Unit 1: Quick Look at Symbolic Logic.

In this Unit, we survey logical operations, logical expressions, the role of truth tables and De Morgan's laws.

Section 1. Logical Variables and Operations.

In this section, we shall consider the symbolic logic as an arithmetic of two letters: "T" and "F". Following the custom, the letter T stands for the word "true", and the letter F stands for the word "false". We refer to the letters T and F as the **truth values**. In this context, we shall survey the five logical operations: negation, conjunction, disjunction, conditional and biconditional.

A **logical variable** p is a variable allowed to take the value "true" or "false" (but not both at the same time). Also, we use the symbol "$\equiv$" to stand for the phrase "equivalent to". The symbol "$\equiv$" acts just like "$=$" sign, but it is customary to use "equivalent to ($\equiv$)" instead of "equal to ($=$)" in the symbolic logic.

As noted earlier, the symbolic logic is an arithmetic of the two letters "T" and "F". Since there are only two letters, we can define an operation on truth values by listing all the possible outcomes in a table. We refer to such a table as the **truth table** of an operation. So, we will define the five operations by giving the truth tables. The first operation is the negation operation. It only takes one of the two letters as an input; the result of negation is the remaining letter.

Negation.

Consider the logical variable p (so, p stands either for "true" or "false"). We define the **negation** of p, denoted $\sim p$ (read "not p"), to be a truth value according to the following rules:

i. If p stands for the truth value T, then $\sim p$ stands for the truth value F, i.e., $\sim T \equiv F$.

ii. If p stands for the truth value F, then $\sim p$ stands for the truth value T, i.e., $\sim F \equiv T$.

So, the truth table on the right defines the negation operation.

p	$\sim p$
T	F
F	T

The rest of five operations take two truth values as inputs, then crunches them down to one truth value.

Conjunction (AND operation).

Let us define an operation among truth values T and F called the conjunction (often referred to as the "AND" operation). For this, let p and q be logical variables. We then define the truth value of the **conjunction** $p \wedge q$ (read "p and q") according to the truth table on the right. For example, if p stands for F while q stands for T, then the truth value of $p \wedge q$ is F, i.e., $F \wedge T \equiv F$. Also, if p stands for T while q stands for T, then the truth value of $p \wedge q$ is T, i.e., $T \wedge T \equiv T$. Note that we some time use "but" as a substitute for "and" in English.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR operation).

Let us define an operation among truth values T and F called the disjunction (often referred to as the "OR" operation). For this, let p and q be logical variables. We then define the truth value of the **disjunction** $p \vee q$ (read "p or q") according to the truth table on the right. For example, if p stands for F while q stands for T, then the truth value of $p \vee q$ is T, i.e., $T \vee F \equiv T$. Also, if p stands for F while q stands for F, then the truth value of $p \vee q$ is F, i.e., $F \vee F \equiv F$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional.

Let us define another operation among truth values called the conditional. As before, let p and q be logical variables. We then define the truth value of the **conditional** $p \to q$ (read "p implies q", or "if p, then q") according to the truth table on the right. For example, $T \to F \equiv F$ and $F \to T \equiv T$.

There are two expressions associated with the conditional $p \to q$. The expression $q \to p$ is called the **converse** of the conditional $p \to q$. Also, the expression $(\sim q) \to (\sim p)$ is called the **contrapositive** of the conditional $p \to q$. The converse and contrapositive come up again and again in connection with the conditional.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional (if and only if).

Let us define an operation among truth values called the biconditional. As before, let p and q be logical variables. The truth value of the **biconditional** $p \leftrightarrow q$ (read "p if and only if q") is defined by the truth table on the right. This operation is know as "biconditional", since the expression $p \leftrightarrow q$ is equivalent to the expression $(p \rightarrow q) \wedge (q \rightarrow p)$. We will discuss the equivalence of two expressions in the next section.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Section 2. Equivalent Logical Expressions.

A **logical expression** is an expression consisting of finitely many logical variables and finitely many operations defined in the previous section. For example, $p \vee (q \rightarrow r)$ and $(p \vee q) \rightarrow (s \vee t)$ are logical expressions. At first, let us consider the problem of constructing the truth table of a logical expression. In order to do this, we need to think about the order of operations. We should mention here that there are three conventions in this regards:

i. We evaluate inside of the parenthesis "( )" first.

ii. We apply the negation operation $\sim$ before other operations.

iii. We evaluate the operations (other than $\sim$) in order from the left to the right.

For example, how should we evaluate $\sim p \vee q$? We have two possibilities: $(\sim p) \vee q$ and $\sim (p \vee q)$. The second convention tells us that we must apply the negation operation before any other operations. So, $\sim p \vee q$ should be interpreted as $(\sim p) \vee q$.

Now, how should we evaluate $p \vee q \wedge r$? Again, we have two possibilities: $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$. Our convention is that we operate in order from the left to the right. So, $p \vee q \wedge r$ is interpreted as $(p \vee q) \wedge r$.

Similarly, how should we evaluate $p \rightarrow q \vee r$? We have two possibilities: $(p \rightarrow q) \vee r$ and $p \rightarrow (q \vee r)$. Our convention is, the same as above, that we operate in order from the left to right. So, $p \vee q \rightarrow r$ should be interpreted as $(p \rightarrow q) \vee r$.

Although the convention determines the order of evaluations, it is better practice to avoid confusion by supplying the parentheses, especially for the third convention. So, in this note, we will supply "unnecessary" parentheses.

Example 1. Let us consider the construction of the truth table for the expression $\sim (p \rightarrow q)$. At first, we make the two columns for p and q to list all the possible combinations of truth values:

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$
T	T		
T	F		
F	T		
F	F		

Now, the supplied parenthesis tell us that we need to evaluate $p \rightarrow q$ for each pair of values of p and q before the negation. For each row, a pair of the truth values for p and q are determined in the first and second column, then the result of $p \rightarrow q$ is recorded in the third column:

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$
T	T	T	
T	F	F	
F	T	T	
F	F	T	

Finally, we apply the negation to the results of $p \rightarrow q$, then form the fourth column:

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Example 2. Let us consider the construction of the truth table for the expression $p \vee (q \wedge r)$. At first, we make the three columns for p, q and r to list all the possible combinations of truth values:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Now, the supplied parenthesis tell us that we need to evaluate the expression $q \wedge r$ for each pair of values of q and r first. The results are collected in the fourth column:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	
T	T	F	F	
T	F	T	F	
T	F	F	F	
F	T	T	T	
F	T	F	F	
F	F	T	F	
F	F	F	F	

Final, we apply the "p OR" to the results of $q \wedge r$, then form the fifth column:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Tryout problem 1. Construct the truth tables for the logical expressions: **a.** $p \wedge (q \rightarrow p)$. **b.** $(p \vee q) \rightarrow (p \wedge r)$.

Now, we can compare two logical expressions in terms of their truth tables. Here is the definition of equivalence of two logical expressions.

Definition 2.1. Two logical expressions are said to be **equivalent**, if two expressions have the same truth tables.

Remark. To show the equivalence of two logical expressions, we can construct the truth tables corresponding to each of the expressions, then compare the corresponding truth values.

Example 3. Let us check the claim that the biconditional $p \leftrightarrow q$ is equivalent to the expression $(p \to q) \wedge (q \to p)$ made in the last section by comparing the truth tables (see below). Note that the last columns of the two truth tables are the same. Thus, two expressions are equivalent expressions; we denote this fact by $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

□

Example 4. Let us consider the truth tables for the expressions $(\sim p) \vee q$ and $p \to q$ below. Note that the last columns of the truth tables associated with the two expressions are identical. Thus, the two expressions are equivalent; we denote this fact by $(\sim p) \vee q \equiv p \to q$.

p	q	$\sim p$	$(\sim p) \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

□

Example 5. Let us consider the truth tables of the conditional $p \to q$ and its converse $q \to p$ below. Note that the last columns of two truth tables are different. Thus, two expressions are not equivalent expressions. We denote this fact by $p \to q \not\equiv q \to p$. For the conditional, we need to worry about the order in which the logical variables p and q to appear.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

□

Example 6. Let us consider the truth tables of the conditional $p \to q$ and its contrapositive $(\sim q) \to (\sim p)$ below. Note that the last columns of two truth tables are the same. Thus, two expressions are equivalent expressions. We thus have $p \to q \equiv (\sim q) \to (\sim p)$, i.e., the conditional and its contrapositive are equivalent.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim q$	$\sim p$	$(\sim q) \to (\sim p)$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

□

Summary. According to the above examples, the conditional $p \to q$ is equivalent to the expressions $(\sim p) \vee q$ and $(\sim q) \to (\sim p)$.

Tryout Problem 2. Determine whether the following pairs of (logical) expressions are equivalent or not.
a. $p \to (q \wedge r)$ and $(p \to q) \wedge r$. **b.** $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$.

Section 3. Negating Logical Expression.

Let A and B be logical expressions. We say that B is a **negation** of A, if B and $\sim A$ are equivalent. In this section, we shall study a process of negating a logical expressions. At first, let us consider the negations of the expressions $p \vee q$ and $p \wedge q$.

Let us compare the truth tables of the expressions $\sim (p \vee q)$ and $(\sim p) \wedge (\sim q)$ below. We see that the last columns of the two truth tables are the same. Thus, the two expressions are equivalent.

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Similarly, consider the truth tables for the expressions $\sim (p \wedge q)$ and $(\sim p) \vee (\sim q)$ below. We see that they are also equivalent.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The two equivalences

$$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q) \quad \text{and} \quad \sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$$

are called the **De Morgan's laws**. Let us consider the negation of the conditional $p \to q$.

Example 1. Consider the expressions $\sim (p \to q)$ and $p \wedge (\sim q)$. According to the truth tables below, we have $\sim (p \to q) \equiv p \wedge (\sim q)$.

p	q	$p \to q$	$\sim (p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

p	q	$\sim q$	$p \wedge (\sim q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

□

According to the above example, the expression $p \wedge (\sim q)$ is the negation of the conditional $p \to q$. A common mistake in negating the conditional $p \to q$ is to claim that the expression $(\sim p) \to (\sim q)$ is the negation of the conditional. According the following example, the expression is not a negation of the conditional.

Example 2. Consider the expressions $\sim (p \to q)$ and $(\sim p) \to (\sim q)$. Since the last columns of the truth tables below are not the same, we have $\sim (p \to q) \not\equiv (\sim p) \to (\sim q)$. Thus, the expression $(\sim p) \to (\sim q)$ is not a negation of the conditional $p \to q$.

p	q	$p \to q$	$\sim (p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

p	q	$\sim p$	$\sim q$	$(\sim p) \to (\sim q)$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

□

Example 3. Consider the expressions $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (\sim p \wedge q)$. Since the last columns of the truth tables below are the same, we have $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$. Thus, the expression $(p \wedge \sim q) \vee (\sim p \wedge q)$ is a negation of the biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	F	T	T
F	T	T	F	T	F	T
F	F	T	T	F	F	F

□

Summary. Here is the list of the negations of the five operations.
1. $\sim(\sim p) \equiv p$. **2.** $\sim(p \vee q) \equiv \sim p \wedge \sim q$. **3.** $\sim(p \wedge q) \equiv \sim p \vee \sim q$. **4.** $\sim(p \rightarrow q) \equiv p \wedge \sim q$.
5. $\sim(p \leftrightarrow q) \equiv (\sim p) \leftrightarrow q$.
Note that **2** and **3** are the De Morgan's law.

In the earlier example, we have shown that the negation of the conditional $p \rightarrow q$ is equivalent to the expression $p \wedge (\sim q)$. We can also derive at this negation using the De Morgan's law.

Example 4. Note that $p \rightarrow q$ is equivalent to $(\sim p) \vee q$ (cf. Section 2). So, the negation $\sim(p \rightarrow q)$ is equivalent to $\sim(\sim p \vee q)$. We then have $\sim(\sim p \vee q) \equiv \sim(\sim p) \wedge (\sim q)$ by the De Morgan's law. Now, $\sim(\sim p) \wedge (\sim q) \equiv p \wedge (\sim q)$; so,

$$\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv \sim(\sim p) \wedge (\sim q) \equiv p \wedge (\sim q). \quad \square$$

Example 5. Let us negate the expression $p \vee (q \wedge r)$ using the above list. This means we like to distribute $\sim$ in the expression $\sim[p \vee (q \wedge r)]$. Note that, by applying the the De Morgan's law twice, we have

$$\sim[p \vee (q \wedge r)] \equiv \sim p \wedge \sim (q \wedge r) \equiv \sim p \wedge (\sim q \vee \sim r). \quad \square$$

Example 6. Let us negate the expression $p \rightarrow (q \wedge r)$. This means we like to simplify the expression $\sim[p \rightarrow (q \wedge r)]$. Since $\sim(a \rightarrow b) \equiv a \wedge \sim b$, we have

$$\sim[p \rightarrow (q \wedge r)] \equiv p \wedge \sim (q \wedge r).$$

By applying the De Morgan's law, we then have

$$p \wedge \sim (q \wedge r) \equiv p \wedge (\sim q \vee \sim r).$$

So,

$$\sim[p \rightarrow (q \wedge r)] \equiv p \wedge (\sim q \vee \sim r). \quad \square$$

Tryout Problem 3. Negate the following expressions [Note: distribute the negation signs so that the negations only occur in front of variables; for example, we like to have $\sim q \wedge \sim r$ instead of $\sim(q \vee r)$].
a. $p \rightarrow (q \wedge \sim r)$. **b.** $\sim p \rightarrow (q \vee r)$. **c.** $p \wedge (q \rightarrow r)$.