac{e-1}{e} \approx $252,848.$

### 9.3. Graphical and Numerical Methods.

**Class Time** AB 3 periods; BC 2 periods. Very important.

## **Key Points**

- $\dot{y} = \frac{dy}{dt}$   $\dot{y} = F(t, y)$  describes the slope  $\dot{y}(t)$  in terms of the location (t, y(t)).
- The slope field for  $\dot{y} = F(t, y)$ .

#### Lecture Material

Have students draw slope fields by hand to help them understand what they represent. Of course, slope fields are better drawn by computer. Have students also sketch solutions to differential equations on computer-generated print-outs of slope fields, by starting at the initial point and following the slope field in both directions. Both of these sketching methods have been tested on AP calculus exams.

A solution of the differential equation  $\dot{y} = F(t, y)$  is referred to as an integral curve, and the equation  $\dot{y} = F(t, y)$  may be thought of as describing the slope of an integral curve y = y(t) at the point (t, y(t)) in terms of the location (t, y(t)).

A slope field for the equation  $\dot{y} = F(t, y)$  represents the differential equation graphically by sketching a short line segment with slope  $F(t_j, y_k)$  and midpoint  $(t_j, y_k)$ , for an array of points  $(t_j, y_k)$ ,  $1 \leq j \leq n, 1 \leq k \leq m$ . Example 1 discusses the use of isoclines to construct slope fields. Also see Exercises 5 and 6. The graph of an integral curve follows the slope field as in Figure 2.

Euler's Method is a BC topic that is not tested on the AB exam. However, it may be included in AB courses.

Euler's method produces a numerical approximation to an initial value problem

$$y' = F(t, y)$$
  $y(t_0) = y_0$ 

by constructing a sequence of points  $(t_k, y_k)$  recursively. The polygon with vertices  $(t_k, y_k), 0 \le k \le N$ , approximates the graph of the solution y(t).

For a fixed time step h, let  $t_k = t_0 + k h$  and suppose that we have obtained corresponding points  $(t_k, y_k)$  for j = 0, 1, ..., n. Since y'(t) = F(t, y(t)), a linear approximation gives

$$y(t_{n+1}) = y(t_n + h) \approx y(t_n) + y'(t_n)h \approx y_n + F(t_n, y_n)h$$

Thus we define the next point in the sequence by

$$y_{n+1} = F(t_n, y_n)h$$

See Examples 3 and 4 and Exercises 14–19. Computer algebra systems are particularly useful in constructing slope fields and implementing Euler's method.

**Example** The semicircle  $y = \sqrt{1 - x^2}$  solves the initial value problem

$$y' = -x/y$$
  $y(-.9) = \sqrt{0.19}$ 

Use Euler's method with a time step h = 0.01 to construct a numerical approximation and compare the graphs,  $-0.9 \le x \le 0.9$ .

**Solution** Set h = 0.01 and define

$$x_k = -0.9 + kh$$
, and  $y_k = y_{k-1} - x_{k-1}/y_{k-1}h$ 

for  $0 \le k \le 180$ . The graphs are compared on the accompanying slide.

## **Discussion Topics/Class Activities**

Compare graphs of exact solutions of Exercises 14 and 19 with numerical approximations obtained by Euler's methods for various values of the time step h.

## Suggested Problems

Exercises 1, 2, 3, 8, 9, 10, 11 (slope fields and integral curves); BC only: 13, 15, 19 (Euler's method)

# Worksheet 9.3. Graphical and Numerical Methods

1. Use the slope field for  $y' = t + y^2$  below to sketch the graphs of the solutions with initial conditions (a) y(-3) = 1 and (b) y(-3) = -3.



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- **2.** Consider the initial value problem y' = t y, y(0) = 0.
  - a. Verify that  $y(t) = e^{-t} + t 1$  is a solution.
  - b. For a given time step h, let  $t_k = kh$  and let  $y_k$  be the approximation to  $y(t_k)$  given by Euler's method. So the sequence  $y_k$ ,  $0 \le k$  is define recursively by

 $y_0 = 0$  and  $y_k = (t_{k-1} - y_{k-1})h = (k-1)h^2 + y_{k-1}(1-h)$  if  $k \ge 1$ . Show that  $y_1 = 0$  and that for  $k \ge 2$ ,  $y_k = -1 + (1-h)^k + hk$ .

c. Use h = 0.1 and plot the polygon with vertices  $(t_k, y_k)$ , k = 0, 1, 2, ..., 10, on the graph of  $y(t) = e^{-t} + t - 1$  below.



# Solutions to Worksheet 9.3

1. Use the slope field for  $y' = t + y^2$  below to sketch the graphs of the solutions with initial conditions (a) y(-3) = 1 and (b) y(-3) = -3.



- **2.** Consider the initial value problem y' = t y, y(0) = 0.
  - a. Verify that  $y(t) = e^{-t} + t 1$  is a solution. If  $y(t) = e^{-t} + t - 1$ , then  $y'(t) = -e^{-t} + 1 = -e(^{-t} + t - 1) + t = t - y(t)$ .
  - b. For a given time step h, let  $t_k = kh$  and let  $y_k$  be the approximation to  $y(t_k)$  given by Euler's method. So the sequence  $y_k$ ,  $0 \le k$  is defined recursively by

$$y_0 = 0$$
 and  $y_k = (t_{k-1} - y_{k-1})h = (k-1)h^2 + y_{k-1}(1-h)$  if  $k \ge 1$ .

- Show that  $y_1 = 0$  and that for  $k \ge 2$ ,  $y_k = -1 + (1-h)^k + hk$ . If  $y_k = -1 + (1-h)^k + hk$ , then  $y_{k+1} = kh^2 + y_k(1-h) = kh^2 + (-1 + (1-h)^k + hk)(1-h) = -1 + (1-h)^{k+1} + h(k+1)$  and the formula follows by induction.
- c. Use h = 0.1 and plot the polygon with vertices  $(t_k, y_k)$ , k = 0, 1, 2, ..., 10, on the graph of  $y(t) = e^{-t} + t - 1$  below.



### 9.4. The Logistic Equation.

**Class Time** AB 0 periods; BC 1 period. Important.

### **Key Points**

- The logistic equation y' = ky (1 <sup>y</sup>/<sub>A</sub>).
  Long-term behavior of solutions of a logistic equation.

### Lecture Material

Unconstrained populations grow exponentially,  $P(t) = P_0 e^{kt}$ ; see Section 5.8. But in practice, and especially over longer periods of time, population growth is limited by the capacity of the environment and competition for resources. A logistic equation is often used in this case. The general logistic equation is

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right),\tag{*}$$

where k > 0 is the growth rate as in the exponential model (in units time<sup>-1</sup>) and the constant A is the *carrying capacity* of the model. The logistics equation is solved by separating variables and applying a partial fractions decomposition (Section 7.5):

$$kt + \ln C = \int \frac{1}{y\left(1 - \frac{y}{A}\right)} \frac{dy}{dt} dt = \int \left(\frac{A}{y} - \frac{1}{y - A}\right) dy$$
$$= \ln \left|\frac{y}{y - A}\right|.$$

Thus  $\frac{y}{y-A} = Ce^k t$  for some constant C. Solving algebraically for y yields the solution to (\*) given by

$$y = \frac{A}{1 - e^{-kt}/C},$$

and if  $y_0 = y(0)$ , then  $C = \frac{y_0}{y_0 - A}$ .

Equation (\*) also has two steady-state solutions (equilibria), y = A and y = 0. Assume at first that A, k > 0. Then

(1) If  $0 < y_0 < A$  then  $C = \frac{y_0}{y_0 - A} < 0$  and (\*) implies that y'(t) > 0 for all t. Thus  $y(t) = \frac{A}{1 + e^{-kt}/|C|}$  increases to A as  $t \to \infty$  and  $\lim_{t \to -\infty} y(t) = 0$ . In terms of populations, this says that if the initial population  $y_0$  is less than the capacity A, then the population will grow with limiting value A.

- (2) If  $y_0 > A$ , then  $C = \frac{y_0}{y_0 A} > 1$  and  $y(t) = \frac{A}{1 e^{-kt}/C}$  has a vertical asymptote at  $t_0 = -\frac{1}{k} \ln C < 0$ ; in particular, y(t) is defined for all  $t \ge 0$ . (\*) implies that y'(t) < 0, and so  $y(t) = \frac{A}{1 - e^{-kt}/C}$  decreases to A while  $\lim_{t \to -\infty} y(t) = 0$ . Thus a logistic model of a population predicts that in the case that the initial population exceeds the capacity, then the population y(t) will decline to A in the limit.
- (3) If  $y_0 < 0$  (which does not occur in population models), then 0 < C < 1, and  $y(t) = \frac{A}{1 e^{-kt}/C}$  has a vertical asymptote at  $t_0 = -\frac{1}{k} \ln C$ , positive. In this case, y(t) decreases to  $-\infty$  as  $t \to t_0 +$ ,  $\lim_{t \to t_0 -} y(t) = \infty$  and y(t) decreases to A on the interval  $(t_0, \infty)$  as  $t \to \infty$ .

Thus the steady-state solution y = A is a stable equilibrium in the sense that every non-constant solution approaches it for large values of t, while y = 0 is unstable since all other solutions diverge away from it as  $t \to \infty$ .

Asymptotic behavior of solutions is illustrated in the accompanying slide. Examples 1 and 2 are basic. Also see Exercises 2 and 3. Exercises 8 and 9 model the spread of a rumor (or of a disease) through a population.

The case that k < 0 is considered in Exercise 12 with specific examples given in Exercises 11 and 1. When k < 0 and  $0 < A < y_0$ , the solution y(t) has a vertical asymptote at  $t_0 = -1/k \ln\left(\frac{y_0}{y_0 - A}\right) > 0$ .

The AP Calculus BC exams have not required students to actually solve a logistic differential equation. Students are expected to know the shape and behavior of the solution:

- The horizontal asymptotes are when  $\frac{dy}{dx} = 0$ . That is, at y = 0 and y = A (Figure 2).
- The population is increasing fastest at  $y = \frac{1}{2}A$ . This is the location of the point of inflection of the solution.

## **Discussion Topics/Class Activities**

Exercise 12 outlines an analysis of population dynamics modeled by a logistic equation with negative growth rate (k < 0).

## Suggested Problems

Exercises 3, 5, 7, 8, 9 (routine), 13 (abstract), 15 (CAS)

# Worksheet 9.4. The Logistic Equation

1. Find the solutions of  $y' = \frac{1}{2}y(1-\frac{y}{4})$  with initial values (a) y(0) = 1 and (b) y(0) = 6. and (c) y(0) = -1. Sketch their graphs on the slope field below and discuss their asymptotic behavior.



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**2.** Find the solutions of  $y' = -\frac{1}{3}y(3-y)$  with initial values (a) y(0) = 2 and (b) y(0) = -2. and (c) y(0) = 3.5. Sketch their graphs on the slope field below and discuss their asymptotic behavior.



## Solutions to Worksheet 9.4

1. Find the solutions of  $y' = \frac{1}{2}y\left(1 - \frac{y}{4}\right)$  with initial values

a. 
$$y(0) = 1$$
:  $y = \frac{4e^{t/2}}{3 + e^{t/2}}, y \nearrow 4$  as  $t \to \infty$ .  
b.  $y(0) = 6$ :  $y = \frac{12e^{t/2}}{-1 + 3e^{t/2}}, y \searrow 4$  as  $t \to \infty$ . and  
 $4e^{t/2}$ 

c. y(0) = -1:  $y = \frac{4e^{t/2}}{-5 + e^{t/2}}$  has a vertical asymptote at  $t = 2 \ln 5 \approx 3.22$ .



2. Find the solutions of  $y' = -\frac{1}{3}y(3-y)$  with initial values a. y(0) = 2:  $y = \frac{6}{2+e^t}$ ,  $t \searrow 0$  as  $t \to \infty$ . b. y(0) = -2:  $y = \frac{6}{2-5e^t}$ ,  $t \nearrow 0$  as  $t \to \infty$ , and c. y(0) = 3.5:  $y = \frac{21}{e^t - 7}$ , has a vertical asymptote at  $t = \ln 7 \approx 1.95$ .



#### 9.5. First-Order Linear Equations.

# Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS.

## **Key Points**

- $a(x)y' + b(x)y = c(x), a(x) \neq 0.$
- Integrating factors.

#### Lecture Material

Suppose that a(x), b(x) and c(x) are continuous functions on an interval I with  $a(x) \neq 0$  for all x in I. Then we may divide through by a(x) to obtain an equivalent equation

$$y' + A(x)y = B(x) \tag{(*)}$$

where A(x) = b(x)/a(x) and B(x) = c(x)/a(x).

The solution of (\*) is obtained by again multiplying through by a non-vanishing function  $\alpha(x)$  so that

$$\alpha(x)y'(x) + \alpha(x)A(x)y(x) = (\alpha(x)y(x))'.$$

By the product rule, it follows that  $\alpha'(x) = A(x)\alpha(x)$ , and we solve this equation for  $\alpha$  by separating variables:

$$\int \frac{1}{\alpha(x)} \alpha'(x) \, dx = \int A(x) \, dx$$
$$\alpha(x) = e^{\int A(x) \, dx}.$$
(\*\*)

or

We obtain the general solution of (\*) to be

$$y(x) = \frac{1}{\alpha(x)} \left( \int \alpha(x) B(x) dx + C \right).$$

In practice, the integrals involved may be impossible to evaluate in closed form and so numerical methods must be used.

Examples 1 and 2 are routine, as are Exercises 5–18 (finding general solutions) and Exercises 19–25 (initial value problems). Example 3 and Exercises 28–30 present an important application–mixing solutions. Exercises 33–35 show how such equations arise in electrical circuits.

### **Discussion Topics/Class Activities**

Exercises 36 and 37 are good applications that show how systems of differential equations naturally arise.

### Suggested Problems

Exercises 5, 8, 14 (general solutions), 19, 21, 23, 25 (initial value problems), 28, 29 (mixing), 33 (electrical circuit)
# Worksheet 9.5. First-Order Linear Equations

1. Solve the initial value problem  $y' = \cos x - y \tan x$ ,  $-\pi/2 < x < \pi/2$  with y(0) = 0. Sketch the graph on the slope field below.



2. A stream feeds into a lake at a rate of 1000 m<sup>3</sup> per day. Assume that the stream is polluted with a toxin whose concentration is 5 grams/m<sup>3</sup>. Assume further that the lake has volume  $10^6$  m<sup>3</sup> and that water flows out of the lake at the same rate of 1000 m<sup>3</sup> per day. Set up and solve a differential equation for the concentration c(t) of toxin in the lake t after it begins to flow into the lake. Assume that c(0) = 0 and, for simplicity, that the toxin is continuously mixed in the lake. What is the limiting concentration as t gets large?

## Solutions to Worksheet 9.5

1. Solve the initial value problem  $y' = \cos x - y \tan x$ ,  $-\pi/2 < x < \pi/2$  with y(0) = 0. Sketch the graph on the slope field below.



2. A stream feeds into a lake at a rate of 1000 m<sup>3</sup> per day. Assume that the stream is polluted with a toxin whose concentration is 5 grams/m<sup>3</sup>. Assume further that the lake has volume  $10^6$  m<sup>3</sup> and that water flows out of the lake at the same rate of 1000 m<sup>3</sup> per day. Set up and solve a differential equation for the concentration c(t) of toxin in the lake t after it begins to flow into the lake. Assume that c(0) = 0 and, for simplicity, that the toxin is continuously mixed in the lake. What is the limiting concentration as t gets large?

If y(t) and c(t) denote respectively the amount (grams) and concentration (g/m<sup>3</sup>) of toxin t days after it begins to enter the lake, then  $c(t) = y(t)/10^6$  and  $y'(t) = 5 * 10^3 - c(t) * 10^3$  or  $c'(t) = 5 * 10^{-3} - 10^{-3}c(t)$  with c(0) = 0. Thus  $c(t) = 5(1 - e^{-10^{-3}t})$  and  $c(t) \to 5$  as  $t \to \infty$ .

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# Chapter 9 AP Problems

- 1. Let y = f(x) be the solution to the differential equation  $\frac{dy}{dx} = \frac{x}{y^2}$  with initial condition f(0) = 1. Then f(x) =
  - A. 2xB.  $\sqrt[3]{3x^2 + \frac{1}{3}}$ C.  $\sqrt[3]{3x + 1}$ D.  $\sqrt[3]{3x^2 + 1}$ E.  $\sqrt[3]{\frac{3}{2}x^2 + 1}$
- 2. Below is a slope field for which of the following differential equations?



A. 
$$\frac{dy}{dx} = \frac{x}{y}$$
  
B. 
$$\frac{dy}{dx} = \frac{y}{x}$$
  
C. 
$$\frac{dy}{dx} = -\frac{x}{y}$$
  
D. 
$$\frac{dy}{dx} = -\frac{y}{x}$$
  
E. 
$$\frac{dy}{dx} = -\frac{x^2}{y}$$

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**3.** The logistic equation  $P(t) = \frac{400}{1 - 2e^{-0.4t}}$  represents the population of squirrels in a 2,000 acre forest. The population is growing fastest when the population is

A. 50

- B. 100
- C. 200
- D. 300
- E. 400

# Solutions to Chapter 9 AP Problems

- 1. Let y = f(x) be the solution to the differential equation  $\frac{dy}{dx} = \frac{x}{y^2}$  with initial condition f(0) = 1. Then f(x) =
  - A. 2x
  - B.  $\sqrt[3]{3x^2 + \frac{1}{3}}$ C.  $\sqrt[3]{3x + 1}$
  - D.  $\sqrt[3]{3x^2+1}$

E. 
$$\sqrt[3]{\frac{3}{2}x^2 + 1}$$

# E [THIS PROBLEM CORRESPONDS WITH SECTION 9.1]

2. Below is a slope field for which of the following differential equations?



A. 
$$\frac{dy}{dx} = \frac{x}{y}$$
  
B. 
$$\frac{dy}{dx} = \frac{y}{x}$$
  
C. 
$$\frac{dy}{dx} = -\frac{x}{y}$$
  
D. 
$$\frac{dy}{dx} = -\frac{y}{x}$$
  
E. 
$$\frac{dy}{dx} = -\frac{x^2}{y}$$

C [THIS PROBLEM CORRESPONDS WITH SECTION 9.3]

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- **3.** The logistic equation  $P(t) = \frac{400}{1 2e^{-0.4t}}$  represents the population of squirrels in a 2,000 acre forest. The population is growing fastest when the population is
  - A. 50
  - B. 100
  - C. 200
  - D. 300
  - E. 400

# C [THIS PROBLEM CORRESPONDS WITH SECTION 9.4]

## Ray Cannon's Chapter 10 Overview

Chapter 10 covers a major topic that is solely in the BC course description and is a very difficult topic to master, namely, infinite series. Be sure that plenty of time is allocated for this chapter; it cannot be rushed through. The first section deals with sequences, and what it means for a sequence to converge. Section 10.2 starts the difficult subject of convergence of infinite series with attention to two special kinds of series: telescoping and geometric. Section 10.3 deals with series with positive terms, obtaining the integral test, thus the p-series test, and both the direct comparison test and the limit comparison test.

The next section treats the alternating series test with the corresponding error bound. Section 10.5 has both the ratio test, which is required, and the root test, which is optional. Now the student is prepared for Section 10.6 dealing with power series, the radius of convergence, and the interval of convergence. This section also includes the topic of using power series to solve differential equations; while this topic is not specifically mentioned in the course description, it does give students practice in formal manipulation of series. The last section of this chapter, 10.7, deals with Taylor series in general, and Maclaurin series in particular, and provides more practice in producing new Taylor series by formal manipulation of known series. There is a wealth of material in this chapter and students seldom feel comfortable with the topic of infinite series going into the exam. "The series question" is often the one with the lowest mean; if a student does well with series, almost assuredly the student will get a good score on the exam.

#### 10.1. Sequences.

**Class Time** AB 0 periods; BC 2 periods. Essential.

#### **Key Points**

- A sequence is a function f whose domain is a subset of the integers. We write  $a_n = f(n)$  for the  $n^{th}$  term and denote the sequence by  $\{a_n\}$  or just  $a_n$ .
- We say  $a_n$  approaches a limit L, and write  $\lim_{n\to\infty} a_n = L$  or  $a_n \to L$  if for every  $\epsilon > 0$ , there exists a number  $N_0$  such that

$$|a_n - L| < \epsilon$$
 for all  $n > N_0$ 

If no such limit exists, we say that  $a_n$  diverges.

- If  $a_n = f(n)$  is defined by f(x) and  $\lim_{x \to \infty} f(x) = L$ , then  $\lim_{n \to \infty} a_n = L$  A geometric sequence is a sequence of the form  $a_n = cr^n$ , where c and r are constants.
- The Basic Limit Laws and the Squeeze Theorem apply to sequences.
- If f is a continuous function and lim<sub>n→∞</sub> a<sub>n</sub> = L, then lim<sub>n→∞</sub> f(a<sub>n</sub>) = f(L).
  The sequence a<sub>n</sub> is bounded above by M if a<sub>n</sub> ≤ M for every n and bounded below by M if  $a_n \leq M$  for every n. Also,  $a_n$  is bounded if it is bounded above and below.
- A sequence  $a_n$  is monotonic if  $a_n \leq a_{n+1}$  or  $a_n \geq a_{n+1}$ . Thus a sequence is monotonic if it is term by term increasing or decreasing.
- Theorem: Every bounded monotonic sequence converges.

### Lecture Material

A sequence is a function f(n) whose domain is a subset of integers. The value  $a_n = f(n)$ is a term and n is the index. We usually write  $\{a_n\}$  or simply  $a_n$  to represent a sequence. Work Exercises 2 and 4 to ensure that students understand the basic definitions. A sequence  $a_n$  converges to a limit L, written  $\lim a_n = L$  or  $a_n \to L$  if, for every  $\epsilon > 0$ , there is a number M such that  $|a_n - L| < \epsilon$  for all n > M. Figure 2 may help illustrate the formal definition of this limit. If a sequence does not converge, we say that it diverges. A useful tool for determining if sequences converge is the following result (Theorem 2): If  $\lim_{x \to \infty} f(x)$  exists, then  $a_n = f(n)$  converges to the same limit:  $\lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x)$ .  $n{
ightarrow}\infty$ Work Exercises 16 and 18 to illustrate the use of this result. A geometric sequence is a sequence of the form  $a_n = cr^n$ , where c and r are constants - r is the common ratio. Show that

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } 0 < r < 1\\ 1 & \text{if } r = 1\\ \text{diverges to } \infty & \text{if } r > 1 \end{cases}$$

and work Exercise 20. Now point out that the Basic Limit Laws, as well as the Squeeze Theorem, hold for sequences. This then implies the following result (Theorem 4): If f(x) is continuous and  $\lim_{n\to\infty} a_n = L$  exists, then  $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$ . Work Exercises 28, 35 and 68 to illustrate these results.

A sequence  $a_n$  is bounded above if there is a number M such that  $a_n \leq M$  for all n, and M is an upper bound. Bounded below and lower bounds are defined analogously. We say that  $a_n$  is bounded if it is both bounded above and below, and unbounded otherwise. If a sequence is convergent, it is bounded (Theorem 5), and if it is monotonically increasing and bounded above or monotonically decreasing and bounded below, then it is convergent, and converges to a number less than (greater than) or equal to the upper (lower) bound (Theorem 6). Work Exercise 12 to illustrate these results.

#### Discussion Topics/Class activities

Work Exercise 87, which gives a mean different from the typical arithmetic mean. Also discuss Exercise 80 to illustrate its use.

#### Suggested Problems

Exercises 3, 5, 7, 15, 17, 27, 35, 37, 67

# Worksheet 10.1. Sequences

1. Calculate the first four terms of the sequence  $b_n = \cos \pi n$ , starting with n = 1.

**2.** Calculate the first four terms of the sequence  $b_n = 2 + (-1)^n$ , starting with n = 1.

**3.** Use Theorem 2 to determine the limit of the sequence  $b_n = \frac{3n+1}{2n+4}$  or state that the sequence diverges.

4. Use Theorem 2 to determine the limit of the sequence  $c_n = 4(2^n)$  or state that the sequence diverges.

5. Determine the limit of the sequence  $y_n = \frac{e^n}{2^n}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

6. Determine the limit of the sequence  $a_n = \frac{\sqrt{n}}{\sqrt{n+4}}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

7. Determine the limit of the sequence  $b_n = e^{n^2 - n}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

8. Determine the limit of the sequence  $b_n = \frac{3-4^n}{2+7\cdot 4^n}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

**9.** Show that  $a_n = \frac{3n^2}{n^2 + 2}$  is strictly increasing. Find an upper bound.

## Solutions to Worksheet 10.1

1. Calculate the first four terms of the sequence  $b_n = \cos \pi n$ , starting with n = 1. Setting n = 1, 2, 3, 4 into the formula for  $b_n$  gives:

> $b_1 = \cos \pi \cdot 1 = \cos \pi = -1$   $b_2 = \cos \pi \cdot 2 = \cos 2\pi = 1$   $b_3 = \cos \pi \cdot 3 = \cos 3\pi = -1$  $b_4 = \cos \pi \cdot 4 = \cos 4\pi = 1$

2. Calculate the first four terms of the sequence  $b_n = 2 + (-1)^n$ , starting with n = 1. The first four terms of  $\{b_n\}$  are obtained by setting n = 1, 2, 3, 4 in the formula for  $b_n$ :

> $b_1 = 2 + (-1)^1 = 2 - 1 = 1$   $b_2 = 2 + (-1)^2 = 2 + 1 = 3$   $b_3 = 2 + (-1)^3 = 2 - 1 = 1$  $b_4 = 2 + (-1)^4 = 2 + 1 = 3$

**3.** Use Theorem 2 to determine the limit of the sequence  $b_n = \frac{3n+1}{2n+4}$  or state that the sequence diverges.

Using Theorem 2 and the asymptotic behavior of rational functions we get:

$$\lim_{n \to \infty} \frac{3n+1}{2n+4} = \lim_{x \to \infty} \frac{3x+1}{2x+4} = \frac{3}{2}$$

4. Use Theorem 2 to determine the limit of the sequence  $c_n = 4(2^n)$  or state that the sequence diverges.

By Theorem 2,

$$\lim_{n \to \infty} 4 \cdot 2^n = \lim_{x \to \infty} 4 \cdot 2^x = \infty.$$

Thus, the sequence  $4(2^n)$  diverges.

5. Determine the limit of the sequence  $y_n = \frac{e^n}{2^n}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

 $\frac{e^n}{2^n} = \left(\frac{e}{2}\right)^n$  and  $\frac{e}{2} > 1$ . By the Limit of Geometric Sequences, proved in Example 4, we conclude that  $\lim_{n \to \infty} \left(\frac{e}{2}\right)^n = \infty$ . Thus, the given sequence diverges to  $\infty$ .

6. Determine the limit of the sequence  $a_n = \frac{\sqrt{n}}{\sqrt{n+4}}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

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We compute the limit using Theorem 2:

$$\lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+4}} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x+4}} = \lim_{x \to \infty} \frac{\frac{\sqrt{x}}{\sqrt{x}}}{\frac{\sqrt{x}}{\sqrt{x}} + \frac{4}{\sqrt{x}}} = \lim_{x \to \infty} \frac{1}{1 + \frac{4}{\sqrt{x}}} = \frac{1}{1+0} = 1$$

7. Determine the limit of the sequence  $b_n = e^{n^2 - n}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

Since  $\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 \left(1 - \frac{1}{x}\right) = \infty$ , and since the exponential function  $e^x$  is increasing, we have  $\lim_{x \to \infty} e^{x^2 - x} = \infty$ . Then, by Theorem 2 also  $\lim_{n \to \infty} e^{n^2 - n} = \infty$ , that is, the sequence  $e^{n^2 - n}$  diverges.

8. Determine the limit of the sequence  $b_n = \frac{3-4^n}{2+7\cdot 4^n}$  or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

Dividing the numerator and denominator by  $4^n$  yields:

$$a_n = \frac{3-4^n}{2+7\cdot 4^n} = \frac{\frac{3}{4^n} - \frac{4^n}{4^n}}{\frac{2}{4^n} + \frac{7\cdot 4^n}{4^n}} = \frac{\frac{3}{4^n} - 1}{\frac{2}{4^n} + 7}$$

We now compute the limit using Limit Laws for Sequences and the limit of the geometric sequence (see Example 4)  $\lim_{n\to\infty} \frac{1}{4^n} = \lim_{n\to\infty} \left(\frac{1}{4}\right)^n = 0$ . We obtain:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\frac{3}{4^n} - 1}{\frac{2}{4^n} + 7} = \frac{\lim_{n \to \infty} \left(\frac{3}{4^n} - 1\right)}{\lim_{n \to \infty} \left(\frac{2}{4^n} + 7\right)} = \frac{3\lim_{n \to \infty} \frac{1}{4^n} - \lim_{n \to \infty} 1}{2\lim_{n \to \infty} \frac{1}{4^n} - \lim_{n \to \infty} 7}$$
$$= \frac{3 \cdot 0 - 1}{2 \cdot 0 + 7} = -\frac{1}{7}$$

9. Show that  $a_n = \frac{3n^2}{n^2 + 2}$  is strictly increasing. Find an upper bound.

We consider the function  $f(x) = \frac{3x^2}{x^2 + 2}$ . Differentiating f yields:

$$f'(x) = \frac{6x(x^2+2) - 3x^2 \cdot 2x}{(x^2+2)^2} = \frac{12x}{(x^2+2)^2}$$

f'(x) > 0 for x > 0, hence f is strictly increasing on this interval. It follows that  $a_n = f(n)$  is also strictly increasing. Next, we show that M = 3 is an upper bound for  $a_n$  by writing

$$a_n = \frac{3n^2}{n^2 + 2} \le \frac{3n^2 + 6}{n^2 + 2} = \frac{3(n^2 + 2)}{n^2 + 2} = 3.$$

That is,  $a_n \leq 3$  for all n.

Class Time AB 0 periods; BC 2 periods. Essential.

#### **Key Points**

• An infinite series is an expression of the form

$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

We call  $a_n$  the general term of the series.

• The  $N^{th}$  partial sum of an infinite series S is the finite sum:

$$S_N = \sum_{n=1}^N a_n = a_1 + a_2 + \dots + a_N$$

If  $\lim_{N\to\infty} S_N$  exists, we say that S is convergent and  $S = \lim_{N\to\infty} S_N$ . Otherwise, S is divergent.

- A Divergence Test: If  $a_n$  does not approach 0, then  $\sum_{n=1}^{n} a_n$  diverges. A series though, may diverge even if  $a_n$  approaches 0.
- A Geometric series with common ratio r satisfying |r| < 1 converges:

$$\sum_{n=M}^{\infty} cr^N = cr^M + cr^{M+1} + \dots = \frac{cr^M}{1-r} = \frac{\text{first term}}{1-r}$$

A geometric series diverges if the common ratio  $r \geq 1$ .

#### Lecture Material

An infinite series is a sum of the form  $\sum_{n=1}^{\infty} a_n$ . The  $n^{th}$  partial sum is defined as  $S_N =$ 

 $\sum_{n=1}^{N} a_n$ , and an infinite series is said to converge to S if  $\lim_{N \to \infty} S_N = S$ , and is said to diverge otherwise. Work Exercise 4 to ensure that students are familiar with these basic concepts. Prove the Divergence Test (Theorem 3), which states that if  $a_n$  does not converge to 0, then  $S = \sum_{n=1}^{\infty} a_n$  diverges. Work Exercises 11 (a telescopic series) and 16. Now show the students the trick to computing partial sums of geometric series to show that Theorem 1

holds: A geometric series with common ratio r converges if |r| < 1 and diverges if  $|r| \ge 1$ . Furthermore:

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r} \quad \text{and} \quad \sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}, \qquad |r| < 1$$

Also point out now that infinite series can be added, subtract, and multiplied by a constant in the usual fashion provided that the series all converge (Theorem 2). Now work Exercises 24, 32, and 38.

### Discussion Topics/Class activities

Work Exercise 51, which gives a clever way of determining the derivative of  $f(x) = x^N$  that does not use the Binomial Theorem (54 and 55 are also interesting).

#### Suggested Problems

Exercises 1–5 odd (computational), 9, 11 (computational), 15, 17, 21, 23, 27, 31, 37, 41, 45 (computational)

# Worksheet 10.2. Infinite Series

1. Compute the partial sums  $S_2$ ,  $S_4$ , and  $S_6$  of  $\sum_{k=1}^{\infty} (-1)^k k^{-1}$ .

2. Calculate  $S_3$ ,  $S_4$ , and  $S_5$ , and then find the sum of  $S = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$  using the identity  $\frac{1}{4n^2 - 1} = \frac{1}{2} \left( \frac{1}{2n - 1} - \frac{1}{2n + 1} \right)$ 

**3.** Use Theorem 3 to prove that the series  $\cos \frac{1}{2} + \cos \frac{1}{3} + \cos \frac{1}{4} + \cdots$  diverges.

4. Use the formula for the sum of a geometric series to find the sum  $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$  or state that the series diverges.

5. Use the formula for the sum of a geometric series to find the sum  $\sum_{i=0}^{\infty} \frac{7 \cdot 3^n}{11^n}$  or state that the series diverges.

6. Use the formula for the sum of a geometric series to find the sum  $\sum_{i=0}^{\infty} \frac{8+2^n}{5^n}$  or state that the series diverges.

# Solutions to Worksheet 10.2

1. Compute the partial sums  $S_2$ ,  $S_4$ , and  $S_6$  of  $\sum_{k=1}^{\infty} (-1)^k k^{-1}$ .  $S_2 = (-1)^1 \cdot 1^{-1} + (-1)^2 \cdot 2^{-1} = -1 + \frac{1}{2} = -\frac{1}{2}$   $S_4 = (-1)^1 \cdot 1^{-1} + (-1)^2 \cdot 2^{-1} + (-1)^3 \cdot 3^{-1} + (-1)^4 \cdot 4^{-1}$   $= S_2 - \frac{1}{3} + \frac{1}{4} = -\frac{1}{2} - \frac{1}{3} + \frac{1}{4} = -\frac{7}{12}$   $S_6 = S_4 + a_5 + a_6 = -\frac{7}{12} + (-1)^5 \cdot 5^{-1} + (-1)^6 \cdot 6^{-1}$  $= -\frac{7}{12} - \frac{1}{5} + \frac{1}{6} = -\frac{37}{60}$ 

**2.** Calculate  $S_3$ ,  $S_4$ , and  $S_5$ , and then find the sum of  $S = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$  using the identity

$$\frac{1}{4n^2 - 1} = \frac{1}{2} \left( \frac{1}{2n - 1} - \frac{1}{2n + 1} \right)$$
$$S_N = \sum_{n=1}^N \frac{1}{2} \left( \frac{1}{2n - 1} - \frac{1}{2n + 1} \right) = \frac{1}{2} - \frac{1}{2(2N + 1)}.$$
 Thus  $S_3 = \frac{3}{7}, S_4 = \frac{4}{9}, S_5 = \frac{5}{11}$  and  $S = \lim_{N \to \infty} S_N = \frac{1}{2}.$ 

**3.** Use Theorem 3 to prove that the series  $\cos \frac{1}{2} + \cos \frac{1}{3} + \cos \frac{1}{4} + \cdots$  diverges.

The general term is  $a_n = \cos \frac{1}{n+1}$ . Since  $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \cos \frac{1}{n+1} = \cos \left( \lim_{n \to \infty} \frac{1}{n+1} \right) = \cos 0 = 1$ , the general term does not converge to zero. Thus, Theorem 3 implies that the series diverges.

4. Use the formula for the sum of a geometric series to find the sum  $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$  or state that the series diverges.

In summation notation we have:

$$1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots = \sum_{\substack{n=0\\1}}^{\infty} 1 \cdot \left(\frac{1}{5}\right)$$

The common ratio is  $r = \frac{1}{5}$ , so |r| < 1. By the Theorem on the Sum of Geometric series,

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the series converges to the following sum:

$$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1 - \frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

5. Use the formula for the sum of a geometric series to find the sum  $\sum_{i=0}^{\infty} \frac{7 \cdot 3^n}{11^n}$  or state that

the series diverges.

The series  $\sum_{n=0}^{\infty} \frac{7 \cdot 3^n}{11^n} = \sum_{n=0}^{\infty} 7\left(\frac{3}{11}\right)^n$  is a geometric series with the common ratio  $r = \frac{3}{11}$ 

. Since |r| < 1, the series converges. We find the sum by setting a = 7,  $r = \frac{3}{11}$  in the formula of Theorem 3. This gives:

$$\sum_{n=0}^{\infty} \left(\frac{3}{11}\right)^n = \frac{7}{1-\frac{3}{11}} = \frac{7}{\frac{8}{11}} = \frac{77}{8} = 9\frac{5}{8}$$

6. Use the formula for the sum of a geometric series to find the sum  $\sum_{n=0}^{\infty} \frac{8+2^n}{5^n}$  or state that the series diverges. We rewrite the series as the sum of the following two geometric series:

$$\sum_{n=0}^{\infty} \frac{8+2^n}{5^n} = \sum_{n=0}^{\infty} \frac{8}{5^n} + \sum_{n=0}^{\infty} \frac{2^n}{5^n} = \sum_{n=0}^{\infty} 8 \cdot \left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n \quad (1)$$

The common ratios of the series are  $r = \frac{1}{5}$  and  $r = \frac{2}{5}$  respectively, hence by the Theorem on the sum of a geometric series, the series converges. For the first series, a = 8 and  $r = \frac{1}{5}$  and for the second a = 1 and  $r = \frac{2}{5}$ . Hence the series converges to the following sums:  $\sum_{n=1}^{\infty} (1)^n = 8 = 8$ 

$$\sum_{n=0}^{\infty} 8 \cdot \left(\frac{1}{5}\right)^n = \frac{8}{1-\frac{1}{5}} = \frac{8}{\frac{4}{5}} = 10$$
$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{3}{5}} = \frac{5}{3} = 1\frac{2}{3}$$
Substituting these values in (1), we derive:
$$\sum_{n=0}^{\infty} \frac{8+2^n}{5^n} = 10 + 1\frac{2}{3} = 11\frac{2}{3}$$

**Class Time** AB 0 periods; BC 2 periods. Essential. **Key Points** 

- The partial sums  $S_N$  of a positive series  $S = \sum_{n=1}^{\infty} a_n$  form an increasing sequence.
- Dichotomy Theorem: a positive series  $S = \sum_{n=1}^{\infty} a_n$  converges if its partial sums  $S_N$  are bounded. Otherwise, S diverges to infinity.
- The Integral Test: If f is positive, decreasing, and continuous, then  $S = \sum_{n=1}^{\infty} f(n)$  converges (diverges) if  $\int_{M}^{\infty} f(x) dx$  converges (diverges) for some M > 0.
- *p*-Series: The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if p > 1 and diverges otherwise.
- The Comparison Test: Let  $0 \le a_n \le b_n$ . If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  also converges. If  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  also diverges.

• Limit Comparison Test: Let 
$$a_n, b_n \ge 0$$
 and assume that  $\lim_{n \to \infty} \frac{a_n}{b_n}$  exists and is non-zero. Then  $\sum_{n=1}^{\infty} b_n$  converges if and only if  $\sum_{n=1}^{\infty} a_n$  converges.

### Lecture Material

All of the series  $\sum_{n=1}^{\infty} a_n$  considered in the section are positive series. That is,  $a_n \ge 0$  for all n. In terms of convergence, there are only two possible ways that a series can behave (the Dichotomy Theorem 1). Let  $S = \sum_{n=1}^{\infty} a_n$  be a positive series. If the partial sums  $S_n$  are bounded above, then the S converges. If the partial sums  $S_n$  are not bounded above, then S diverges to infinity. We now consider convergence tests, one of the most important of which is the Integral Test (Theorem 2). Let  $a_n = f(n)$  where f(x) is positive, decreasing, and continuous for  $x \ge 1$ . If  $\int_1^{\infty} f(x) dx$  converges, then  $\sum_{i=1}^{\infty} a_n$  converges. Also, if  $\int_{1}^{\infty} f(x)dx$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges. The Integral Test can be justified graphically as in Figures 2 and 3. Work Exercise 10 to illustrate the use of the Integral Test. Work Example 1 and discuss the harmonic series. Now point out that the Integral Test can be used to determine the convergence of *p*-series, and obtain the following result (Theorem 3): The infinite series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if p > 1 and diverges otherwise. Next, state the Comparison Test: Assume that there exists M > 0 such that  $a_n \ge b_n \ge 0$  for all  $n \ge M$ . If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} b_n$  converges, while if  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  also diverges. Work Exercises 20 and 28 to illustrate the use of the Comparison Test. The final convergence test of this section is the Limit Comparison Test which states that if  $a_n$  and  $b_n$  are *positive* sequences such that  $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} b_n$  converges. Work Exercises 40 and 44 to illustrate the use of the Limit Comparison Test.

Discussion Topics/Class activities Solve Exercise 89.

#### Selected Problems

Exercises 1–31 every other odd, 39, 43, 47, 49–77 (mixed review of all methods)

# Worksheet 10.3. Convergence of Series with Positive Terms

1. Use the Integral Test to determine if the infinite series  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  is convergent.

2. Use the Comparison Test to determine if the infinite series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2^n}}$  is convergent.

3. Use the Comparison Test to determine if the infinite series  $\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$  is convergent.

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4. Use the Limit Comparison Test to determine the convergence or divergence of the infinite series  $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$ .

5. Use the Limit Comparison Test to determine the convergence or divergence of the infinite series  $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3 - 1}}$ .

### Solutions to Worksheet 10.3

1. Use the Integral Test to determine if the infinite series  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  is convergent.

We use the Integral Test with  $f(x) = \frac{1}{x(\ln x)^2}$ . f is positive and continuous for  $x \ge 2$ . We show that f is decreasing in this interval. Differentiating f gives:

$$f'(x) = -\frac{1}{(x(\ln x)^2)^2} (x(\ln x)^2)'$$
  
=  $-\frac{1}{x^2(\ln x)^4} \left( 1 \cdot (\ln x)^2 + x \cdot 2(\ln x) \cdot \frac{1}{x} \right)$   
=  $-\frac{1}{x^2(\ln x)^4} ((\ln x)^2 + 2\ln x)$ 

Since  $\ln x > 0$  for x > 1, f'(x) is negative for x > 1, hence f is decreasing for  $x \ge 2$ . We compute the improper integral  $\int_{2}^{\infty} \frac{1}{x(\ln x)^2} dx$  making the substitution  $u = \ln x$ ,  $du = \frac{1}{x} dx$ . We obtain:

$$\int_{2}^{\infty} \frac{1}{x(\ln x)^{2}} dx = \lim_{R \to \infty} \int_{2}^{R} \frac{1}{x(\ln x)^{2}} dx = \lim_{R \to \infty} \int_{\ln 2}^{\ln R} \frac{1}{u^{2}} du = \lim_{R \to \infty} -\frac{1}{u} \Big|_{\ln 2}^{\ln R}$$
$$= -\lim_{R \to \infty} \left(\frac{1}{\ln R} - \frac{1}{\ln 2}\right) = -\left(0 - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

The integral converges, hence the given series also converges.

2. Use the Comparison Test to determine if the infinite series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2^n}}$  is convergent.

For  $n \ge 1$  we have:  $\frac{1}{\sqrt{n}+2^n} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$  (1) The series  $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$  converges since it is a geometric series with  $r = \frac{1}{2}$ . The Comparison Test and inequality (1) imply that the series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2^n}$  also converges.

- 3. Use the Comparison Test to determine if the infinite series  $\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$  is convergent.
  - Since  $3^{-n} > 0$  for all *n*, we have:  $\frac{2}{3^n + 3^{-n}} \leq \frac{2}{3^n} = 2 \cdot \left(\frac{1}{3}\right)^n$  (1) The series  $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$  is a geometric series with  $r = \frac{1}{3}$ , hence it converges. Thus, by Linearity of Infinite Series, the series  $\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^n = 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$  converges. Inequality (1) and the Comparison Theorem imply that the series  $\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$  converges.
- 4. Use the Limit Comparison Test to determine the convergence or divergence of the infinite series  $\sum_{n=2}^{\infty} \frac{n^2}{n^4 1}$ . Let  $a_n = \frac{n^2}{n^4 - 1}$ . Since for large n,  $\frac{n^2}{n^4 - 1} \approx \frac{n^2}{n^4} = \frac{1}{n^2}$ , we apply the Limit Comparison Test with  $b_n = \frac{1}{n^2}$ :  $\frac{a_n}{b_n} = \frac{\frac{n^2}{n^4 - 1}}{\frac{1}{n^2}} = \frac{n^4}{n^4 - 1}$ We compute the following limit:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^4}{n^4 - 1} = \lim_{n \to \infty} \frac{\frac{n^4}{n^4}}{\frac{n^4}{n^4} - \frac{1}{n^4}}$$
$$= \lim_{n \to \infty} \frac{1}{1 - \frac{1}{n^4}} = \frac{1}{1 - 0} = 1 \neq 0$$

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The series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is a convergent *p*-series, hence  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  also converges. The Limit Comparison Test implies that the series  $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$  converges.

5. Use the Limit Comparison Test to determine the convergence or divergence of the infinite series  $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3 - 1}}$ . Let  $a_n = \frac{n}{\sqrt{n^3 - 1}}$ . We observe that for large n,  $\frac{n}{\sqrt{n^3 - 1}} \approx \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n^2}}$ . Therefore, we apply the Limit Comparison test with  $b_n = \frac{1}{\sqrt{n}}$ .

We compute the quotient  $\frac{a_n}{b_n}$ :  $\frac{a_n}{b_n} = \frac{\frac{n}{\sqrt{n^3-1}}}{\frac{1}{\sqrt{n}}} = \frac{n\sqrt{n}}{\sqrt{n^3-1}} = \sqrt{\frac{n^3}{n^3-1}}$ We now compute the limit:  $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \sqrt{\frac{n^3}{n^3-1}} = \lim_{n \to \infty} \sqrt{\frac{1}{1-\frac{1}{n^3}}} = \sqrt{\frac{1}{1-0}} = 1 \neq 0$ The series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  is a divergent *p*-series, hence  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$  also diverges. The Limit Comparison Test implies the series  $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3-1}}$  diverges.

#### 10.4. Absolute and Conditional Convergence.

Class Time AB 0 periods; BC 1 period. Essential.

### **Key Points**

- An infinite series \$\sum\_{n=1}^{\infty} a\_n\$ is absolutely convergent if the series \$\sum\_{n=1}^{\infty} |a\_n|\$ converges.
  Theorem: If \$\sum\_{n=1}^{\infty} |a\_n|\$ converges, then \$\sum\_{n=1}^{\infty} a\_n\$ also converges.
- $\sum_{n=1}^{\infty} a_n$  is conditionally convergent if it converges but is not absolutely convergent.
- Alternating Series Test (also known as the Leibniz Test): If  $a_n$  is a positive sequence such that  $\lim_{n\to\infty} a_n = 0$ , then the alternating series  $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges with  $0 \le S \le a_1$ , and the partial sums satisfy  $S_{2N} \le S \le S_{2N+1}$ .
- The absolute value of the error in using N terms to approximate an alternating series is less than the absolute value of the first omitted term.
- We have two ways to analyze non-positive series: either show absolute convergence or use the Alternating Series Test, if applicable.

### Lecture Material

A series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent if  $\sum_{n=1}^{\infty} |a_n|$  converges. The following theorem is not difficult to see: If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then  $\sum_{n=1}^{\infty} a_n$  converges. A series  $\sum_{n=1}^{\infty} a_n$  is conditionally convergent if it is convergent but not absolutely convergent. An alternating series, that is, a series  $\sum_{n=1}^{\infty} a_n$  where the terms alternate in sign, may be conditionally convergent. We have the following test (Alternating Series Test): Let  $a_n$  be a non-increasing positive series such that  $\lim_{n \to \infty} a_n = 0$  (so  $a_1 \ge a_2 \ge \cdots \ge 0$ ). Then the alternating series  $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$  converges. Furthermore,  $0 \le S \le a_1$  and  $S_{2N} \le S \le S_{2N+1}$  for all N. Now work Exercises 2, 4, and 6. The Alternating Series Test gives information about the error involved when approximating an alternating series with a partial sum. This leads to Theorem 3 which is illustrated by Example 5.

#### **Discussion Topics/Class activities**

Show your class the solution to Exercise 34.

### Selected Problems

Exercises 1–9 odd (computational), 17–31 odd (computational)

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# Worksheet 10.4. Absolute and Conditional Convergence

1. Show that the following series converges conditionally:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2/3}} = \frac{1}{1^{2/3}} - \frac{1}{2^{2/3}} + \frac{1}{3^{2/3}} - \frac{1}{4^{2/3}} + \cdots$$

2. Determine if the series  $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3 + 1}$  converges absolutely, conditionally, or not at all.

**3.** Determine if the series  $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$  converges absolutely, conditionally, or not at all.

### Solutions to Worksheet 10.4

1. Show that  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \frac{5}{16} - \dots$  converges conditionally.

We first show that the series converges, using Leibniz Test for Alternating Series. The terms  $a_n = \frac{n}{n^2 + 1}$  tend to zero since  $\lim_{n \to \infty} \frac{n}{n^2 + 1} = \lim_{n \to \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} = 0$ . We show that  $a_n$  forms a decreasing sequence, by showing that the function  $f(x) = \frac{x}{x^2 + 1}$  is decreasing on  $x \ge 1$ . We differentiate f:  $f'(x) = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$ We see that f'(x) < 0 for x > 1, so f is decreasing for x > 1. By the continuity at x = 1,

We see that f'(x) < 0 for x > 1, so f is decreasing for x > 1. By the continuity at x = 1, we conclude that f is decreasing on  $x \ge 1$ . This implies that the sequence  $\{a_n\}$  is decreasing , that is:

$$a_1 \ge a_2 \ge a_3 \ge \dots$$

We now apply Leibniz Test to conclude that the given alternating series converges.

To prove conditional convergence, we must show that the positive series  $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$  diverges. We do so using the Limit Comparison Test, with the divergent harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$ . This gives:  $\lim_{n \to \infty} \frac{\frac{n^2+1}{n}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{n^2+1} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1 \neq 0$ Thus, the positive series  $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$  also diverges.

Since the given series converges but diverges absolutely, this series converges conditionally.

2. Determine if the series  $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3 + 1}$  converges absolutely, conditionally, or not at all. We compute the limit:

$$\lim_{n \to \infty} \frac{n^4}{n^3 + 1} = \lim_{n \to \infty} \frac{n}{1 + \frac{1}{n^3}} = \infty$$

It follows that the general term  $\frac{(-1)^n n^4}{n^3 + 1}$  of the series does not tend to zero, hence this series diverges by the Divergence Test.

3. Determine if the series  $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$  converges absolutely, conditionally, or not at all. The positive series is  $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ . Since  $\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$  for  $n \geq 1$ , the Comparison Test and the convergence of the *p*-series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  imply that the series  $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$  converges. Hence, the given series is absolutely convergent. **Class Time** AB 0 periods; BC 1 period. Essential.

### **Key Points**

- The Ratio Test: Assume the limit  $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists. Then  $\sum_{n=1}^{\infty} a_n$  converges absolutely if  $\rho < 1$ , and diverges if  $\rho > 1$ . If  $\rho = 1$ , the test is inconclusive.
- The Root Test: Assume the limit  $L = \lim_{n \to \infty} \sqrt[n]{a_n}$  exists. Then  $\sum_{n=1}^{\infty}$  converges if L < 1 and diverges if L > 1. If L = 1, then the test is inconclusive.

#### Lecture Material

First introduce the Ratio Test (Theorem 1) which states that if the limit  $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then

(1) If  $\rho < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges absolutely. (2) If  $\rho > 1$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

(3) If  $\rho = 1$ , the test is inconclusive, so the series may converge or diverge.

Work Exercises 2, 4, and 6 to illustrate the use of the Ratio Test. Now introduce the Root Test (Theorem 2), which states that if the limit  $L = \lim_{n \to \infty} \sqrt[n]{a_n}$  exists, then

- (1) If L < 1, then \$\sum\_{n=1}^{\infty}\$ a\_n converges absolutely.</li>
   (2) If L > 1, then \$\sum\_{n=1}^{\infty}\$ a\_n diverges.

(3) If L = 1, the test is inconclusive, so the series may converge or diverge.

Work Exercises 36, 38, and 40 to illustrate the use of the Root Test.

### **Discussion Topics/Class activities**

Work Exercise 57, which gives a proof of the Root Test.

#### Selected Problems

Exercises 1–17 odd (computational), 35–51 odd (computational)

# Worksheet 10.5. The Ratio and Root Tests

1. Apply the Ratio Test to determine the convergence or divergence of  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{5^n}$ , or state that the Ratio Test is inconclusive.

2. Apply the Ratio Test to determine the convergence or divergence of  $\sum_{n=1}^{\infty} \frac{3n+2}{5n^3+1}$ , or state that the Ratio Test is inconclusive.

3. Apply the Ratio Test to determine the convergence or divergence of  $\sum_{n=1}^{\infty} \frac{2^n}{n}$ , or state that the Ratio Test is inconclusive.

4. Use the Root Test to determine the convergence or divergence of  $\sum_{n=0}^{\infty} \frac{1}{10^n}$ , or state that the Root Test is inconclusive.

5. Use the Root Test to determine the convergence or divergence of  $\sum_{n=0}^{\infty} \left(\frac{k}{k+10}\right)^k$ , or state that the Root Test is inconclusive.

6. Prove that 
$$\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$$
 diverges. Hint: Use that  $2^{n^2} = (2^n)^n$  and  $n! \le n^n$ .

# Solutions to Worksheet 10.5

1. Apply the Ratio Test to determine the convergence or divergence of  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{5^n}$ , or state that the Ratio Test is inconclusive.

Let 
$$a_n = \frac{(-1)^{n-1}n}{5^n}$$
. Then,  $|a_n| = \frac{n}{5^n}$  and  $|a_{n+1}| = \frac{n+1}{5^{n+1}}$  so we have:  
 $|\frac{a_{n+1}}{a_n}| = \frac{\frac{n+1}{5^{n+1}}}{\frac{n}{5^n}} = \frac{5^n}{5^{n+1}} \cdot \frac{n+1}{n} = \frac{n+1}{5^n}$   
 $\rho = \lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = \lim_{n \to \infty} \frac{n+1}{5^n} = \lim_{n \to \infty} \frac{1+\frac{1}{n}}{5} = \frac{1}{5}$   
Since  $\rho < 1$ , the series converges absolutely.

2. Apply the Ratio Test to determine the convergence or divergence of  $\sum_{n=1}^{\infty} \frac{3n+2}{5n^3+1}$ , or state that the Ratio Test is inconclusive

In this series, 
$$a_n = \frac{3n+2}{5n^3+1}$$
 and  $a_{n+1} = \frac{3(n+1)+2}{5(n+1)^3+1} = \frac{3n+5}{5(n+1)^3+1}$ . Hence,  

$$\frac{a_{n+1}}{a_n} = \frac{3n+5}{5(n+1)^3+1} \cdot \frac{5n^3+1}{3n+2} = \frac{3n+5}{3n+2} \cdot \frac{5n^3+1}{5(n+1)^3+1}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left( \frac{3+\frac{5}{n}}{3+\frac{2}{n}} \cdot \frac{5+\frac{1}{n^3}}{5(1+\frac{1}{n})^3+\frac{1}{n^3}} \right) = 1 \cdot 1 = 1$$
In this case  $\rho = 1$  so the Ratio Test is inconclusive.

Notice that the series converges by the Limit Comparison Test if we compare it with the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

3. Apply the Ratio Test to determine the convergence or divergence of  $\sum_{n=1}^{\infty} \frac{2^n}{n}$ , or state that the Ratio Test is inconclusive

In this series 
$$a_n = \frac{2^n}{n}$$
,  $a_{n+1} = \frac{2^{n+1}}{n+1}$ . Hence,  

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2n}{n+1}$$

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2n}{n+1} = 2$$
Since,  $\rho > 1$ , the series diverges.

4. Use the Root Test to determine the convergence or divergence of  $\sum_{n=0}^{\infty} \frac{1}{10^n}$ , or state that the Root Test is inconclusive.

Let  $a_n = \frac{1}{10^n}$ . Then  $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{10^n}} = \frac{1}{10}$ , hence  $\lim_{n \to \infty} \sqrt[n]{a_n} = \frac{1}{10}$ . Since the limit is less than 1, the series converges by the Root Test.

5. Use the Root Test to determine the convergence or divergence of  $\sum_{n=0}^{\infty} \left(\frac{k}{k+10}\right)^k$ , or state that the Root Test is inconclusive.

In this series 
$$a_k = \left(\frac{k}{k+10}\right)^k$$
. Hence,  
 $L = \lim_{k \to \infty} \sqrt[k]{a_k} = \lim_{k \to \infty} \sqrt[k]{\left(\frac{k}{k+10}\right)^k} = \lim_{k \to \infty} \frac{k}{k+10} = \lim_{k \to \infty} \frac{1}{1+\frac{10}{k}} = 1$   
Since  $L = 1$ , the Root Test is inconclusive.

6. Prove that  $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$  diverges. Hint: Use that  $2^{n^2} = (2^n)^n$  and  $n! \le n^n$ . Since  $n! \le n^n$ , we have:

$$\frac{2^{n^2}}{n!} \ge \frac{2^{n^2}}{n^n} \quad (1)$$

We now show that the series  $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n}$  diverges. We compute the following limit:

$$L = \lim_{n \to \infty} \sqrt[n]{\frac{2^{n^2}}{n^n}} = \lim_{n \to \infty} \sqrt[n]{\left(\frac{2^n}{n}\right)^n} = \lim_{n \to \infty} \frac{2^n}{n}$$
  
We compute the resulting limit using the

We compute the resulting limit using the Limit of Sequence Defined by a Function and L'Hopital's Rule. This gives:

$$L = \lim_{n \to \infty} \frac{2^n}{n} = \lim_{x \to \infty} \frac{2^x}{x} = \lim_{x \to \infty} \frac{2^x \ln 2}{1} = \infty$$
  
Thus, the Root Test implies that the series  $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n}$  diverges.

We now apply the Comparison Test and inequality (1) to conclude that the series  $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$  also diverges.

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#### 10.6. Power Series.

**Class Time** AB 0 periods; BC 2 periods. Essential.

#### **Key Points**

• A power series is an infinite series of the form  $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ . We refer to

c as the center of F(x).

- A power series has three possible types of convergence behavior:
  - (1) F(x) converges only for x = c, or
  - (2) F(x) converges for all x, or
  - (3) There exists R > 0 such that F(x) converges absolutely for |x c| < R and diverges for |x c| > R.

*R* is the radius of convergence of F(x). Convergence at the endpoints  $c \pm R$  must be checked separately. If Case (1) occurs, we set R = 0 and if (2) occurs, we set  $R = \infty$ .

- If  $r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then  $R = r^{-1}$  with the conventions that R = 0 if  $r = \infty$  and  $R = \infty$  if r = 0.
- The power series F(x) is differentiable on (c-R, c+R), and can be differentiated and integrated term by term on the interval (c-R, c+R). In particular,

$$F'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$
 and  $\int F(x) dx = a + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ 

where a is an arbitrary constant.

• The power series expansion  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  is valid for |x| < 1, and is useful

for deriving expansions of other related functions by substitution, integration, or differentiation.

#### Lecture Material

A power series centered at the point c is an infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

In terms of convergence, the primary question is for what values of x does F(x) converge? Clearly F(x) always converges for x = c. Now discuss Theorem 1, which gives the three possibilities for the convergence of F(x): Let  $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ . Then

- (1) F(x) converges only for x = c, or
- (2) F(x) converges for all x, or
- (3) There is a number  $0 < R < \infty$  such that F(x) converges absolutely if |x c| < R and diverges if |x c| > R. F(x) may or may not converge at the endpoints |x c| = R.

If (1) occurs, set R = 0, and in case (2), set  $R = \infty$ . We say that R is the radius of convergence of F(x). Note that the endpoints of the interval must be considered separately. A useful tool for finding R is the following form of the Ratio Test: Let  $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ , and assume that  $r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ . Then F(x) has radius of convergence  $R = r^{-1}$  ( $R = \infty$  if r = 0 and R = 0 if  $r = \infty$ ). Work Exercise 3, which will not only give practice at using the Ratio Test, but will also demonstrate the different behaviors occur at the endpoints. An important power series is the geometric series. As  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$  if |r| < 1, by substituting x for r we have that  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ , |x| < 1. Other functions can be represented as a power series using this formula. Work Exercise 40 to illustrate this. Differentiation and integration of power series can be done term by term in the interval of convergence (Theorem 2): If  $F(x) = \sum_{n=0}^{\infty} (x-c)^n$  has radius of convergence R > 0, then F(x) is differentiable on (c - R, c + R), and its derivative and antiderivative may be computed term by term. Precisely, if c - R < x < c + R, then

$$F'(x) = \sum_{n=1}^{\infty} na_n (x-c)^{n-1}, \text{ and}$$
$$\int F(x)dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}, \text{ where } A \text{ is a constant.}$$

#### Discussion Topics/Class activities

Work Exercise 66, which shows that a power series is continuous in its interval of convergence.

#### Selected Problems

Exercises 1, 3 (computational), 7 – 25 every other odd (computational), 35, 37, 39

## Worksheet 10.6. Power Series

1. Show that the following three power series have the same radius of convergence. Then show that (a) diverges at both endpoints, (b) converges at one endpoint but diverges at the other, and (c) converges at both endpoints.

(a) 
$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$$
 (b)  $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n9^n}$  (c)  $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^29^n}$ 

2. Use the formula for geometric series to expand the function  $\frac{1}{1+3x}$  in a power series with center c = 0 and determine the set of x for which the expansion is valid.

### Solutions to Worksheet 10.6

1. Show that the following three power series have the same radius of convergence. Then show that (a) diverges at both endpoints, (b) converges at one endpoint but diverges at the other, and (c) converges at both endpoints.

(a) 
$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$$
(b) 
$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n9^n}$$
(c) 
$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^{2}9^n}$$
Series (a):  $a_n = \frac{1}{9^n}$ . Hence,  
 $r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{1}{9^{n+1}}}{\frac{1}{9^n}} \right| = \lim_{n \to \infty} \frac{9^n}{9^{n+1}} = \frac{1}{9}$ 
The radius of convergence is, thus,  $R = r^{-1} = 9$   
Series (b): $a_n = \frac{1}{n \cdot 9^n}$ .  
Hence,  
 $r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{1}{(n+1)9^{n+1}}}{\frac{1}{n} \cdot 9^n} \right|$   
 $= \lim_{n \to \infty} \left( \frac{n}{n+1} \cdot \frac{9^n}{9^{n+1}} \right) = \lim_{n \to \infty} \left( \frac{1}{9} \cdot \frac{n}{n+1} \right) = \frac{1}{9}$ 

 $= \lim_{n \to \infty} \left( \frac{1}{n+1} \cdot \frac{1}{9^{n+1}} \right) = \lim_{n \to \infty} \left( \frac{1}{9} \cdot \frac{1}{n+1} \right) =$ The radius of convergence is, thus,  $R = r^{-1} = 9$ 

Series (c):
$$a_n = \frac{1}{n^2 \cdot 9^n}$$
.  
Hence,

Hence,  

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\overline{(n+1)^2 9^{n+1}}}{\frac{1}{n^2 \cdot 9^n}} \right|$$

$$= \lim_{n \to \infty} \left( \frac{n^2}{(n+1)^2} \cdot \frac{9^n}{9^{n+1}} \right) = \lim_{n \to \infty} \left( \frac{1}{9} \cdot \left( \frac{n}{n+1} \right)^2 \right) = \frac{1}{9}$$
The radius of convergence is, thus,  $R = r^{-1} = 9$ .

We see that the three series have the same radius of convergence R = 9. The interval of convergence is |x-5| < 9, that is (-4, 14). We check the convergence of each series at the endpoints x = 14 and x = -4.

Series (a): 
$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$$
  
For  $x = 14$ , the series  $\sum_{n=1}^{\infty} \frac{(14-5)^n}{9^n} = \sum_{n=1}^{\infty} 1$  diverges by the Divergence Test.

For x = -4, the series  $\sum_{n=1}^{\infty} \frac{(-4-5)^n}{9^n} = \sum_{n=1}^{\infty} (-1)^n$  diverges by this test. Series (b):  $\sum_{n=1}^{\infty} \frac{(x-5)^5}{n \cdot 9^n}$ For x = 14, the series  $\sum_{n=1}^{\infty} \frac{(14-5)^n}{n \cdot 9^n} = \sum_{n=1}^{\infty} \frac{1}{n}$  is the divergent harmonic series. For x = -4, the series  $\sum_{n=1}^{\infty} \frac{(-4-5)^n}{n \cdot 9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges conditionally by Leibniz Test. Series (c):  $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 \cdot 9^n}$ For x = 14, the series  $\sum_{n=1}^{\infty} \frac{(14-5)^n}{n^2 \cdot 9^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is a convergent *p*-series. For x = -4, the series  $\sum_{n=1}^{\infty} \frac{(-4-5)^n}{n^2 \cdot 9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  converges abolutely, since its associated positive series is  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  which is a convergent *p*-series. 2. Use the formula for geometric series to expand the function  $\frac{1}{1+3x}$  in a power series with center c = 0 and determine the set of x for which the expansion is valid. The formula for geometric series implies that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$
  
We replace  $x$  by  $(-3x)$ , to obtain the following expansion:  
$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 3^n x^n$$
  
This expansion is valid for  $|-3x| < 1$ , or  $|x| < \frac{1}{3}$ .

For  $x = \frac{1}{3}$  the series diverges by the Divergence Test.

Class Time AB 0 periods; BC 2 periods. Essential.

#### **Key Points**

• The Taylor series of f(x) centered at x = c is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

The partial sum  $T_N(x)$  of T(x) is the  $N^{th}$  Taylor polynomial.

- If f(x) has an expansion  $\sum_{n=0}^{\infty} a_n (x-c)^n$  as a power series on an interval (c-R, c+R)
- with R > 0, then this power series is the Taylor series centered at x = c.
- When c = 0, T(x) is the MacLaurin Series of f(x).
- The equality f(x) = T(x) holds if and only if the remainder, defined as  $R_N(x) = f(x) T_N(x)$ , tends to 0 as  $N \to \infty$ .
- Suppose f(x) is infinitely differentiable on an interval I = (c R, c + R) with R > 0 and assume there exists M > 0 such that  $|f^{(k)}(x)| < M$  for  $x \in I$ . Then f(x) is represented by its Taylor series. That is, f(x) = T(x) for  $x \in I$ .
- When determining a Taylor series of a function, it is often useful to start with known Taylor series and apply one of the operations multiplication, substitution, differentiation, or integration.

#### Lecture Material

Begin by deriving the formula for the Taylor Series of f(x) centered at x = c, which is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n.$$

In the special case when c = 0, the Taylor series is also known as the MacLaurin series, and is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Also point out that the power series expansion is unique if the radius of convergence is positive (Theorem 1): If f(x) is represented by a power series F(x) centered at c on an interval (c - R, c + R) with R > 0, then F(x) is the Taylor series of f(x) centered at x = c. Work Exercise 2 to ensure that the students are familiar with these ideas. The previous result tells us that if we wish to represent a function as a power series, the Taylor series is the only way in which this can be done. However, there is no guarantee that the Taylor series converges. We do, though, have the following result (Theorem 2): Let f(x) be infinitely differentiable on I = (c - R, c + R) with R > 0. Assume there is a constant M such that for all  $k \ge 0$ ,  $|f_{\infty}^{(k)}(x)| \le M$  for all  $x \in I$ . Then f(x) is represented by its

Taylor series on I:  $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$  for all  $x \in I$ . Now point out that a Taylor

series is a power series and thus it can be differentiated and integrated term by term within its interval of convergence, and two Taylor series may be multiplied or one may be substituted into another. Work Exercises 4, 5, and 16 to illustrate the latter ideas (as well as to ensure the students are comfortable with the MacLaurin series of some common functions such as  $e^x$  and  $\sin x$ ). Then work Exercises 32 and 38 (on finding Taylor series). Now point out that Taylor series yield a generalization of the Binomial Theorem. Namely Theorem 3, which states: For any exponent a, the Taylor expansion of  $(1 + x)^a$  is valid for |x| < 1:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \dots + \binom{a}{n}x^n + \dotsb$$

Now work Exercise 24.

For the AP exam students should memorize the Maclaurin series for  $e^x$ ,  $\sin(x)$ ,  $\cos(x)$  and  $\frac{1}{1-x}$  (p. 599).

Graph a function and its first few Taylor polynomials to show the interval of convergence  $(\ln(x) \text{ and } \sin(x) \text{ are good examples})$ .

#### **Discussion Topics/Class activities**

Exercise 91 is an interesting application of power series to show that e is irrational.

#### Selected Problems

Exercises 1, 3, 5, 7, 15, 19, 23, 29, 33, 35, 49, 53, 63, 65

## Worksheet 10.7. Taylor Series

**1.** Write out the first 4 terms of the Taylor series of f(x) centered at c = 3 if

f(3) = 1 f'(3) = 2 f''(3) = 12 f'''(3) = 3

**2.** Find the MacLaurin series of  $f(x) = \sin(2x)$ .

**3.** Find the MacLaurin series of  $f(x) = e^{4x}$ .

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4. Find the MacLaurin series of  $f(x) = x^2 e^{x^2}$ .

**5.** Find the Taylor series of  $\sqrt{x}$  centered at c = 4.

**6.** Find the Taylor series of  $\frac{1}{1-4x}$  centered at c = -2.

7. Write out the first 5 terms of the binormial series for  $f(x) = (1+x)^{1/3}$ .

## Solutions to Worksheet 10.7

1. Write out the first 4 terms of the Taylor series of f(x) centered at c = 3 if

$$f(3) = 1 \qquad f'(3) = 2 \qquad f''(3) = 12 \qquad f'''(3) = 3$$
  
The first 4 terms of the Taylor series are:  
$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \dots$$
$$= 1 + 2(x-3) + \frac{12}{2!}(x-3)^2 + \frac{3}{3!}(x-3)^3 + \dots$$
$$= 1 + 2(x-3) + 6(x-3)^2 + \frac{1}{2}(x-3)^3 + \dots$$

**2.** Find the MacLaurin series of  $f(x) = \sin(2x)$ .

Replacing x by 2x in the MacLaurin series for  $\sin x$  gives:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ for all } x$$
$$\sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}, \text{ for all } x$$

3. Find the MacLaurin series of  $f(x) = e^{4x}$ . We substitute 4x in the MacLaurin series for  $e^x$ , obtaining:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
, for all  $x$   
 $e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n x^n}{n!}$ , for all  $x$ 

4. Find the MacLaurin series of  $f(x) = x^2 e^{x^2}$ . We first substitute  $x^2$  in the series for  $e^x$ :

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} \text{ for all } x$$

$$e^{x^{2}} = \sum_{n=0}^{\infty} \frac{(x^{2})^{n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$
We now multiply by  $x^{2}$  to obtain:
$$x^{2}e^{x^{2}} = x^{2}\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

5. Find the Taylor series of  $\sqrt{x}$  centered at c = 4.

We rewrite the function<u>in the form</u>

$$\sqrt{x} = \sqrt{4 + (x - 4)} = \sqrt{4\left(1 + \frac{x - 4}{4}\right)} = 2\sqrt{1 + \frac{x - 4}{4}} \quad (1)$$

We find the MacLaurin series of  $\sqrt{1+x}$  by setting  $a = \frac{1}{2}$  in the Binomial series:

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \left(\frac{2}{n}\right) x^n \text{, for } |x| < 1$$
Replacing  $x$  by  $\frac{x-4}{4}$  we obtain for  $|\frac{x-4}{4}| < 1$  or  $|x-4| < 4$ :
$$\sqrt{1+\frac{x-4}{4}} = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}_n\right) \left(\frac{x-4}{4}\right)^n$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}_n\right) \cdot \frac{1}{4^n} (x-4)^n$$
(2)

Combining (1) and (2) yields:

$$\sqrt{x} = 2 + \sum_{n=1}^{\infty} \left( \frac{1}{2} \\ n \right) \cdot \frac{2}{4^n} (x-4)^n \quad (3)$$

We compute the coefficients in this series for  $n \ge 1$  we get:

$$\begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix} \cdot \frac{2}{4^n} = \underbrace{\frac{1}{2} \left(\frac{1}{2} - 1\right) \cdot \dots \cdot \left(\frac{1}{2} - (n - 1)\right) \cdot 2}{n! \cdot 4^n} \\ = \underbrace{\frac{1 \cdot (1 - 2) (1 - 4) \cdot \dots \cdot (1 - 2n + 2) \cdot 2}{n! \cdot 2^n \cdot 4^n} \\ = \frac{1 (1 - 2) (1 - 4) \dots (1 - 2n + 2) \cdot 2}{n! \cdot 2^n \cdot 4^n} \\ = \frac{(-1)^{n-1} \cdot (1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 3))}{n! \cdot 2^{n-1} \cdot 4^n} \\ = \frac{(-1)^{n-1} \cdot (1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 3))}{n! 2^{n-1} \cdot 4^n \cdot (2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n - 2))} \\ = \frac{(-1)^{n-1} \cdot (1 \cdot 2n - 2)!}{n! 2^{n-1} \cdot 4^n \cdot 2^{n-1} (1 \cdot 2 \cdot 3 \cdot \dots \cdot (n - 1))} \\ = \frac{(-1)^{n-1} (2n - 2)! \cdot n}{2^{4n-2} \cdot (n!)^2}$$

$$\sqrt{x} = 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2n-2)! \cdot n}{2^{4n-2} (n!)^2} (x-4)^n$$
  
**6.** Find the Taylor series of  $\frac{1}{2}$  centered at  $c = -2$ .

Find the Taylor series of 
$$\frac{3x-2}{3x-2}$$
 centered at  $c = -2$ .  
We rewrite the following as follows:  
 $\frac{1}{3x-2} = \frac{1}{-5+3(x+1)} = \frac{1}{-5\left(1-\frac{3(x+1)}{5}\right)} = -\frac{1}{5}\frac{1}{1-\frac{3(x+1)}{5}}$   
We replace  $x$  by  $\frac{3(x+1)}{5}$  in the MacLaurin series for  $\frac{1}{1-x}$ . We get for  $|\frac{3(x+1)}{5}| < 1$   
or  $|x+1| < \frac{5}{3}$ :  
 $\frac{1}{3x-2} = -\frac{1}{5}\sum_{n=0}^{\infty} \left(\frac{3(x+1)}{5}\right)^n = -\frac{1}{5}\sum_{n=0}^{\infty} \frac{3^n}{5^n}(x+1)^n$   
 $= -\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}}(x+1)^n$   
Thus,

$$\frac{1}{3x-2} = -\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} (x+1)^n \text{, for } |x+1| < \frac{5}{3}$$

7. Write out the first four terms of the binomial series for  $f(x) = (1+x)^{1/3}$ . Using the formula for the binomial series with  $a = \frac{1}{3}$ , we obtain the following first four terms:

•

$$1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{81}$$

## Chapter 10 AP Problems

No calculator allowed.

- 1. Which of the following sequences is/are <u>not</u> bounded?
  - I.  $a_n = \sqrt{2n} \sqrt{n+1}$ II.  $b_n = 2 - \frac{5}{n}$ III.  $c_n = 2\sin(\pi n^2)$ A. I only B. I and II only C. II only D. II and III only
  - E. I, II and III
- **2.** Consider the series defined by  $S_k = \sum_{n=1}^k \frac{1}{n^2 + 3n + 2}$ . a. Evaluate  $S_2$  and  $S_3$ .
  - b. Using the method of partial fractions, rewrite the expression  $\frac{1}{n^2 + 3n + 2}$  as a sum/difference of two fractions.

- c. Using the result of part (b), evaluate  $S = \sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$ .
- **3.** Which of the following series converges?

I. 
$$\sum_{n=1}^{\infty} \frac{1}{n+2\sqrt{n}}$$
II. 
$$\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5-1}}$$
III. 
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

- A. I and II only
- B. II and III only
- C. I and III only
- D. II only
- E. III only

4. Which of the following series converges absolutely?

I. 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1.1^n}$$
  
II. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cos(\pi n)}$$
  
III. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n + .5^n}$$

- A. I only
- B. II only
- C. III only
- D. I and II only
- E. I and III only
- 5. For which of the following series is the Ratio Test inconclusive?

A. 
$$\sum_{n=1}^{\infty} \frac{1}{n!}$$
  
B. 
$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$
  
C. 
$$\sum_{n=1}^{\infty} \frac{3n}{2n^3 + 1}$$
  
D. 
$$\sum_{n=1}^{\infty} \frac{n!}{n^3}$$
  
E. 
$$\sum_{n=1}^{\infty} \frac{e^n}{(n-1)!}$$

- 6. Consider the function F defined by the power series  $\sum_{n=0}^{\infty} (-1)^n \cdot x^n$ .
  - a. Write the first three nonzero terms and the general term for  $F(x^2)$ .
  - b. Determine the interval of convergence of  $F(x^2)$ . Show the work that leads to your answer.

c. Given that 
$$F(x^2) = \frac{1}{a + bx^c}$$
, find the values of  $a, b, and c$ .

- d. Evaluate the improper integral  $\int_0^\infty F(x^2) \, dx$ . Show the work that leads to your conclusion.
- 7. In the Maclaurin series expansion of  $f(x) = 8(x+4)^{3/2}$ , what is the coefficient of the  $x^3$  term?

A. 
$$-3$$
  
B.  $-\frac{3}{8}$   
C.  $-\frac{1}{16}$   
D.  $\frac{3}{8}$   
E.  $\frac{3}{2}$ 

## Solutions to Chapter 10 AP Problems

1. Which of the following sequences is/are <u>not</u> bounded?

I.  $a_n = \sqrt{2n} - \sqrt{n+1}$ II.  $b_n = 2 - \frac{5}{n}$ III.  $c_n = 2\sin(\pi n^2)$ A. I only B. I and II only C. II only D. II and III only

## E. I, II and III A [THIS PROBLEM CORRESPONDS WITH SECTION 10.1]

2. Consider the series defined by  $S_k = \sum_{n=1}^k \frac{1}{n^2 + 3n + 2}$ . a. Evaluate  $S_2$  and  $S_3$ .

$$S_2 = \frac{1}{4}; S_3 = \frac{3}{10}$$

b. Using the method of partial fractions, rewrite the expression  $\frac{1}{n^2 + 3n + 2}$  as a sum (or difference) of two fractions.

 $\frac{1}{n+1} - \frac{1}{n+2}$ 

c. Using the result of part (b), evaluate  $S = \sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$ .

# $\frac{1}{2}$ [THIS PROBLEM CORRESPONDS WITH SECTION 10.2]

**3.** Which of the following series converges?

I. 
$$\sum_{n=1}^{\infty} \frac{1}{n+2\sqrt{n}}$$
II. 
$$\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5-1}}$$
III. 
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

- A. I and II only
- B. II and III only
- C. I and III only
- D. II only
- E. III only

## B [THIS PROBLEM CORRESPONDS WITH SECTION 10.3]

4. Which of the following series converges absolutely?

I. 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1.1^n}$$
  
II. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cos(\pi n)}$$
  
III. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n + .5^n}$$

A. I only

- B. II only
- C. III only
- D. I and II only
- E. I and III only

## E [THIS PROBLEM CORRESPONDS WITH SECTION 10.4]

5. For which of the following series is the Ratio Test inconclusive?

A. 
$$\sum_{n=1}^{\infty} \frac{1}{n!}$$
  
B. 
$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$
  
C. 
$$\sum_{n=1}^{\infty} \frac{3n}{2n^3 + 1}$$
  
D. 
$$\sum_{n=1}^{\infty} \frac{n!}{n^3}$$
  
E. 
$$\sum_{n=1}^{\infty} \frac{e^n}{(n-1)!}$$

C [THIS PROBLEM CORRESPONDS WITH SECTION 10.5]

- 6. Consider the function F defined by the power series  $\sum_{n=0}^{\infty} (-1)^n \cdot x^n$ .
  - a. Write the first three nonzero terms and the general term for  $F(x^2)$ .

$$1 - x^2 + x^4 - \dots + (-1)^n \cdot x^{2n}$$

b. Determine the interval of convergence of  $F(x^2)$ . Show the work that leads to your answer.

$$\left|\frac{(-1)^{n+1}x^{2n+2}}{(-1)^n x^{2n}}\right| < 1$$
$$|x^2| < 1$$
$$-1 < x < 1$$

The series diverges for x = -1 and 1.

For x = -1, the series is  $\sum_{n=1}^{\infty} 1^n = 1 + 1 + 1 + \cdots$  which diverges. For x = 1, the series is  $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + \cdots$  and the partial sums do not converge.

Alternate solution: The series is geometric with a common ratio of  $-x^2$  and will converge for  $|x^2| < 1$  or -1 < x < 1 only.

c. Given that 
$$F(x^2) = \frac{1}{a + bx^c}$$
, find the values of  $a, b, and c$ .

This is a geometric series with common ratio  $-x^2$ . Thus  $F(x^2) = \frac{1}{1+x^2}$  and a = 1, b = 1, c = 2.

d. Evaluate the improper integral  $\int_0^\infty F(x^2) dx$ . Show the work that leads to your conclusion.

$$\int_{0}^{\infty} F(x^{2}) dx = \int_{0}^{\infty} \frac{1}{1+x^{2}} dx = \lim_{b \to \infty} \int_{0}^{b} \frac{1}{1+x^{2}} dx = \lim_{b \to \infty} \tan^{-1} x \Big|_{0}^{b} = \frac{\pi}{4}$$
  
[THIS PROBLEM CORRESPONDS WITH SECTION 10.6]

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7. In the Maclaurin series expansion of  $f(x) = 8(x+4)^{3/2}$ , what is the coefficient of the  $x^3$  term?



# B [THIS PROBLEM CORRESPONDS WITH SECTION 10.7]

#### Ray Cannon's Chapter 11 Overview

Chapter 11 has no topic from the AB course description, and deals with the calculus of curves defined in ways other than simply as the graph of a function y = f(x). Section 11.1 starts by treating a curve as giving the location of a particle, (x(t), y(t)), thought of as a function of time, and shows how to find the familiar tangent line in terms of the derivatives of the coordinate functions. Section 11.2 introduces the concept of speed of a particle, and makes the connection between speed and the length of the curve. This is a connection many students fail to make when taking the BC exam; they fail to understand length of a curve as distance the particle travels.

Section 11.3 introduces the concept of polar coordinates, and then Section 11.4 presents the calculus involved with polar coordinates.

Sections 11.5 through 11.7 cover the notion of vectors in the plane, both the algebra of vectors and the geometry, and finally in Section 11.7 the calculus, both differentiation and antidifferentiation, of vector functions.

## 11. Parametric Equations, Polar Coordinates, and Vector Functions

#### 11.1. Parametric Equations.

Class Time AB 0 periods; BC 2 periods. Essential.

Ideally, most of this section is review of precalculus work. BC students should understand how a parametric equation defines a path in the plane. The first derivative of a parametric equation is new material.

#### **Key Points**

- A path traced by a point P = (x, y), where x and y are functions of a parameter t, is called a parametric or parametrized path or curve. We write c(t) = (f(t), g(t)) or c(t) = (x(t), y(t)).
- Note that the path c(t) = (x(t), y(t)) and the curve that it traces may be different, as the path  $(\cos t, \sin t)$  moves around the unit circle infinitely many times as t varies from 0 to  $\infty$ .
- Parametrizations are not unique. In fact, every path can parametrized in infinitely many ways.
- The slope of the tangent line at a point on a parametrized curve c(t) = (x(t), y(t)) is the derivative

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

provided that  $x'(t) \neq 0$ 

• Note that the derivatives dy/dt and dx/dt are not the same as the derivative dy/dx, which is the slope of the tangent line.

#### Lecture Material

Begin by defining a parametric equation. Work Exercise 1 to demonstrate how to plot points with a parametric curve. Explain the difference between the underlying curve of the parametric equation and the path of the parametric curve using the parametrization  $x = \cos t$  and  $y = \sin t$  of the the unit circle. Also show by working Exercise 2 that some parametric equations can also be written in the form y = f(x) by eliminating the parameter. Show the parametric form of a line given in Example 4, as well as the parametrization of a circle (with center at origin and at (a, b) - this follows Example 4 in the text). Also explain how to graph parametric curves by working Exercise 18, and how to obtain parametric equations for common curves by working Exercise 26. Now show that the slope of the line tangent to the curve c(t) = (x(t), y(t)), where x(t) and y(t) are differentiable and  $x'(t) \neq 0$ , is  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$ , and illustrate this result by working

Exercise 49.

#### Discussion Topics/Class activities

Work Exercise 85 with your students. This is an exercise on how to find the area under a parametrized curve (that has nice properties); or discuss Bézier curves and do Example 10.

#### Suggested Problems

Exercises 4 (graphical), 7 – 13 odd (computational), 15 – 19 odd (graphical), 21 – 37 odd (computational), 45 – 57 odd (computational)

## Worksheet 11.1. Parametric Equations

- 1. Find the coordinates at  $t = 0, \frac{\pi}{4}$ , and  $\pi$  of a particle moving along the path  $c(t) = (\cos 2t, \sin^2 t)$ .
- **2.** Express  $x = t^{-1}$ ,  $y = t^{-2}$ , in the form y = f(x) by eliminating the parameter.
- **3.** Graph the curve x = 2 + 4t, y = 3 + 2t, and draw an arrow specifying the direction corresponding to motion.



4. Find the parametric equations for the curve  $y = 8x^2 - 3x$ .

5. Find an equation y = f(x) for the parametric curve  $c(t) = (\frac{1}{2}t, \frac{1}{4}t^2 - t)$ , and compute  $\frac{dy}{dx}$  in two ways: using Equation 7 and by differentiating f(x).

#### Solutions to Worksheet 11.1

- 1. Find the coordinates at  $t = 0, \frac{\pi}{4}$ , and  $\pi$  of a particle moving along the path  $c(t) = (\cos 2t, \sin^2 t)$ . Setting t = 0,  $t = \frac{\pi}{4}$  and  $t = \pi$  in  $c(t) = (\cos 2t, \sin^2 t)$  we obtain the following coordinates of the particle: t = 0 :  $(\cos 2 \cdot 0, \sin^2 0) = (1, 0)$   $t = \frac{\pi}{4}$  :  $\left(\cos \frac{2\pi}{4}, \sin^2 \frac{\pi}{4}\right) = \left(0, \frac{1}{2}\right)$  $t = \pi$  :  $(\cos 2\pi, \sin^2 \pi) = (1, 0)$
- 2. Express  $x = t^{-1}$ ,  $y = t^{-2}$ , in the form y = f(x) by eliminating the parameter. From  $x = t^{-1}$ , we have  $t = x^{-1}$ . Substituting in  $y = t^{-2}$  we obtain:  $y = t^{-2} = (x^{-1})^{-2} = x^2 \Rightarrow y = x^2, x \neq 0.$
- **3.** Graph the curve x = 2 + 4t, y = 3 + 2t, and draw an arrow specifying the direction corresponding to motion.

We find the function by eliminating the parameter. Since x = 2 + 4t we have  $t = \frac{x-2}{4}$ ,

hence  $y = 3 + 2\left(\frac{x-2}{4}\right)$  or  $y = \frac{x}{2} + 2$ . Also, since 2 + 4t and 3 + 2t are increasing functions the direction of motion is the direction of increasing t.

4. Find the parametric equations for the curve  $y = 8x^2 - 3x$ . Letting t = x yields the parametric representation:  $c(t) = (t, 8t^2 - 3t)$ . Another representation is obtained as follows:

We first complete the square in  $y = 8x^2 - 3x$  to obtain:

$$y = 2\left(4x^2 - 2 \cdot \frac{3}{4}x + \frac{9}{16}\right) - \frac{9}{8} = 2\left(2x - \frac{3}{4}\right)^2 - \frac{9}{8}$$
  
We now let  $t = 2x - \frac{3}{4}$ . Then  $x = \frac{4t + 3}{2}$  and  $y = 2t^2 - \frac{8}{9}$  yielding the parametrization:  
 $c(t) = \left(\frac{4t + 3}{2}, 2t^2 - \frac{8}{9}\right).$ 

5. Find an equation y = f(x) for the parametric curve  $c(t) = (\frac{1}{2}t, \frac{1}{4}t^2 - t)$ , and compute  $\frac{dy}{dx}$  in two ways: using Equation 7 and by differentiating f(x). Since  $x = \frac{1}{2}t$  we have t = 2x. Substituting in  $y = \frac{1}{4}t^2 - t$  yields:

 $y = \frac{1}{4}(2x)^2 - 2x = x^2 - 2x.$ We differentiate  $y = x^2 - 2x$  and then substitute  $x = \frac{1}{2} \cdot 3 = \frac{3}{2}$  in the derivative:  $\frac{dy}{dx} = 2x - 2 \Rightarrow \frac{dy}{dx}|_{x=\frac{3}{2}} = 2 \cdot \frac{3}{2} - 2 = 1.$ Now, we find  $\frac{dy}{dx}$  using Eq. (7). Thus,  $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(\frac{1}{4}t^2 - t)'}{(\frac{1}{2}t)'} = \frac{\frac{1}{2}t - 1}{\frac{1}{2}} = t - 2 \Rightarrow \frac{dy}{dx}|_{t=3} = 3 - 2 = 1.$ 

#### 11.2. Arc Length and Speed.

Class Time AB 0 periods; BC 1 period. Essential.

#### **Key Points**

• The length s along a path c(t) = (x(t), y(t)) for  $a \le t \le b$  is

$$s = \operatorname{arc length} = \int_{a}^{b} \sqrt{x'(t)^{2} + y'(t)^{2}} dt$$

- The arc length s is equal to the distance traveled over a given time interval  $[t_0, t_1]$ , while the net displacement is the distance between the starting point  $c(t_0)$  and the endpoint  $c(t_1)$ .
- The speed of a particle with trajectory c(t) = (x(t), y(t)) is  $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$ .

#### Lecture Material

Begin by reminding the students about the formula for calculating the arc length obtained in Section 9.1, arc length  $= \int_a^b \sqrt{1 + f'(x)^2} dx$ . Then generalize this formula for parametric equations (which conceptually is the same as the preceding formula, and is obtained by using line segments to approximate the arc length, as in Figure 1) to obtain the following formula: Let c(t) = (x(t), y(t)) and assume that x'(t) and y'(t) exist and are continuous. Then the length s of c(t) for  $a \le t \le b$  is equal to

$$s = \int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Now work Exercise 4 to illustrate the use of this result. As the distance traveled by a particle can be interpreted as the length of the curve traced by the particle, if c(t) = (x(t), y(t)) is the path of a particle, the distance traveled by the particle is s as above. Hence the speed of the particle is the derivative of s, and so by the Fundamental Theorem of Calculus, the speed at time t of a particle with trajectory c(t) = (x(t), y(t)) is

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

Work Exercise 16 to illustrate this. Finally, remind the students that the distance a particle travels in an interval is not the same as its net change in position, or displacement.

#### Discussion Topics/Class activities

Exercise 36 is an application that a large number of students will find interesting.

#### Suggested Problems

Exercises 1 - 13 odd (computational), 17 - 23 odd (computational), 25 (graphing calculator)

## Worksheet 11.2. Arc Length and Speed

**1.** Find the length of the path of (1 + 2t, 2 + 4t) over the interval  $1 \le t \le 4$ .

2. Determine the speed s(t) of a particle with trajectory  $(t^3, t^2)$  at time t = 2 (in units of meters and seconds).

#### Solutions to Worksheet 11.2

**1.** Find the length of the path of (1 + 2t, 2 + 4t) over the interval  $1 \le t \le 4$ .

We have x = 1 + 2t and y = 2 + 4t, hence x' = 2 and y' = 4. Using the Formula for Arc Length we obtain: 4

$$S = \int_{1}^{1} \sqrt{2^2 + 4^2} dt = \int_{1}^{1} \sqrt{20} dt = \sqrt{20} (4 - 1) = 6\sqrt{5}$$

2. Determine the speed s(t) of a particle with trajectory  $(t^3, t^2)$  at time t = 2 (in units of

meters and seconds). We have  $x(t) = t^3$ ,  $y(t) = t^2$  hence  $x'(t) = 3t^2$ , y'(t) = 2t. The speed of the particle at time t is thus,  $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{9t^4 + 4t^2} = t\sqrt{9t^2 + 4}.$ At time t = 2 the speed is:  $\frac{ds}{dt}|_{t=2} = 2\sqrt{9 \cdot 2^2 + 4} = 2\sqrt{40} = 4\sqrt{10} \approx 12.65 \text{m/s}.$ 

#### 11.3. Polar coordinates.

#### Class Time AB 0 periods; BC 1–2 periods. Essential.

Ideally, most of this section is a review of precalculus work. BC students should be able to find the intersection of two polar curves and calculate the area enclosed by one curve or between two curves. Area is discussed in Section 11.4.

#### **Key Points**

• A point P in the plane has polar coordinates  $(r, \theta)$ , where r is the distance from P to the origin and  $\theta$  is the angle between the positive x-axis and the segment  $\overline{OP}$  in the counterclockwsie direction. The rectangular coordinates of P are:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

- If P has rectangular coordinates (x, y), then:  $r = \sqrt{x^2 + y^2}$ ,  $\tan \theta = \frac{y}{x}$ . Note that  $\theta$  must be chosen so that  $(r, \theta)$  lies in the proper quadrant. If x = 0, then  $\theta = \pi/2$  if y > 0 while  $\theta = -\pi/2$  if y < 0.
- A point does not have a unique polar representation, as the points  $(r, \theta)$  and  $(r, \theta + 2n\pi)$  represent the same point for all integers n. Also, the origin has polar coordinates  $(0, \theta)$  for all  $\theta$ .
- Negative radial coordinates can be represented as  $(-r, \theta) = (r, \theta + \pi)$ .
- Note the following common polar equations:

curve	polar equation
circle of radius $R$ , center at the origin	r = R
line through origin of slope $m = \tan \theta_0$	$\theta = \theta_0$
line where $P = (d, \alpha)$ is point closest to the origin	$r = d \sec(\theta - \alpha)$
$(x-a)^2 + y^2 = a^2$	$r = 2a\cos\theta$
$x^{2} + (y - a)^{2} = a^{2}$	$r = 2a\sin\theta$

#### Lecture Material

Polar coordinates are an alternative way of plotting points in the usual Cartesian plane. The coordinate of a point P are  $(r, \theta)$ , where r is the distance from P to the origin O, and  $\theta$  is the angle between  $\overline{OP}$  and the positive x-axis. Use Figure 2 to show how to plot a point in polar coordinates, and use Figure 1 to show that the following formulas for converting rectangular coordinates to polar coordinates and vice versa hold:

Polar to Rectangular	Rectangular to Polar
$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\cos\theta$	$\tan \theta = \frac{y}{x}$

Work Exercises 3 and 6 to demonstrate the use of these formulas. Point out the following facts and conventions:

- the angular coordinate  $\theta$  is not unique as  $(r, \theta)$  and  $(r, \theta + 2\pi n)$  label the same point,
- the origin does not have a well defined angular coordinate, so by convention we assign to the origin the polar coordinates  $(0, \theta)$  for any  $\theta$ ,
- $\theta = \tan^{-1} \frac{y}{x}$  is only valid if  $x \neq 0$ . If x = 0, then P = (0, y) (in rectangular coordinates) lies on the y-axis, so the polar angle  $\theta = \frac{\pi}{2}$  if y > 0 or  $-\frac{\pi}{2}$  if y < 0.
- By convention, negative radial coordinates are allowed. For r > 0,  $(-r, \theta)$  is defined to be the reflection of  $(r, \theta)$  through the origin, as in Figure 4. Hence  $(-r, \theta)$  and  $(r, \theta + \pi)$  label the same point.

Now show the derivations of the polar equations given in the table in the Key Points above (note that the derivation of  $r = d \sec(\theta - \alpha)$  is given in Example 5), and then work Exercises 12, 14, 18, and 20.

#### Discussion Topics/Class activities

Exercise 49, which shows how to find  $\frac{dy}{dx}$  in polar coordinates.

#### Suggested Problems

Exercises 1, 2, 4, 5 (computational), 13–21 odd (computational), 31 (graphical), 33

## Worksheet 11.3. Polar Coordinates

- 1. Use a calculator to convert from rectangular to polar coordinates (make sure your choice of  $\theta$  gives the correct quadrant):
  - (a) (2,3)
  - (b) (4, -7)
  - (c) (-3, -8)
  - (d) (-5,2)
- 2. Convert from polar to rectangular coordinates:(a) (0,0)
  - (b)  $(-4, \frac{\pi}{3})$
  - (c)  $(0, \frac{\pi}{6})$
- **3.** Convert r = 7 to an equation in rectangular coordinates.

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4. Convert  $r = 2\sin\theta$  to an equation in rectangular coordinates.

5. Convert  $x^2 + y^2 = 5$  to an equation in polar coordinates.

6. Convert  $y = x^2$  to an equation in polar coordinates.
## Solutions to Worksheet 11.3

- 1. Use a calculator to convert from rectangular to polar coordinates (make sure your choice of  $\theta$  gives the correct quadrant):
  - (a) (2,3)

(b) (4, -7)

(c) (-3, -8)

(d) (-5,2)

Part a:

The point (2,3) is in the first quadrant, x = 2 and y = 3. Hence:

$$\theta = \tan^{-1} \left(\frac{3}{2}\right) \approx 0.98$$

$$r = \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.6$$

$$\Rightarrow (r, \theta) \approx (3.6, 0.98).$$
Part b:

The point (4, -7) is in the fourth quadrant with x = 4 and y = -7. Thus,  $\theta = \tan^{-1}\left(\frac{-7}{4}\right) = 2\pi - 1.05 \approx 5.2$   $r = \sqrt{4^2 + (-7)^2} = \sqrt{65} \approx 8.1$ Part c:

The point (-3, -8) is in the third quadrant, with x = -3 and y = -8. Thus,  $\theta = \tan^{-1}\left(\frac{-8}{-3}\right) = \tan^{-1}\left(\frac{8}{3}\right) = \pi + 1.2 \approx 4.3$   $r = \sqrt{(-3)^2 + (-8)^2} \approx 8.5$ Part d: The point (-5, 2) is in the second quadrant, with x = -5 and y = 2. Hence:

The point (-5, 2) is in the second quadrant, with 
$$x = -5$$
 and  $y = 2$ . The  $\theta = \tan^{-1}\left(\frac{2}{-5}\right) = \pi - 0.38 \approx 2.8$   
 $r = \sqrt{(-5)^2 + 2^2} = \sqrt{29} \approx 5.4$ 
 $\Rightarrow (r, \theta) \approx (5.4, 2.8)$ .

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2. Convert from polar to rectangular coordinates:

(a) (0,0)

(b) 
$$(-4, \frac{\pi}{3})$$

(c)  $(0, \frac{\pi}{6})$ 

Part a:

 $r = 0, \theta = 0$  are polar coordinates of the origin (x, y) = (0, 0). Part b:

The point  $(r, \theta) = \left(-4, \frac{\pi}{3}\right)$  has rectangular coordinates,

 $x = r \cos \theta = -4 \cos \frac{\pi}{3} = -4 \cdot \frac{1}{2} = -2$  $y = r \sin \theta = -4 \sin \frac{\pi}{3} = -4 \cdot \frac{\sqrt{3}}{2} = -2\sqrt{3} \Rightarrow (x, y) = (-2, -2\sqrt{3})$ Part c:

The point  $\left(0,\frac{\pi}{6}\right)$  is the origin (x,y) = (0,0). Recall that the polar coordinates of the origin are  $(0, \theta)$  for any  $\theta$ .

**3.** Convert r = 7 to an equation in rectangular coordinates. r = 7 describes the points having distance 7 from the origin, that is, the circle with radius 7 centered at the origin.

The equation of the circle in rectangular coordinates is:  $x^2 + y^2 = 7^2 = 49$ 

- 4. Convert  $r = 2\sin\theta$  to an equation in rectangular coordinates. We multiply the equation by r and substitute  $r^2 = x^2 + y^2$ ,  $r \sin \theta = y$ . This gives:  $r^2 = 2r \sin \theta$ ,  $x^2 + y^2 = 2y$ . Transfering sides and completing the square yield:  $x^{2} + y^{2} - 2y = 0, x^{2} + (y - 1)^{2} = 1.$ Thus,  $r = 2\sin\theta$  is the equation of a circle of radius 1 centered at (0, 1).
- 5. Convert  $x^2 + y^2 = 5$  to an equation in polar coordinates. We make the substitution  $x^2 + y^2 = r^2$  to obtain;  $r^2 = 5$  or  $r = \sqrt{5}$ .
- 6. Convert  $y = x^2$  to an equation in polar coordinates. Substituting  $y = r \sin \theta$  and  $x = r \cos \theta$  yields:  $r\sin\theta = r^2 \cos^2\theta$ Then, dividing by  $r\cos^2\theta$  we obtain:  $\frac{\sin\theta}{\cos^2\theta} = r$ ,  $r = \tan\theta \sec\theta$ .

#### 11.4. Area and Arc Length in Polar Coordinates.

### Class Time AB 0 periods; BC 1 period. Essential.

BC students should be able to find the intersection of two polar curves and calculate the area enclosed by one curve or between two curves.

## **Key Points**

• Polar coordinates are used to find the area bounded by the polar curve  $r = f(\theta)$ and two rays  $\theta = \alpha$  and  $\theta = \beta$ . The area is equal to

$$\frac{1}{2}\int_{\alpha}^{\beta}f(\theta)^2d\theta$$

• The area between the polar curves  $r = f_1(\theta)$  and  $r = f_2(\theta)$ , where  $f_1(\theta) \ge f_2(\theta)$ , is

$$\frac{1}{2}\int_{\alpha}^{\beta} (f_1(\theta)^2 - f_2(\theta)^2)d\theta$$

#### Lecture Material

First derive the formula for determining area in polar coordinates. This is done by dividing the area under consideration into "triangular strips" (see Figure 1), calculating the area of the strips, and then using the usual limiting procedure. The resulting formula is: If  $f(\theta)$  is a continuous function, then the area bounded by a curve in polar form  $r = f(\theta)$  and the rays  $\theta = \alpha$  and  $\theta = \beta$  is

$$\frac{1}{2}\int_{\alpha}^{\beta}r^{2}d\theta = \frac{1}{2}\int_{\alpha}^{\beta}f(\theta)^{2}d\theta.$$

Point out that the formula above give the actual area, and remains valid if  $f(\theta)$  is negative. Illustrate this formula by working Exercise 4. Next, the area between two curves in polar coordinates is exactly as the students expect: The area between  $r = f_1(\theta)$  and  $r = f_2(\theta)$ with  $f_1(\theta) \leq f_2(\theta)$  in the section  $\alpha \leq \theta \leq \beta$  is given by

$$\frac{1}{2}\int_{\alpha}^{\beta}(f_1(\theta)^2 - f_2(\theta)^2)d\theta.$$

Work Exercise 12 to illustrate the use of this formula.

The formula for arc length is not tested on the BC exam and may be omitted.

#### Selected Problems

Exercises 1–35 every other odd (computational and graphical)

# Worksheet 11.4. Area and Arc Length in Polar Coordinates

1. Compute the area of the shaded region  $0 \le r \le 4\cos\theta$ ,  $-\pi/4 \le \theta \le \pi/3$ , as an integral in polar coordinates.



**2.** Find the area of the intersection of the circles  $r = \sin \theta$  and  $r = \cos \theta$ .



3. Calculate the total length of the circle  $r = 4 \sin \theta$  using an integral in polar coordinates.

# Solutions to Worksheet 11.4

1. Compute the area of the shaded region  $0 \le r \le 4\cos\theta$ ,  $-\pi/4 \le \theta \le \pi/3$ , as an integral in polar coordinates.



2. Find the area of the intersection of the circles  $r = \sin \theta$  and  $r = \cos \theta$ .



3. Calculate the total length of the circle  $r = 4 \sin \theta$  using an integral in polar coordinates. We use the Formula for the Arc Length:

$$S = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta \quad (1)$$

In this case,  $f(\theta) = 4\sin\theta$  and  $f'(\theta) = 4\cos\theta$ , hence:

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(4\sin\theta)^2 + (4\cos\theta)^2} = \sqrt{16\left(\sin^2\theta + \cos^2\theta\right)} = 4 \quad (2)$$

The circle is traced as  $\theta$  is varied from 0 to  $\pi$ . Substituting  $\alpha = 0, \ \beta = \pi$  and (2) in (1) yields:

$$S = \int_{0}^{\pi} 4d\theta = 4\pi$$

### 11.5. Vectors in the Plane.

Class Time AB 0 periods; BC 1–2 periods. Essential.

This material can be taught in precalculus and reviewed here. If students have not worked previously with vectors, then more time will be needed here.

### **Key Points**

- A vector  $\mathbf{v} = \overrightarrow{PQ}$  is determined by a basepoint or tail P and a terminal point or head Q. The length of  $\mathbf{v} = \overrightarrow{PQ}$ , denoted ||v||, is the distance from P to Q.
- The vector  $\mathbf{v} = \overrightarrow{OP}$  based at the origin O with head P = (a, b) is the position vector of P.
- Two vectors **v** and **w** are equivalent if they are translates of each other. That is, if **v** and **w** have the same magnitude and direction, but possibly different basepoints.
- The vector  $\mathbf{v} = \overrightarrow{PQ}$ , where  $P = (a_1, b_1)$  and  $Q = (a_2, b_2)$ , has components  $a = a_2 a_1$  and  $b = b_2 b_1$ .
- Two vectors are equivalent if and only if they have the same components. If **v** has components a and b, we write  $\mathbf{v} = \langle a, b \rangle$ . This notation is somewhat ambiguous as equivalent vectors have the same components, but in practice rarely causes confusion. We use the standard convention that all vectors are based at the origin unless otherwise stated.
- If  $\mathbf{v} = \langle a, b \rangle$ , then  $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ .
- Vector addition is defined geometrically by the parallelogram law. If  $\lambda$  is a scalar, then the scalar multiple  $\lambda \mathbf{v}$  is the vector of length  $|\lambda| ||\mathbf{v}||$  in the same directions as  $\mathbf{v}$  if  $\lambda > 0$  and in the opposite direction if  $\lambda < 0$ . For  $\lambda = 0$ , we set  $0\mathbf{v} = \mathbf{0}$ . In components, we have:

$$\langle a, b \rangle + \langle d, c \rangle = \langle a + d, b + c \rangle$$
 and  $\lambda \langle a, b \rangle = \langle \lambda a, \lambda b \rangle.$ 

- Non-zero vectors  $\mathbf{v}$  and  $\mathbf{w}$  are parallel if  $\mathbf{w} = \lambda \mathbf{v}$  for some scalar  $\lambda \neq 0$ . They point in the same direction if  $\lambda > 0$  and in opposite directions if  $\lambda < 0$ .
- A unit vector is a vector of length one. If  $\mathbf{v} \neq \mathbf{0}$ , the unit vector pointing in the direction  $\mathbf{v}$  is  $\mathbf{e}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$ .
- Every vector can be expressed as a linear combination of the standard basis vectors  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$ . If  $\mathbf{v} = \langle a, b \rangle$ , then  $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ .
- If  $\mathbf{v} = \langle a, b \rangle$  is non-zero and makes an angle  $\theta$  with the positive x-axis, then  $a = \|\mathbf{v}\| \cos \theta$  and  $b = \|\mathbf{v}\| \sin \theta$ . The unit vector pointing in the direction of  $\mathbf{v}$  is

$$\mathbf{e}_{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \langle \cos \theta, \sin \theta \rangle.$$

• The Triangle Inequality states that for any two vectors  $\mathbf{v}$  and  $\mathbf{w}$ ,  $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ . Equality holds if and only if  $\mathbf{w} = \lambda \mathbf{v}$  where  $\lambda \geq 0$  or  $\mathbf{v} = 0$ .

#### Lecture Material

Introduce the basic terminology and notation dealing with two-dimensional vectors. The length or magnitude of a vector  $\mathbf{v}$ , denoted  $\|\mathbf{v}\|$ , is the distance from the head P of  $\mathbf{v}$  to the tail Q of  $\mathbf{v}$ . Thus if  $P = (a_1, b_1)$  and  $Q = (a_2, b_2)$ , then applying the usual distance formula, we have that

$$\|\mathbf{v}\| = \|\overrightarrow{PQ}\| = |P-Q| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_2)^2}$$

The components of a vector  $\mathbf{v} = PQ$  where  $P = (a_1, b_1)$  and  $Q = (a_2, b_2)$  are the quantities  $a = a_2 - a_1$  (the *x*-component) and  $b = b_2 - b_1$  (the *y*-component). The components are denoted by  $\langle a, b \rangle$ . Now work Exercise 8.

Two vectors  $\mathbf{v}$  and  $\mathbf{w}$  of nonzero length are parallel if the lines through  $\mathbf{v}$  and  $\mathbf{w}$  are parallel (so parallel vectors either point in the same or opposite directions). A vector  $\mathbf{v}$  undergoes a translation when it is moved parallel to itself without changing its length or direction. The resulting vector is a translate of  $\mathbf{v}$ . Work Exercise 23. Two vectors  $\mathbf{v}$  and  $\mathbf{w}$  are equivalent if  $\mathbf{w}$  is a translate of  $\mathbf{v}$ . Note that every vector can be translated so that its tail is at the origin. We will follow the convention that *all vectors are based at the origin unless otherwise stated*. We may thus write  $\mathbf{v} = \langle a, b \rangle$ . Note that the zero vector (whose head and tail coincide) is the vector  $\mathbf{0} = \langle 0, 0 \rangle$  and has length 0. Now work Exercise 28.

We now turn to vector algebra. Define the scalar multiple  $\lambda \mathbf{v}$  as the vector of length  $|\lambda| ||\mathbf{v}||$  pointing in the same direction as  $\mathbf{v}$  if  $\lambda > 0$ , and in the opposite direction if  $\lambda < 0$ . If  $\lambda = 0$ , we set  $0\mathbf{v} = \mathbf{0}$ . Hence  $||\lambda \mathbf{v}|| = |\lambda| ||\mathbf{v}||$ . The vector sum  $\mathbf{v} + \mathbf{w}$  is only defined when  $\mathbf{v}$  and  $\mathbf{w}$  have the same basepoint, and can be described in two ways (see Figure 8). First, translate  $\mathbf{w}$  to the equivalent vector  $\mathbf{w}'$  whose tail coincides with head of  $\mathbf{v}$ . The sum  $\mathbf{v} + \mathbf{w}$  is then the vector pointing from the tail of  $\mathbf{v}$  to the head of  $\mathbf{w}'$  (Figure 8 (a)). Second, the vector sum  $\mathbf{v} + \mathbf{w}$  is the vector pointing from the basepoint to the opposite vertex of the parallelogram formed by  $\mathbf{v}$  and  $\mathbf{w}$  (this is the Parallelogram Law and is illustrated in Figure 8 (b)). If  $\mathbf{v} = \langle a, b \rangle$  and  $\mathbf{w} = \langle c, d \rangle$  are represented as components, then we simply have:

• 
$$\mathbf{v} + \mathbf{w} = \langle a + c, b + d \rangle,$$

• 
$$\mathbf{v} - \mathbf{w} = \langle a - c, b - d \rangle$$

- $\lambda \mathbf{v} = \langle \lambda a, \lambda b \rangle,$
- $\mathbf{v} = \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$  and  $\mathbf{v} \mathbf{v} = \mathbf{0}$ .

Now work Exercises 10, and 20, and also point out that vector operations obey the usual laws of algebra, namely the commutative, associative, and distributive laws (Theorem 1).

A linear combination of vectors  $\mathbf{v}$  and  $\mathbf{w}$  is a vector of the form  $r\mathbf{v} + s\mathbf{w}$ , where r and s are scalars. Point out using Figure 13 (a) that every vector can be expressed as a linear combination of  $\mathbf{v}$  and  $\mathbf{w}$ , provided that  $\mathbf{v}$  and  $\mathbf{w}$  are not parallel. Work Exercise 55 to illustrate this fact.

A unit vector is a vector of length one. If  $\mathbf{v} = \langle a, b \rangle$ , then  $\mathbf{e}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$  is the unit vector pointing in the same direction as  $\mathbf{v}$ . If  $\mathbf{v}$  makes an angle of  $\theta$  with the positive *x*-axis, then  $\mathbf{e}_{\mathbf{v}} = \langle \cos \theta, \sin \theta \rangle$  and  $\mathbf{v}$  has components  $\langle \|\mathbf{v}\| \cos \theta, \|\mathbf{v}\| \sin \theta \rangle$ . The unit vectors  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$  are the standard basis vectors, and clearly every vector can be written as a linear combination of these vectors. Work Exercise 52. With all of these ideas in hand, it would probably be worthwhile to work a practical example such as Exercise 60 to demonstrate that vectors can be useful.

Finally, state the Triangle Inequality: For any two vectors  $\mathbf{v}$  and  $\mathbf{w}$ ,  $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ . Equality holds if and only if  $\mathbf{w} = \lambda \mathbf{v}$  where  $\lambda \geq 0$  or  $\mathbf{v} = \mathbf{0}$ . Justify the Triangle Inequality using Figure 20.

### Discussion Topics/Class activities

Work Exercise 65, which will point out that vectors can also be useful in geometry.

#### Selected Problems

Exercises 1–3 odd (computational and graphical), 5–13 odd (computational) 17–21 odd (graphical), 23 (computational), 25–27 odd (computational), 29–45 odd (computational), 49–59 odd (computational)

# Worksheet 11.5. Vectors in the Plane

- **1.** Find the components of  $\overrightarrow{PQ}$ , where P = (1, -4) and Q = (3, 5).
- 2. Let  $\mathbf{v} = \langle 6, 9 \rangle$ . Which of the following vectors are parallel to  $\mathbf{v}$  and which point in the same direction?
  - (a)  $\langle 12, 18 \rangle$
  - (b)  $\langle 3,2\rangle$
  - (c)  $\langle 2,3\rangle$
  - (d)  $\langle -6, -9 \rangle$
  - (e)  $\langle -24, -27 \rangle$
  - (f)  $\langle -24, -36 \rangle$

**3.** Sketch the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{PQ}$  and determine if they are equivalent. A = (1,4), B = (-6,3), P = (1,4), and Q = (6,3)



- 4. Calculate  $4(\langle 1,1\rangle + \langle 3,2\rangle)$
- 5. Sketch  $\mathbf{v} = \langle 1, 3 \rangle$ ,  $\mathbf{w} = \langle 2, -2 \rangle$ ,  $\mathbf{v} + \mathbf{w}$ ,  $\mathbf{v} \mathbf{w}$ .



6. Express  $\mathbf{u} = \langle 6, -2 \rangle$  as a linear combination  $\mathbf{u} = r\mathbf{v} + s\mathbf{w}$ , where  $\mathbf{v} = \langle 1, 1 \rangle$  and  $\mathbf{w} = \langle 1, -1 \rangle$ . Sketch the vectors  $\mathbf{u}, \mathbf{w}, \mathbf{w}$ , and the parallelogram formed by  $r\mathbf{v}$  and  $s\mathbf{w}$ .



7. Calculate the linear combination  $(-2\mathbf{i} + 9\mathbf{j}) + (3\mathbf{i} - 4\mathbf{j})$ .

8. A plane flying due east at 200 km/hr encounters a 40 km/hr north-easterly wind. The resultant velocity of the plane is the vector sum  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ , where  $\mathbf{v}_1$  is the velocity vector of the plane and  $\mathbf{v}_2$  is the velocity vector of the wind. The angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is  $\frac{\pi}{4}$ . Determine the resultant speed of the plane (the length of  $\mathbf{v}$ ).

# Solutions to Worksheet 11.5

1. Find the components of  $\overrightarrow{PQ}$ , where P = (1, -4) and Q = (3, 5). The components of  $\overrightarrow{PQ}$  are:  $\overrightarrow{PQ} = \langle 3 - 1, 5 - (-4) \rangle = \langle 2, 9 \rangle$ 

- 2. Let  $\mathbf{v} = \langle 6, 9 \rangle$ . Which of the following vectors are parallel to  $\mathbf{v}$  and which point in the same direction?
  - (a)  $\langle 12, 18 \rangle$
  - (b)  $\langle 3, 2 \rangle$
  - (c)  $\langle 2, 3 \rangle$
  - (d)  $\langle -6, -9 \rangle$
  - (e)  $\langle -24, -27 \rangle$
  - (f)  $\langle -24, -36 \rangle$

Two vectors are parallel if they are scalar multiples of each other. The vectors point in the same direction if the multiplying scalar is positive. We use this to obtain the following conclusions:

Part a:

 $\langle 12,18\rangle=2\langle 6,9\rangle=2{\bf v}\Rightarrow$  both vectors point in the same direction. Part b:

 $\langle 3,2\rangle$  is not a scalar multiple of  ${\bf v},$  hence the vectors are not parallel. Part c:  $\begin{smallmatrix}1&&&1\\&&&1\end{smallmatrix}$ 

 $\langle 2,3\rangle = \frac{1}{3}\langle 6,9\rangle = \frac{1}{3}\mathbf{v} \Rightarrow$  both vectors point in the same direction.

Part d: (6, 0) (6)

 $\langle -6, -9 \rangle = -\langle 6, 9 \rangle = -\mathbf{v} \Rightarrow$  parallel to  $\mathbf{v}$  and points in the opposite direction. Part e:

 $\langle -24,-27\rangle$  is not a scalar multiple of  ${\bf v},$  hence the vectors are not parallel. Part f:

 $\langle -24, -36 \rangle = -4\langle 6, 9 \rangle = -4\mathbf{v} \Rightarrow$  parallel to  $\mathbf{v}$  and points in the opposite direction.

- **3.** Sketch the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{PQ}$  and determine if they are equivalent. A = (1, 4), B = (-6, 3), P = (1, 4), and Q = (6, 3)We compute  $\overrightarrow{AB}$  and  $\overrightarrow{PQ}$  and see if they have the same components:  $\overrightarrow{AB} = \langle -6 - 1, 3 - 4 \rangle = \langle -7, -1 \rangle$  $\overrightarrow{PQ} = \langle 6 - 1, 3 - 4 \rangle = \langle 5, -1 \rangle$   $\Rightarrow$  The vectors are not equivalent.
- 4. Calculate  $4(\langle 1, 1 \rangle + \langle 3, 2 \rangle)$  $4(\langle 1, 1 \rangle + \langle 3, 2 \rangle) = \langle 7, 6 \rangle$

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5. Sketch  $\mathbf{v} = \langle 1, 3 \rangle$ ,  $\mathbf{w} = \langle 2, -2 \rangle$ ,  $\mathbf{v} + \mathbf{w}$ ,  $\mathbf{v} - \mathbf{w}$ .

We compute the sum  $\mathbf{v} + \mathbf{w}$  and the difference  $\mathbf{v} - \mathbf{w}$  and then sketch the vectors. This gives:

$$\mathbf{v} + \mathbf{w} = \langle 1, 3 \rangle + \langle 2, -2 \rangle$$
  
=  $\langle 1 + 2, 3 - 2 \rangle = \langle 3, 1 \rangle$   
$$\mathbf{v} - \mathbf{w} = \langle 1, 3 \rangle - \langle 2, -2 \rangle$$
  
=  $\langle 1 - 2, 3 + 2 \rangle = \langle -1, 5 \rangle$ 

**6.** Express  $\mathbf{u} = \langle 6, -2 \rangle$  as a linear combination  $\mathbf{u} = r\mathbf{v} + s\mathbf{w}$ , where  $\mathbf{v} = \langle 1, 1 \rangle$  and  $\mathbf{w} = \langle 1, -1 \rangle$ .

We have  $\langle 6, -2 \rangle = r \langle 1, 1 \rangle + s \langle 1, -1 \rangle$  Thus 6 = r + s and -2 = r - s. Adding gives 4 = 2r. Thus r = 2 and s = 4

- 7. Calculate the linear combination (-2i + 9j) + (3i 4j). (-2i + 9j) + (3i - 4j) = (-2 + 3)i + (9 - 4)j = i + 5j.
- 8. A plane flying due east at 200 km/hr encounters a 40 km/hr north-easterly wind. The resultant velocity of the plane is the vector sum  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ , where  $\mathbf{v}_1$  is the velocity vector of the plane and  $\mathbf{v}_2$  is the velocity vector of the wind. The angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is  $\frac{\pi}{4}$ . Determine the resultant speed of the plane (the length of  $\mathbf{v}$ ).

The resultant speed of the plane is the length of the sum vector  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ . We place the *xy*-coordinate system as shown in the figure, and compute the components of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . This gives:

$$\mathbf{v}_{1} = \langle v_{1}, 0 \rangle$$

$$\mathbf{v}_{2} = \langle v_{2} \cos \frac{\pi}{4}, v_{2} \sin \frac{\pi}{4} \rangle = \langle v_{2} \cdot \frac{\sqrt{2}}{2}, v_{2} \cdot \frac{\sqrt{2}}{2} \rangle$$
We now compute the sum  $\mathbf{v} = \mathbf{v}_{1} + \mathbf{v}_{2}$ :
$$\mathbf{v} = \langle v_{1}, 0 \rangle + \langle \frac{\sqrt{2}v_{2}}{2}, \frac{\sqrt{2}v_{2}}{2} \rangle = \langle \frac{\sqrt{2}}{2}v_{2} + v_{1}, \frac{\sqrt{2}}{2}v_{2} \rangle$$
The resultant speed is the length of  $\mathbf{v}$ , that is:
$$v = \|\mathbf{v}\| = \sqrt{\left(\frac{\sqrt{2}v_{2}}{2}\right)^{2} + \left(v_{1} + \frac{\sqrt{2}v_{2}}{2}\right)^{2}} = \sqrt{\frac{v_{2}^{2}}{2} + v_{1}^{2} + 2 \cdot \frac{\sqrt{2}}{2}v_{2}v_{1} + \frac{v_{2}^{2}}{2}} = \sqrt{v_{1}^{2} + v_{2}^{2} + \sqrt{2}v_{1}v_{2}}$$
Finally, we substitute the given information  $v_{1} = 200$  and  $v_{2} = 40$  in the equation above,

to obtain:  $v = \sqrt{200^2 + 40^2 + \sqrt{2} \cdot 200 \cdot 40} \approx 230 \text{km/hr}$ 

## Class Time NOT TESTED ON EITHER THE AB OR BC EXAMS.

## **Key Points**

- The dot product of  $\mathbf{v} = \langle a_1, b_1 \rangle$  and  $\mathbf{w} = \langle a_2, b_2 \rangle$  is  $\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2$ .
- By convention, the angle  $\theta$  between two vectors is chosen to satisfy  $0 \le \theta \le \pi$ .
- The dot product and the angle  $\theta$  between **v** and **w** are related by the following equations:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta, \quad \cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}, \quad \theta = \cos^{-1} \left( \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \right)$$

- To test for orthogonality,  $\mathbf{v} \perp \mathbf{w}$  if and only if  $\mathbf{v} \cdot \mathbf{w} = 0$ .
- The angle between  $\mathbf{v}$  and  $\mathbf{w}$  is acute if  $\mathbf{v} \cdot \mathbf{w} > 0$  and obtuse if  $\mathbf{v} \cdot \mathbf{w} < 0$ .
- Let  $\mathbf{v} \neq \mathbf{0}$ . Then every vector  $\mathbf{u}$  has a decomposition  $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$ , where  $\mathbf{u}_{\parallel}$  is parallel to  $\mathbf{v}$  and  $\mathbf{u}_{\perp}$  is orthogonal to  $\mathbf{v}$ .  $\mathbf{u}_{\parallel}$  is the projection of  $\mathbf{u}$  along  $\mathbf{v}$  and is denoted  $\operatorname{proj}_{\mathbf{v}}(\mathbf{u})$ .
- Let  $\mathbf{e}_{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$  be the unit vector in the direction of  $\mathbf{v}$ . Then

$$\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = (\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}})\mathbf{e}_{\mathbf{v}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}.$$

and  $\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel}$ .

• Let  $\theta$  be the angle between **u** and **v**. The component of **u** along **v** is the scalar quantity

component of **u** along 
$$\mathbf{v} = \mathbf{u} \cdot \mathbf{e}_{\mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\| \cos \theta$$
.

### Lecture Material

The dot product  $\mathbf{v} \cdot \mathbf{w}$  of two vectors  $\mathbf{v} = \langle a_1, b_1 \rangle$  and  $w = \langle a_2, b_2 \rangle$  is  $\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2$ . Point out that the dot product is extremely important in multivariable calculus, as the students will see later. Discuss the elementary properties of the dot product that are summarized in Theorem 2, and work Exercises 2 and 38. Now derive Theorem 3, which gives the relationship between the dot product and the angle  $\theta$  between two non-zero vectors  $\mathbf{v}$  and  $\mathbf{w}$ :

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$
 or  $\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$ 

Remark that while the angle between two vectors is not unique, we will adopt the standard convention that this angle satisfies  $0 \le \theta \le \pi$ . Of course, we may now solve the preceding equation for  $\theta$  and obtain

$$\theta = \cos^{-1} \left( \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \right).$$

Two non-zero vectors  $\mathbf{v}$  and  $\mathbf{w}$  are perpendicular or orthogonal, written  $\mathbf{v} \perp \mathbf{w}$ , if the angle  $\theta$  between them is  $\pi/2$ . Hence  $\mathbf{v} \perp \mathbf{w}$  if and only if  $\mathbf{v} \cdot \mathbf{w} = 0$ . Similarly, point out that  $\theta$  is obtuse  $(\pi/2 < \theta \leq \pi)$  if  $\mathbf{v} \cdot \mathbf{w} < 0$ . Now work Exercises 14 and 24.

Given two vectors  $\mathbf{v} \neq \mathbf{0}$  and  $\mathbf{w}$ , it is possible to write  $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$ , where  $\mathbf{u}_{\parallel}$  is parallel to  $\mathbf{v}$  and  $\mathbf{u}_{\perp}$  is perpendicular to  $\mathbf{v}$  (see Figure 6). The vector  $\mathbf{u}_{\parallel}$  is the projection of  $\mathbf{u}$  along  $\mathbf{v}$ , and is denoted by  $\operatorname{proj}_{\mathbf{v}}(\mathbf{u})$ . Show that

$$\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = (\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}})\mathbf{e}_{\mathbf{v}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$$

Note that we may find  $\mathbf{u}_{\perp}$  as  $\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel}$  (and verify that  $\mathbf{u}_{\perp}$  is orthogonal to  $\mathbf{v}$ ). The scalar  $\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}}$  is the component of  $\mathbf{u}$  along  $\mathbf{v}$  and  $\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\| \cos \theta$ .

## Discussion Topics/Class activities

Work Exercise 40, where the Law of Cosines is derived.

## Selected Problems

Exercises 1–9 every other odd (computational), 13–17 odd (computational), 23–27 odd (computational), 37, 39, (computational), 41–47 odd (computational), 49–53 odd (computational)

# Worksheet 11.6. Dot Product and the Angle Between Two Vectors

- **1.** Compute the dot product  $\langle 3, -2, 2 \rangle \cdot \langle 1, 0, 1 \rangle$ .
- 2. Simplify the expression  $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) 2\mathbf{v} \cdot \mathbf{w}$
- **3.** Determine if the vectors  $\langle 1, 1, 1 \rangle$  and  $\langle 3, -2, -1 \rangle$  are orthogonal and if not, whether the angle between them is acute or obtuse.
- 4. Find the angle between the vectors  $\langle 1, 1, 5 \rangle$  and  $\langle 1, -1, 5 \rangle$ .
- **5.** Find the projection  $\operatorname{proj}_{\mathbf{v}}(\mathbf{u})$ , where  $u = \langle 2, 0 \rangle$  and  $\mathbf{v} = \langle 4, 3 \rangle$ .
- 6. Find the decomposition  $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$  with respect to  $\mathbf{b}$ , where  $\mathbf{a} = \langle 4, -1, 0 \rangle$  and  $\mathbf{b} = \langle 0, 1, 1 \rangle$ .

## Solutions to Worksheet 11.6

- 1. Compute the dot product  $\langle 3, -2, 2 \rangle \cdot \langle 1, 0, 1 \rangle$ . By the definition of the dot product we have  $\langle 3, -2, 2 \rangle \cdot \langle 1, 0, 1 \rangle = 3 \cdot 1 + (-20 \cdot 0 + 2 \cdot 1 = 5)$ .
- 2. Simplify the expression  $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) 2\mathbf{v} \cdot \mathbf{w}$ Using properties of the dot product we obtain:  $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) - 2\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot (\mathbf{v} + \mathbf{w}) + \mathbf{w} \cdot (\mathbf{v} + \mathbf{w}) - 2\mathbf{v} \cdot \mathbf{w}$   $= \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w}$   $= \|\mathbf{v}\|^2 + \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2 - 2\mathbf{v} \cdot \mathbf{w}$  $= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$
- **3.** Determine if the vectors  $\langle 1, 1, 1 \rangle$  and  $\langle 3, -2, -1 \rangle$  are orthogonal and if not, whether the angle between them is acute or obtuse.

We compute the dot product of the two vetors,  $\langle 1, 1, 1 \rangle \cdot \langle 3, -2, -1 \rangle = 0$ . Thus the vectors are orthogonal.

4. Find the angle between the vectors (1, 1, 5) and (1, -1, 5).

We denote  $\mathbf{v} = \langle 1, 1, 5 \rangle$ ,  $\mathbf{w} = \langle 1, -1, 5 \rangle$ . To use the formula for the cosine of the angle between two vectors we first compute the following values:

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}$$
$$\|\mathbf{w}\| = \sqrt{1^2 + (-1)^2 + 5^2} = \sqrt{27}$$
$$\mathbf{v} \cdot \mathbf{w} = \langle 1, 1, 5 \rangle \cdot \langle 1, -1, 5 \rangle$$
$$= 1 \cdot 1 + 1 \cdot (-1) + 5 \cdot 5 = 25$$
Hence,
$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{25}{\sqrt{27} \cdot \sqrt{27}} = \frac{25}{27}$$

- $\Rightarrow \theta = \cos^{-1}\left(\frac{25}{27}\right) \approx 22.19^{\circ}$ 5. Find the projection  $\operatorname{proj}_{\mathbf{v}}(\mathbf{u})$ , where  $u = \langle 2, 0 \rangle$  and  $\mathbf{v} = \langle 4, 3 \rangle$ .
  - We compute the dot products:  $\mathbf{u} \cdot \mathbf{v} = \langle 2, 0 \rangle \cdot \langle 4, 3 \rangle = 2 \cdot 4 + 0 \cdot 3 = 8$   $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 4^2 + 3^2 = 25$ The projection of  $\mathbf{u}$  along  $\mathbf{v}$  is the following vector:  $\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = \frac{8}{25} \langle 4, 3 \rangle = \langle \frac{32}{25}, \frac{24}{25} \rangle.$

6. Find the decomposition  $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$  with respect to  $\mathbf{b}$ , where  $\mathbf{a} = \langle 4, -1, 0 \rangle$  and  $\mathbf{b} = \langle 0, 1, 1 \rangle$ .

We first compute  $\mathbf{a} \cdot \mathbf{b}$  and  $\mathbf{b} \cdot \mathbf{b}$  to find the projection of  $\mathbf{a}$  along  $\mathbf{b}$ :  $\mathbf{a} \cdot \mathbf{b} = \langle 4, -1, 0 \rangle \cdot \langle 0, 1, 1 \rangle = 4 \cdot 0 + (-1) \cdot 1 + 0 \cdot 1 = -1$   $\mathbf{b} \cdot \mathbf{b} = \|\mathbf{b}\|^2 = 0^2 + 1^2 + 1^2 = 2$ Hence,  $\mathbf{a}_{\parallel} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} = \frac{-1}{2} \langle 0, 1, 1 \rangle = \langle 0, -\frac{1}{2}, -\frac{1}{2} \rangle$ We now find the vector  $\mathbf{a}_{\perp}$  orthogonal to  $\mathbf{b}$  by computing the difference:  $\mathbf{a} - \mathbf{a}_{\parallel} = \langle 4, -1, 0 \rangle - \langle 0, -\frac{1}{2}, -\frac{1}{2} \rangle = \langle 4, -\frac{1}{2}, \frac{1}{2} \rangle$ We, thus, have:  $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp} = \langle 0, -\frac{1}{2}, -\frac{1}{2} \rangle + \langle 4, -\frac{1}{2}, \frac{1}{2} \rangle$ 

## 11.7. Calculus of Vector-Valued Functions.

#### Class Time AB 0 periods; BC 2 periods. Essential.

For the BC Calculus exam, only parametric and vector functions in the plane are required. Students should be able to work with the velocity vector as the derivative of the position vector or the derivative of the curve given in parametric form.

## **Key Points**

- Limits, differentiation and integration of vector-valued functions are performed componentwise.
- The derivative  $\mathbf{r}'(t_0)$  is called the tangent vector or velocity vector at  $r(t_0)$ .
- If  $\mathbf{r}'(t_0) \neq 0$ , then the tangent line at  $t_0$  has vector parameterization

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0) \text{ for all } t \in \mathbb{R}.$$

- If  $\mathbf{R}_1(t)$  and  $\mathbf{R}_2(t)$  are differentiable, vector-valued functions such that  $\mathbf{R}'_1(t) = \mathbf{R}'_2(t)$  on [a, b], then  $\mathbf{R}_2(t) = \mathbf{R}_1(t) + \mathbf{c}_0$  on [a, b] for some constant vector  $\mathbf{c}_0$ .
- The Fundamental Theorem for vector-valued functions: if  $\mathbf{r}(t)$  is continuous on [a, b] and if  $\mathbf{R}(t)$  is an antiderivative of  $\mathbf{r}(t)$ , then

$$\int_{a}^{b} \mathbf{r}(t) \, dt = \mathbf{R}(b) - \mathbf{R}(a)$$

#### Lecture Material

This section is about extending limits, continuity, differentiation and integration to threedimensional vector-valued functions. A vector-valued function  $\mathbf{r}(t)$  approaches a vector v as t approaches  $t_0$ , if  $\lim_{t\to t_0} ||\mathbf{r}(t) - \mathbf{v}|| = 0$ , and we write  $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{v}$ . Use the slide provided to illustrate this definition. Prove Theorem 2 that vector-valued limits are computed componentwise. This is actually the most important concept in this section. Work Exercises 2 and 4.

Thus continuity, differentiation and integration are defined componentwise, that is,  $\mathbf{r}(t)$  is continuous at t if and only if each component function is continuous at t,  $\mathbf{r}(t)$  is differentiable at t if and only if each component function is differentiable at t, and  $\mathbf{r}(t)$  is integrable on [a, b] at t if and only if each component function is integrable on [a, b]. Work Exercises 8, 12, and 40

Review the Sum Rule, Constant Multiple rule and Chain Rule and work Exercises 8 and 12.

If  $\mathbf{r}(t)$  is a vector-valued function then  $\mathbf{r}'(t_0)$  is the vector tangent to the curve parameterized by  $\mathbf{r}(t)$  at the point  $r(t_0)$ . Thus the tangent line at that point is  $\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$ for all  $t \in \mathbb{R}$ . Work Exercise 14. It is interesting to note that  $\mathbf{r}(t)$  has constant length if and only if  $\mathbf{r}(t)$  is orthogonal to  $\mathbf{r}'(t)$ . This is the content of example 6. Example 7 indicates one difference between vector and scalar-valued derivatives by finding the points on a planar curve where the tangent line is horizontal.

An antiderivative of  $\mathbf{r}(t)$  is a vector-valued function  $\mathbf{R}(t)$  such that  $\mathbf{R}'(t) = \mathbf{r}(t)$ . Similar to the scalar-valued case, if two vector-valued functions with the same derivative on an interval just differ by a constant vector. Finally, end with the Fundamental Theorem of Calculus for Vector-Valued Functions and work Exercises 42 and 44.

The position vector  $\langle x(t), y(t) \rangle$  gives the same graph as the parametric curve x = x(t), y = y(t). The derivative of this vector  $\langle x'(t), y'(t) \rangle$  gives the velocity vector of a point moving on the parametric curve; the derivative of the velocity vector  $\langle x''(t), y''(t) \rangle$  is the acceleration vector of the moving point. Use questions from past AP exams to explore these ideas. These ideas are tested every year on the BC exam.

#### Suggested Problems

Exercises 1, 5, 7–15 odd, supplement with questions from the AP exams

# Worksheet 11.7. Calculus of Vector-Valued Functions

1. Evaluate the following limits. a.  $\lim_{t \to \pi} \sin 2t\mathbf{i} + \cos t\mathbf{j} + \tan 4t\mathbf{k}$ 

b. 
$$\lim_{t \to 0} \frac{1}{t+1} \mathbf{i} + \frac{e^t - 1}{t} \mathbf{j} + 4t \mathbf{k}$$

2. Compute the derivatives of the following vector-valued functions with respect to t. a.  $\mathbf{v}(t) = \langle \sin 3t, \cos t \rangle$ 

b. 
$$\mathbf{c}(t) = t^{-1}\mathbf{i} - e^{2t}\mathbf{k}$$

**3.** Evaluate 
$$\int_0^1 \langle 2t, 4t, -\cos t \rangle dt$$
.

# Solutions to Worksheet 11.7

- 1. Evaluate the following limits.
  - a.  $\lim_{\substack{t \to \pi \\ \lim_{t \to \pi} \sin 2t\mathbf{i} + \cos t\mathbf{j} + \tan 4t\mathbf{k} = -\mathbf{j}}$

b. 
$$\lim_{t \to 0} \frac{1}{t+1} \mathbf{i} + \frac{e^t - 1}{t} \mathbf{j} + 4t \mathbf{k}$$
$$\lim_{t \to 0} \frac{1}{t+1} \mathbf{i} + \frac{e^t - 1}{t} \mathbf{j} + 4t \mathbf{k} = \mathbf{i} + \mathbf{j}$$

# **2.** Compute the derivatives of the following vector-valued functions with respect to t.

a. 
$$\mathbf{v}(t) = \langle \sin 3t, \cos t \rangle$$
  
 $\mathbf{v}'(t) = \langle 3 \cos 3t, -\sin t \rangle$ .  
b.  $\mathbf{c}(t) = t^{-1}\mathbf{i} - e^{2t}\mathbf{k}$   
 $\mathbf{c}'(t) = -t^{-2}\mathbf{i} - 2e^{2t}\mathbf{k}$ .

**3.** Evaluate  $\int_0^1 \langle 2t, 4t, -\cos t \rangle dt$ .

The vector valued integration is defined via componentwise integration. Therefore,

$$\begin{aligned} \int_{0}^{1} \langle 2t, 4t, -\cos 3t \rangle \ dt &= \left\langle \int_{0}^{1} 2t \ dt, \int_{0}^{1} 4t \ dt, \int_{0}^{1} -\cos 3t \ dt \right\rangle \\ &= \left\langle t^{2}|_{0}^{1}, 2t^{2}|_{0}^{1}, -\frac{\sin 3t}{3}|_{0}^{1} \right\rangle \\ &= \left\langle 1, 2, -\frac{\sin 3}{3} \right\rangle \end{aligned}$$

# Chapter 11 AP Problems

For 4 and 6, a calculator may be used. For 1, 2, 3 and 5, no calculator allowed.

- 1. Which of the following gives the slope of the line tangent to the graph of the relation given by the parametric equations  $x = 4t^2 6t + 2$  and  $y = -3t^2$ ?
  - A. 8t 6B. -6tC.  $\frac{4t - 3}{-3t}$ D.  $\frac{-3t}{4t - 3}$ E.  $-\frac{3}{4}$

2. Which of the following gives the length of the path described by the parametric equations  $x = \sin^2 t$  and  $y = \tan(2t)$  from  $x = \frac{\pi}{4}$  to  $x = \frac{3\pi}{4}$ ?

A. 
$$\int_{\pi/4}^{3\pi/4} \sqrt{\sin^4 t + \tan^2(2t)} dt$$
  
B. 
$$\int_{\pi/4}^{3\pi/4} \sqrt{4\cos^2 t + 4\sec^2(2t)} dt$$
  
C. 
$$\int_{\pi/4}^{3\pi/4} \sqrt{4\sin^2 t \cos^2 t + 4\sec^4(2t)} dt$$
  
D. 
$$\int_{\pi/4}^{3\pi/4} \sqrt{4\cos^2 t + 4\sec^4(2t)} dt$$
  
E. 
$$\int_{\pi/4}^{3\pi/4} \sqrt{2\sin t \cos t + 2\sec^2(2t)} dt$$

**3.** Which of the following gives the polar equation for xy = 4?

A. 
$$r = 4 \sin \theta \cos \theta$$
  
B.  $r = 4 \sec \theta \csc \theta$   
C.  $r = 2\sqrt{\sin \theta \cos \theta}$   
D.  $r = \frac{4}{\sin \theta \cos \theta}$   
E.  $r = 2\sqrt{\sec \theta \csc \theta}$ 

4. What is the area inside one petal of the rose curve  $r = 4\cos(2\theta)$ ?

- A. 1B. π
- C. 4
- D.  $2\pi$
- E.  $4\pi$

5. Which of the following gives the area inside  $r = 2 + 2\sin\theta$  and outside  $r = 4\sin\theta$ ?

A. 
$$\frac{1}{2} \int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)^2) d\theta$$
  
B. 
$$\int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)^2) d\theta$$
  
C. 
$$\int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)) d\theta$$
  
D. 
$$2 \int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)^2) d\theta$$
  
E. 
$$\int_{\pi/2}^{3\pi/2} ((2+2\sin\theta) - (4\sin\theta)) d\theta$$

**6.** The figure below shows the graph of  $f(y) = \frac{3}{2}y$  and  $g(y) = \sqrt{4 - y^2} + 2$ . Let R be the region bounded by the two curves and the x-axis.



a. Set up and evaluate an integral expression that gives the area of R with respect to y.

b. g(y) is part of the curve  $(x-2)^2 + y^2 = 4$ . Find the polar equation for this curve.

- c. Use the polar equation you found in part b to set up an integral expression that gives the area of R with respect to  $\theta$ .
- 7. Which of the following vectors has length  $2\sqrt{3}$  and makes and angle of  $\frac{\pi}{6}$  with the x-axis?
  - I.  $\langle 3, \sqrt{3} \rangle$ II.  $\langle 2, 2\sqrt{2} \rangle$ III.  $\langle -\sqrt{3}, -1 \rangle$
  - A. I only

- B. I and II only
- C. II only
- D. I and III only
- E. I, II and III

8. Suppose  $\mathbf{v} = \langle 3, -4 \rangle$ . If  $\mathbf{u}$  is a unit vector perpendicular to  $\mathbf{v}$ , then  $\mathbf{u}$  could be

A. 
$$\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$
  
B.  $\left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle$   
C.  $\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$   
D.  $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$   
E.  $\langle 4, 3 \rangle$ 

**9.** If 
$$r(t) = \langle t^3, 2t - 1 \rangle$$
, then  $r'(2) =$ 

A. 4

- B.  $3\sqrt{17}$
- C.  $\langle 8,3\rangle$
- D.  $\langle 12, 2 \rangle$
- E.  $\langle 12, 3 \rangle$

- **10.** The path of a particle satisfies  $\frac{dr}{dt} = \langle 4t^3, e^t \rangle$ . If  $r(0) = \langle 1, 4 \rangle$ , then what is the location of r(2)?
  - A.  $\langle 0, 1 \rangle$ B.  $\langle 1, 3 \rangle$ C.  $\langle 4, e^4 \rangle$
  - D.  $\langle 17, e^2 + 3 \rangle$
  - E.  $\langle 19, e^5 \rangle$

# Solutions to Chapter 11 AP Problems

- 1. Which of the following gives the slope of the line tangent to the graph of the relation given by the parametric equations  $x = 4t^2 6t + 2$  and  $y = -3t^2$ ?
  - A. 8t 6B. -6tC.  $\frac{4t - 3}{-3t}$ D.  $\frac{-3t}{4t - 3}$ E.  $-\frac{3}{4}$ D [THIS PROBLEM CORRESPONDS WITH SECTION 11.1]
- 2. Which of the following gives the length of the path described by the parametric equations  $x = \sin^2 t$  and  $y = \tan(2t)$  from  $x = \frac{\pi}{4}$  to  $x = \frac{3\pi}{4}$ ?
  - A.  $\int_{\pi/4}^{3\pi/4} \sqrt{\sin^4 t + \tan^2(2t)} dt$ B.  $\int_{\pi/4}^{3\pi/4} \sqrt{4\cos^2 t + 4\sec^2(2t)} dt$ C.  $\int_{\pi/4}^{3\pi/4} \sqrt{4\sin^2 t \cos^2 t + 4\sec^4(2t)} dt$ D.  $\int_{\pi/4}^{3\pi/4} \sqrt{4\cos^2 t + 4\sec^4(2t)} dt$ E.  $\int_{\pi/4}^{3\pi/4} \sqrt{2\sin t \cos t + 2\sec^2(2t)} dt$

C [THIS PROBLEM CORRESPONDS WITH SECTION 11.2]

**3.** Which of the following gives the polar equation for xy = 4?

A.  $r = 4 \sin \theta \cos \theta$ B.  $r = 4 \sec \theta \csc \theta$ C.  $r = 2\sqrt{\sin \theta \cos \theta}$ D.  $r = \frac{4}{\sin \theta \cos \theta}$ E.  $r = 2\sqrt{\sec \theta \csc \theta}$ E [THIS PROBLEM CORRESPONDS WITH SECTION 11.3]

- 4. What is the area inside one petal of the rose curve  $r = 4\cos(2\theta)$ ?
  - A. 1B. π
  - C. 4
  - D.  $2\pi$
  - E.  $4\pi$

## D [THIS PROBLEM CORRESPONDS WITH SECTION 11.4]

5. Which of the following gives the area inside  $r = 2 + 2\sin\theta$  and outside  $r = 4\sin\theta$ ?

A. 
$$\frac{1}{2} \int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)^2) d\theta$$
  
B.  $\int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)^2) d\theta$   
C.  $\int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)) d\theta$   
D.  $2 \int_{\pi/2}^{3\pi/2} ((2+2\sin\theta)^2 - (4\sin\theta)^2) d\theta$   
E.  $\int_{\pi/2}^{3\pi/2} ((2+2\sin\theta) - (4\sin\theta)) d\theta$ 

B [THIS PROBLEM CORRESPONDS WITH SECTION 11.4]

6. The figure below shows the graph of  $f(y) = \frac{3}{2}y$  and  $g(y) = \sqrt{4 - y^2} + 2$ . Let R be the region bounded by the two curves and the x-axis.



a. Set up and evaluate an integral expression that gives the area of R with respect to y.

Let A = 1.846153846 (y-value at point of intersection). Then

Area = 
$$\int_0^A (g(y) - f(y)) \, dy = 4.198$$

## [THIS PROBLEM CORRESPONDS WITH SECTION 6.1]

b. g(y) is part of the curve  $(x-2)^2 + y^2 = 4$ . Find the polar equation for this curve.

Substituting  $x = r \cos \theta$  and  $y = r \sin \theta$ , we obtain  $r = 4 \cos \theta$ . [THIS PROBLEM CORRESPONDS WITH SECTION 11.3]

c. Use the polar equation you found in part b to set up an integral expression that gives the area of R with respect to  $\theta$ .

Area 
$$=\frac{1}{2}\int_{0}^{\tan^{-1}(2/3)} (4\cos\theta)^2 d\theta$$
  
[THIS PROBLEM CORRESPONDS WITH SECTION 11.4]

Area Formulas	Circumference
Transformations of sin(x), $cos(x)$ , csc(x), $sec(x)$	Transformations of tan( <i>x</i> ) and cot( <i>x</i> )
Finding an Inverse $f^{-1}(x)$	Verifying Inverses
2 Properties of Inverses	Even/Odd Functions
Examples of Even Functions	Examples of Odd Functions

	Circle: $A = \pi r^2$
$C=2\pi r$	Trapezoid: $A = \frac{h}{2}(b_1 + b_2)$
	Triangle: $A = \frac{1}{2}bh$
$C = \pi d$	Cylinder: $SA = 2\pi r^2 + 2\pi rh$
	Sphere: $SA=4\pi r^2$
$y = a \tan(bx) + c$	$y = a \sin(bx) + c$
Amplitude = $ a $	Amplitude = $ a $
Period = $\frac{\pi}{h}$	Period = $\frac{2\pi}{2\pi}$
Vertical Shift = $c$	b Vertical Shift = c
*Same goes for cot( <i>x</i> )	*Same goes for $\cos(x)$ , $\sec(x)$ , and $\csc(x)$
<ol> <li>You can verify that two functions are inverses algebraically by seeing if their composition equals <i>x</i>. <i>f</i>(<i>f</i><sup>-1</sup>(<i>x</i>)) = <i>x</i> or <i>f</i><sup>-1</sup>(<i>f</i>(<i>x</i>)) = <i>x</i></li> <li>You can verify that two functions are inverses graphically by seeing if they are symmetrical about the line <i>y</i> = <i>x</i>.</li> </ol>	<ol> <li>Interchange x and y in the equation</li> <li>Solve for y. (This will not be possible if y cannot be written as a function of x)</li> <li>Replace y with f<sup>-1</sup>(x)</li> </ol>
Even: Symmetrical about the y-axis $f(-x) = f(x)$ Odd: Symmetrical about the origin 	<ol> <li>If the point (<i>a</i>, <i>b</i>) is on the graph of <i>f</i>(<i>x</i>), the point (<i>b</i>, <i>a</i>) is on the graph of <i>f</i><sup>-1</sup>(<i>x</i>).</li> <li>The slopes of inverse functions are reciprocals.</li> </ol>
<b>Example:</b> $y = x^2$ <i>y</i> is the same whether you plug in <i>x</i> or $-x$ $(-1, -1)$ <b>Example:</b> $y = x^3$ <i>y</i> is the opposite sign when you plug in <i>x</i> or $-x$	Ex: Suppose $f(x)$ and $g(x)$ are inverses 1. If $f(2)=8$ , then $g(8)=2$ If $f'(2)=3/4$ , then $g'(8)=4/3$
$y = \sin x$	$y = x^2$
$y = \frac{x}{2}$	$y = \cos x$
$x^2 + 1$ $y = x^5 - x^3$	$y = \sqrt{9 - x^2}$ (Semi-Circle)

Finding Domain	Function Recognition $y = e^x$ and $y = \ln(x)$
Parallel Lines Perpendicular (Normal) Lines	Strategies for Finding a Limit
Horizontal Asymptote	Vertical Asymptote
A Function Is Continuous If	Types of Discontinuities
Average Rate of Change	Instantaneous Rate of Change

y = $e^x$ Note: These functions are inverses. Notice the symmetry about the line y = x	<ul> <li>You cannot divide by zero</li> <li>You cannot take the square root of a negative number</li> <li>You cannot take the ln or log of a negative number or zero.</li> <li>Example: y = √9 - x<sup>2</sup> Domain: [-3, 3] y = ln(x) Domain: (0, ∞)</li> </ul>
<ol> <li>Direct Substitution</li> <li>Factor and Cancel</li> <li>End Behavior Model (EBM). Use for lim f(x)         <ul> <li>x→±∞</li> </ul> </li> <li>Rationalize the Numerator</li> <li>L'Hôpital's Rule</li> <li>Compare Left &amp; Right Hand Limits (Piecewise Functions)</li> </ol>	Parallel lines have the same slope Perpendicular lines have negative reciprocal slopes
The line $x = a$ is a Vertical Asymptote of the graph of a function if either $\lim_{x \to a^-} f(x) = \pm \infty$ or $\lim_{x \to a^+} f(x) = \pm \infty$ . Vertical Asymptotes exist at values of x that cause the function to be undefined. Ex: $f(x) = \frac{1}{x}$ has a VA at $x = 0$	The line $y = b$ is a Horizontal Asymptote of the graph of a function if either $\lim_{x \to \infty} f(x) = b$ Or $\lim_{x \to -\infty} f(x) = b$ . Ex. $y = \underbrace{\frac{2x}{x+1}}_{x+1}$ has a HA of $y = 2$ .
<ol> <li>Removable (Hole)</li> <li>Jump</li> <li>Infinite (Asymptote)</li> <li>Oscillating</li> <li>Note: If a function is differentiable at a given point it is also continuous at that point.</li> <li>If a function is continuous at a given point it may or may not be differentiable at that point.</li> </ol>	$\lim_{x \to c} f(x) = f(c)$ The limit of $f(x)$ as x approaches c equals the value at c, and they are finite. Ex. $y = \begin{cases} x & x < 3 \\ -x+6 & x \ge 3 \end{cases}$ is continuous at x=3 because $\lim_{x \to 3^{-}} (x) = \lim_{x \to 3^{+}} (-x+6) = f(3)$
Derivative Slope of a Tangent Line On the Calculator: [MATH] [8]	Slope = $\frac{f(b) - f(a)}{b - a}$ Slope of a Secant Line
Intermediate Value Theorem	Mean Value Theorem
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Piecewise Functions Continuous? Differentiable?	Position Velocity Acceleration
Particle Motion	Definition of a Derivative
Implicit Differentiation Tips	Derivative of $e^x$ , $a^x$ , $\ln x $ , and $\log x$
Derivative of Inverse Trigonometric Functions	Trig Derivatives

Basic Rules For Differentiation	Basic Rules For Differentiation (Continued)
Product and Quotient Rule For Differentiation	Derivative of ln[ <i>f</i> ( <i>x</i> )]
First Derivative Test	Second Derivative Test
Second Derivative Test for Relative Extrema	Extreme Value Theorem
Volume Formulas (Often used for Related Rates Problems)	Relative Max Relative Min (Justify your answer)

<ul> <li>4.  d/dx [cf(x)] = c f'(x) The derivative of a function times a constant multiple is the constant multiple times its derivative.</li> <li>5.  d/dx [f(x) + g(x)] = f'(x) + g'(x) The derivative of a sum is the sum of the derivatives.</li> </ul>	1. $\frac{d}{dx}(c) = 0$ 2. $\frac{d}{dx}(x) = 1$ 3. $\frac{d}{dx}(x^n) = nx^{n-1}$ The derivative of x is 1 Power Rule
$\frac{d}{dx} \ln[f(x)] = \frac{f'(x)}{f(x)}$	Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
<ul> <li>The second derivative f''(x) of a function can be used to determine where f(x) is:</li> <li>1. Concave Up (f''(x) is positive)</li> <li>2. Concave Down (f''(x) is negative)</li> <li>3. At an Inflection Point (f''(x) changes sign, + to - or - to +)</li> </ul>	<ul> <li>FIRST determine where f'(x)=0 and where f'(x)DNE. State the domain &amp; draw a sign line.</li> <li><u>The original function, f(x), is</u>: <ol> <li>Increasing when f'(x) is positive</li> <li>Decreasing when f'(x) is negative</li> <li>At a maximum when f'(x) goes from + to -</li> </ol> </li> </ul>
If a function is continuous on a closed interval [a, b] then it has a maximum and a minimum value on the interval. No greater value of fination. No greater value of fination. No smaller value of fination. No smalle	This is an alternate finate finate for a function. 1. If $f'(c) = 0$ , and $f''(c) < 0$ , <i>f</i> has a local max at $x = c$ . 2. If $f'(c) = 0$ , and $f''(c) > 0$ , <i>f</i> has a local min at $x = c$ .
$f(x) \text{ has a relative max at } x = \because$ $f'(x) \text{ goes from + to - at } x = \$ $f(x) \text{ has a relative min at } x = \because$ $f'(x) \text{ goes from - to + at } x = \$	Cube: $V = x^3$ Cylinder: $V = \pi r^2 h$ Cone: $V = (1/3) \pi r^2 h$ Sphere: $V = (4/3) \pi r^3$ Rectangular Prism: $V = LWH$

Newton's Method	RAM
Trapezoidal Rule (Uneven Intervals)	Trapezoidal Rule (Even Intervals)
Average Value of a Function	Integration Techniques
Integration of Basic Trigonometric Functions	Integration of Basic Trigonometric Functions (Continued)
Integration of tan( <i>u</i> ) and cot( <i>u</i> )	Integration of $\frac{1}{u}, e^{u}, a^{u}$

Rectangular Approximation Method (Estimates Area Under a Curve) $\overline{\text{Time(sec)}}$ 1015306090 $\overline{\text{Rate(m/s)}}$ 1520253035LRAM=(5)(15)+(15)(20)+(30)(25)+(30)(30)RRAM = (5)(20)+(15)(25)+(30)(30)+(30)(35)MRAM = (30-10)(20) + (90-30)(30)	<ul> <li>Used for determining the zeros of a function.</li> <li>1. Call the original function f(x). Calculate the derivative f'(x)</li> <li>2. Evaluate x<sub>n+1</sub> = x<sub>n</sub> - f(x<sub>n</sub>)/f'(x<sub>n</sub>) using the initial approximation to find x<sub>2</sub>.</li> <li>3. Repeat!</li> </ul>
$T = \frac{h}{2} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$ $\boxed{\frac{\text{Time(sec)}}{\text{Rate(m/s)}} \frac{10}{15} \frac{15}{20} \frac{25}{25} \frac{30}{30} \frac{35}{35}}{15}$ $T = (5/2) [15 + 2(20) + 2(25) + 2(30) + 35] T = 500 \text{ meters}}$	Time(sec)       10       15       30       60       90         Rate(m/s)       15       20       25       30       35 $T = \frac{1}{2}(15+20)(5) + \frac{1}{2}(20+25)(15)$ (uneven intervalsuse individual trapezoids)
<ol> <li>1. Anti-derivatives</li> <li>2. U-Substitution</li> <li>On the Calculator: [MATH] [9]</li> </ol>	$av(f) = \frac{1}{b-a} \int_{a}^{b} f(x) dx$ $y = f(x)$ $y = f(x)$ $f(x) = \frac{1}{b-a} \int_{a}^{b} f(x) dx$ $y = f(x)$ $f(x) = \frac{1}{b-a} \int_{a}^{b} f(x) dx$ $y = f(x)$ $f(x) = \frac{1}{b-a} \int_{a}^{b} f(x) dx$ $y = f(x)$
$\int \sec u \tan u  du = \sec u + C$ $\int \csc u \cot u  du = -\csc u + C$ $\int \csc^2 u  du = -\cot u + C$	$\int \sin u  du = -\cos u + C$ $\int \cos u  du = \sin u + C$ $\int \sec^2 u  du = \tan u + C$
$\int \frac{du}{u} = \ln  u  + C$ $\int e^{u} du = e^{u} + C$ $\int a^{u} du = \frac{a^{u}}{\ln a} + C$	$\int \tan u  du = \ln  \sec u  + C$ $\int \cot u  du = \ln  \sin u  + C$

Integration of Logarithmic Forms	Integration Power Rule
Fundamental Theorem of Calculus Part I (Example Problem)	Volumes of Revolution (Around the <i>x</i> -axis) Disks!
Volume by Cross Sections	Volumes of Revolution (Around the <i>x</i> -axis) Washers!
Interpretation of $\int_{0}^{b}  R(t)  dt$	Interpretation of $\int_{0}^{b} R(t) dt$
Interpretation of $\frac{1}{b-a}\int_{a}^{b} f'(x) dx$	L'Hôpital's Rule

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$$\int u^{n} du = \frac{u^{n+1}}{n+1} + C, \quad n \neq 1$$

$$\int u^{n} du = \frac{u^{n+1}}{n+1} + C, \quad n \neq 1$$

$$\int u^{n} \ln u \, du = u \ln u - u + C$$

$$\int u^{n} \ln u \, du = u \ln u - u + C$$

$$\int \frac{1}{u \ln u} du = \ln |\ln u| + C$$
If f is the function given by
$$f(x) = \int_{4}^{2r} \sqrt{r^{2} - t} \, dt, \text{ then } f'(2) = V$$
You are asked to evaluate the derivative of an anti-derivative!
$$f'(x) = \sqrt{(2x)^{2} - 2x \cdot 2}$$

$$f'(2) = \sqrt{(4x)^{2} - 4 \cdot 2} = 2\sqrt{12}$$
I. Find the limits of integration
$$V = \pi \int_{a}^{b} R^{2} - r^{2} dx$$
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$$V = \pi \int_{a}^{b} R^{2} - r^{2} dx$$
Suppose  $R(t)$  is the rate, in miles per minute, that a student rides a bike.
$$\int_{a}^{b} |R(t)| dt$$
Interpretation: This is the net distance (in miles) the student travels during the first b minutes.
Suppose that
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \infty$$
then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$
Suppose the interval is the rate at which the water enters the tank (context) in gallons/min (units) over the interval [a, b]

Fundamental Theorem of Calculus – Part I	Fundamental Theorem of Calculus – Part II
General Power Rule & Constant Multiple Rule For Differentiation	Derivatives of Inverses
U-Substitution	Integration of $sec(u)$ and $csc(u)$
Integration of Inverse Trig Forms	Integration of Exponential Forms
Integration of Exponential/ Trigonometric Forms	Integration by Parts Formula

Assume that $f(x)$ is continuous on $[a, b]$ and let $F(x)$ be an antiderivative of $f(x)$ on $[a, b]$ . Then $\int_{a}^{b} f(x)dx = F(b) - F(a)$	Assume that $f(x)$ is a continuous function on $[a, b]$ Then the area function $A(x) = \int_{a}^{x} f(t)dt$ is an antiderivative of $f(x)$ , that is $A'(x) = f(x)$ or equivalently $\frac{d}{dx} \int_{a}^{x} f(t)dt = f(x)$ . Furthermore, A(x) satisfies the initial condition $A(a) = 0$
$g'(x) = \frac{1}{f'(g(x))}$ Where $g(x)$ is the inverse of $f^{-1}(x)$	General Power Rule: $\frac{d}{dx}f(x)^{n} = nf(x)^{n-1}f'(x)$ Constant Multiple Rule: $\frac{d}{dx}f(kx+b) = kf'(kx+b)$
$\int \sec u  du = \ln   \sec u + \tan u   +C$ $\int \csc u  du = \ln   \csc u - \cot u   +C$	If an integral has the form $f(u(x))u'(x)$ , then rewrite the entire integral in terms of $u$ and its differential $du = u'(x) dx$ : $\int f(u(x))u'(x) dx = \int f(u) du$
$\int ue^{au} du = \frac{1}{a^2} (au-1)e^{au} + C$ $\int u^n e^{au} du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au} du$	$\int \frac{du}{\sqrt{a^2 - u^2}} du = \sin^{-1} \frac{u}{a} + C$ $\int \frac{du}{a^2 + u^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$
$\int u(x)v'(x)dx =$ $u(x)v(x) - \int u'(x)v(x)dx$	$\int e^{au} \sin bu  du = \frac{e^{au}}{a^2 + b^2} (a \sin bu - b \cos bu) + C$ $\int e^{au} \cos bu  du = \frac{e^{au}}{a^2 + b^2} (a \cos bu + b \sin bu) + C$