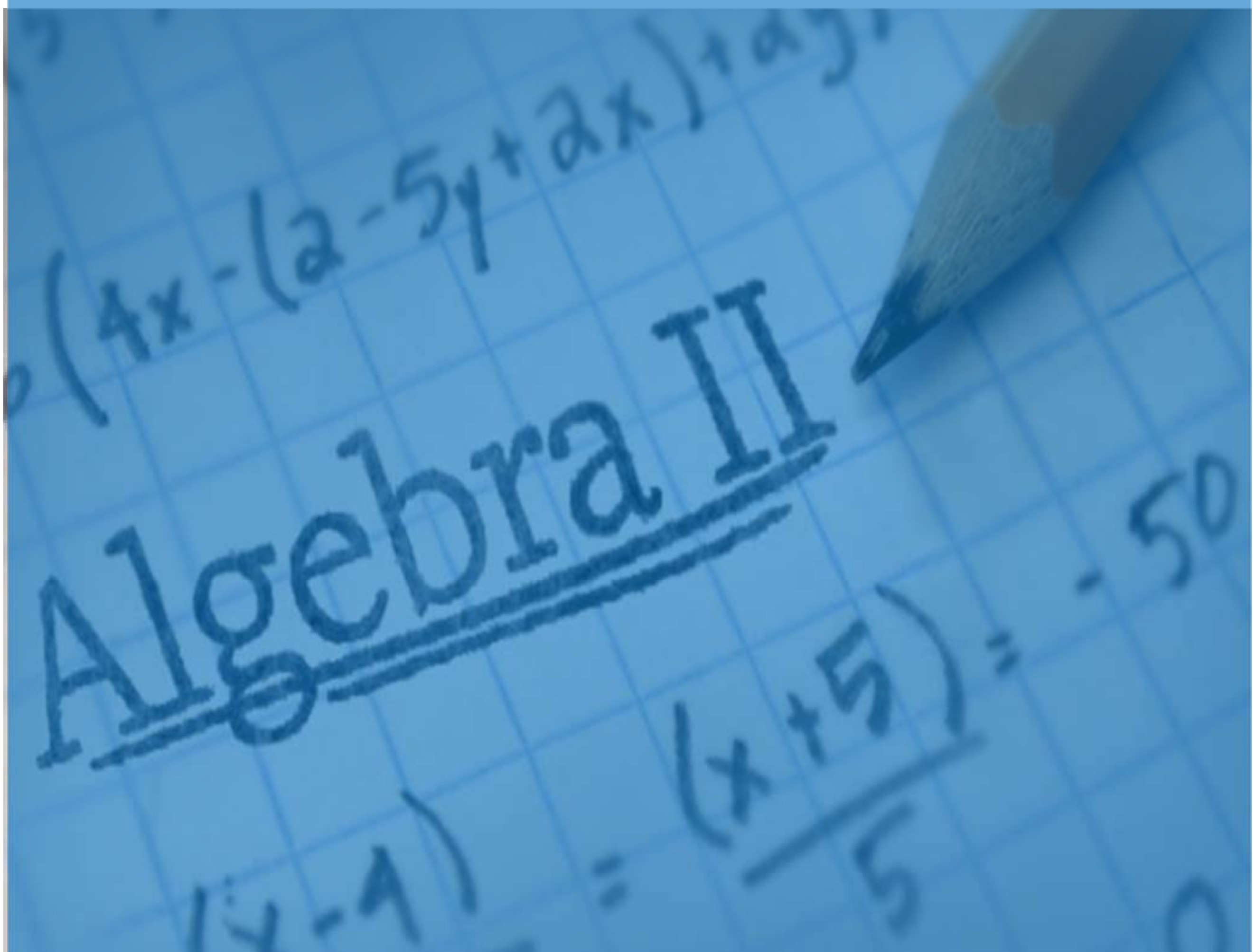
HOUSTON COUNTY
BOARD OF EDUCATION
HIGH-ACHIEVING TEACHERS

MATHEMATICS
DEPARTMENT



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MATHEMATICS DEPARTMENT

Algebra 2

Unit 1: Quadratics Revisited

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Lesson B: Quadratic Graph Characteristics Part 2	(A2.U1.C1.B.____.QuadraticGraphingPart2)
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Lesson D: Vertex and Standard Forms	(A2.U1.C1.D.____.VertexStandard)
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Lesson F: Solving by Factoring	(A2.U1.C1.F.____.FactorToSolve)
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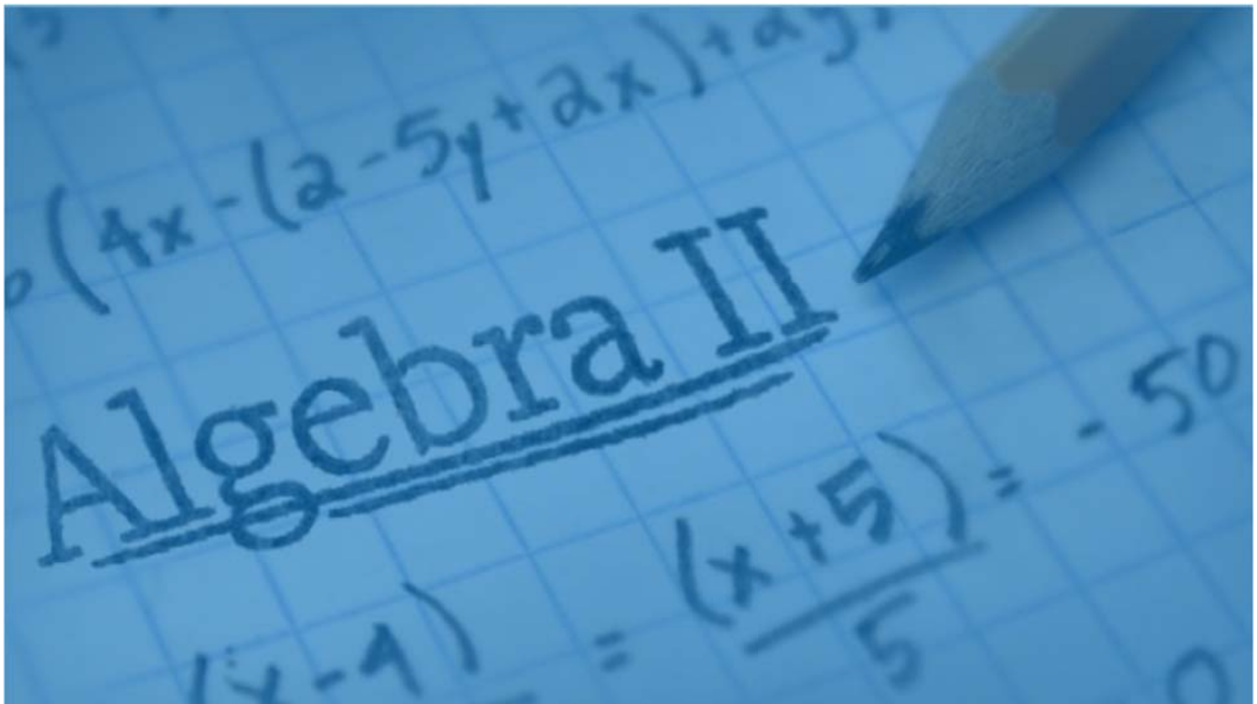
Science Technology Engineering Mathematics



Unit 1
Quadratics Revisited

Algebra 2

Unit 1: Quadratics Revisited

Concept 1: Quadratic Review

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Vocabulary

Term	Definition	Notation	Diagram/Visual
Domain	______ ______ ______		
Range	______ ______ ______		
Turning point	______ ______ ______		
Interval of increase	______ ______ ______		
Interval of decrease	______ ______ ______		

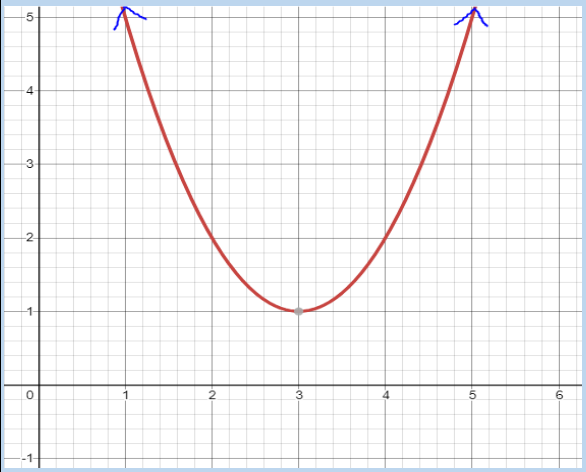
Key Ideas

Term	
Interval notation	
End behavior	

Graph Characteristics, Part 1

Many characteristics about a graph can be determined by observation. In this lesson, we will learn how to identify domain and range of a quadratic function from its graph. Then, we will learn how to write the end behavior. We will use the turning point to write the intervals of increase and decrease.

Example!



The domain of a function is the set of input values x, and the range is the set of output values f(x).

The domain of all quadratic functions is all real values of x because the lowest value of x is $-\infty$ on this graph and the highest value of x is $+\infty$ on this graph. So, we write $(-\infty, +\infty)$ for the domain.

The lowest value of f(x) is 1 in this graph, and the highest value of f(x) is $+\infty$, so we write $[1, +\infty)$ for the range.

The end behavior on the left is written:
As $x \to -\infty, f(x) \to +\infty$.
The end behavior on the right is written:
As $x \to +\infty, f(x) \to +\infty$.

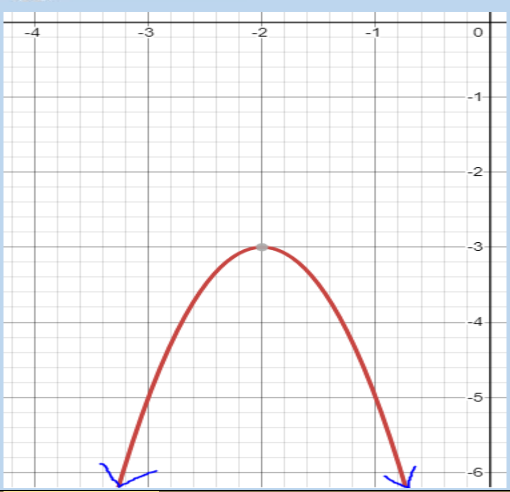
The turning point of this quadratic function is (3, 1).

The interval of decrease is $(-\infty, 3)$.
The interval of increase is $(3, +\infty)$.

Questions To Ponder

When the domain, range, and intervals of increase/decrease are written in interval notation, there is only one bracket [in the example above. Why?

Example!



The lowest value of x is _____ on this graph and the highest value of x is _____ this graph. So, we write __________ for the domain.

The lowest value of f(x) is _____ in this graph, and the highest value of f(x) is _____, so we write __________ for the range.

The end behavior on the left is written:
As ___________, _____________.
The end behavior on the right is written:
As ___________, _____________.

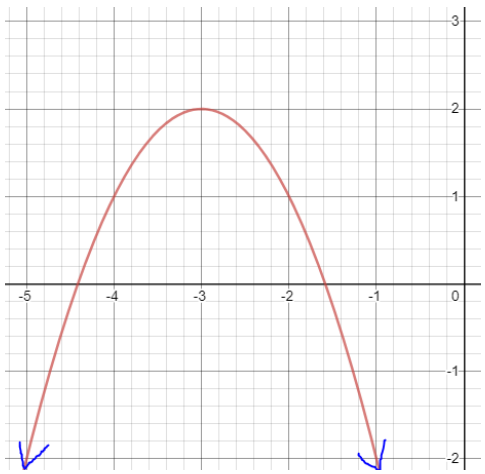
The turning point of this quadratic function is _________.

The interval of decrease is __________.
The interval of increase is __________.

Questions To Ponder

Compare and contrast the previous two examples. How are the direction of opening and the characteristics related?

1.



Domain: **Range:**

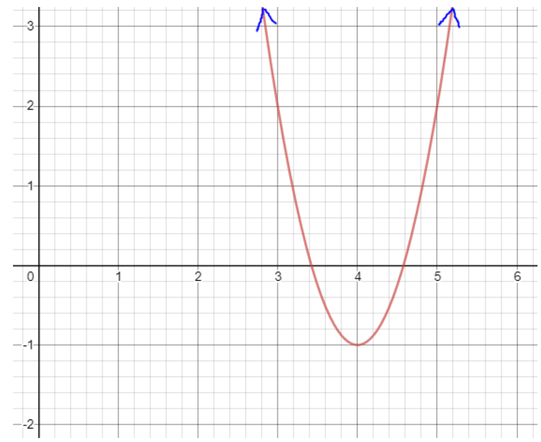
End behavior:

Turning point:

Interval of Increase:

Interval of Decrease:

2.



Domain: **Range:**

End behavior:

Turning point:

Interval of Increase:

Interval of Decrease:

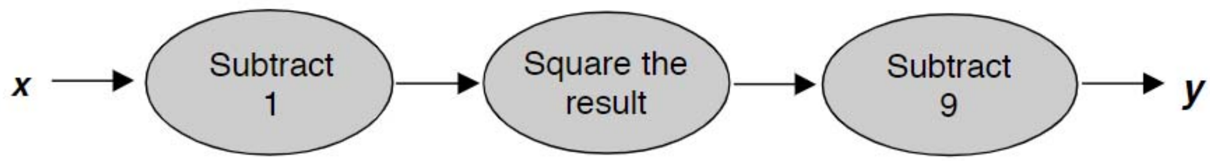
3. A certain quadratic function has a turning point at (-3, 5). What other characteristic can you write without seeing the graph?

4. Complete the table below.

Domain	Range	End behavior on the left	End behavior on the right	Turning point	Interval of increase	Interval of decrease
				(2, 3)	$(2, +\infty)$	$(-\infty, 2)$
	$(-\infty, 7]$					$(-10, +\infty)$
		As $x \to -\infty$, $f(x) \to +\infty$.		$(-6, -11)$		

QUADRATIC GRAPHING TASK

This is the quadratic number machine.



1. a. Show that, if x is 5, y is 7. ____

   b. What is y if x is 0? ____

   c. Use algebra to show that, for this machine, $y = x^2 - 2x - 8$.

The diagram on the next page shows the graph of the machine's quadratic function $y = x^2 - 2x - 8$ and the graphs of $y = 3$ and $y = x$.

2. a. Which point on the diagram shows the minimum value of y? ____________________________________

b. Which point(s) on the diagram show(s) the solution(s) to the equation $3 = x^2 - 2x - 8$?

____

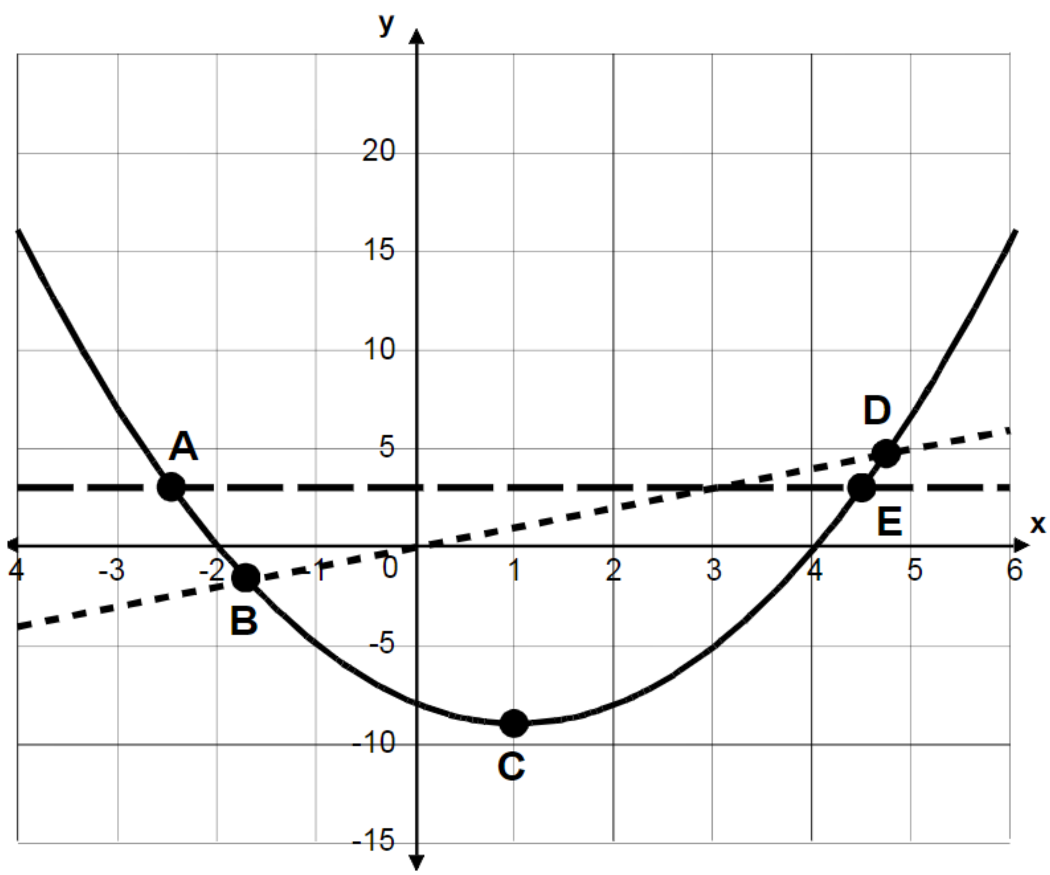
c. Which point(s) on the diagram show(s) the solution(s) to the equation $x = x^2 - 2x - 8$?

____

3. a. Use the graph to solve the equation $x^2 - 2x - 8 = 0$. Mark the solutions on the graph.

x = __________ or x = ____________

b. Use algebra to solve the same equation.



Graph Characteristics

1. $f(x) = -(x+3)^2 + 3$

a = *b* = *c* =

Opens Up or Down?

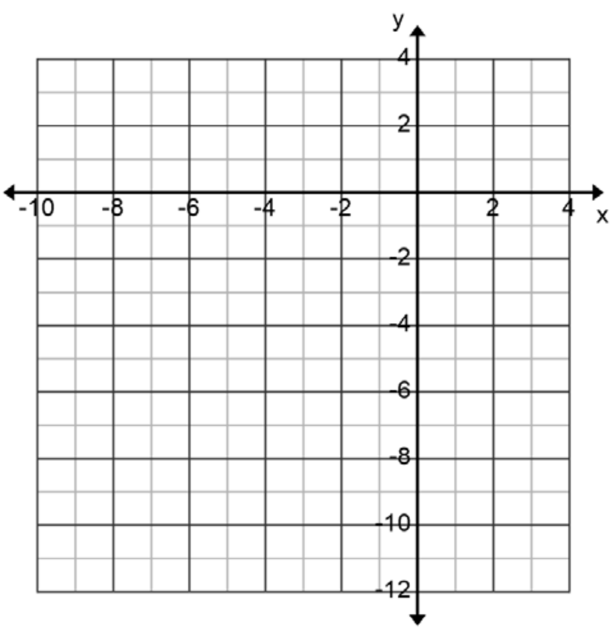
Is the vertex a Max or Min?

y-intercept:

Axis of symmetry is *x* = ____________

Vertex: (___________, __________)

Domain: Range:



2. $f(x) = \frac{1}{2}x^2 - 4x + 4$

a = *b* = *c* =

Opens Up or Down?

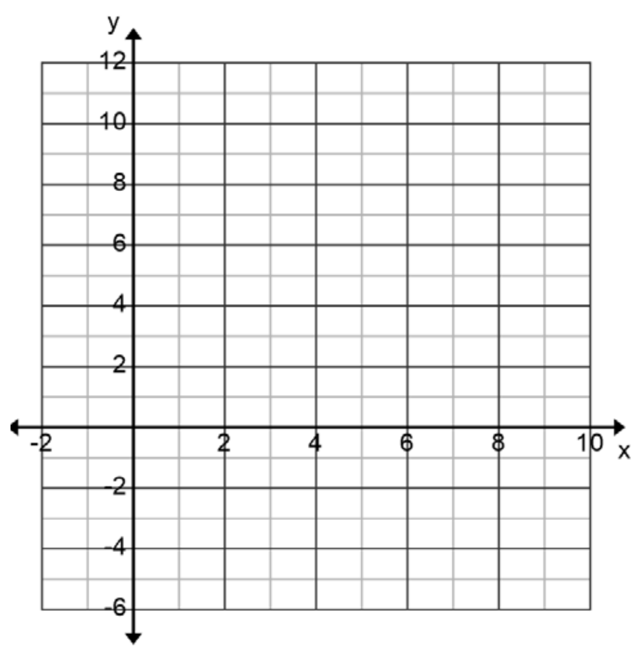
Is the vertex a Max or Min?

y-intercept:

Axis of symmetry is *x* = ____________

Vertex: (___________, __________)

Domain: Range:



3. $f(x) = -2(x-2)^2 + 3$

a = *b* = *c* =

Opens Up or Down?

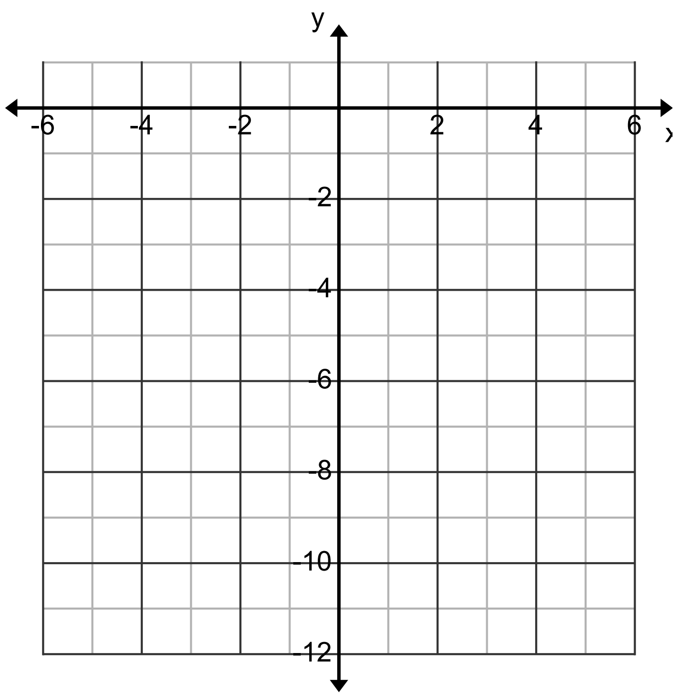
Is the vertex a Max or Min?

y-intercept:

Axis of symmetry is *x* = ____________

Vertex: (___________, __________)

Domain: Range:



4. $f(x) = 2x^2 - 1$

a = *b* = *c* =

Opens Up or Down?

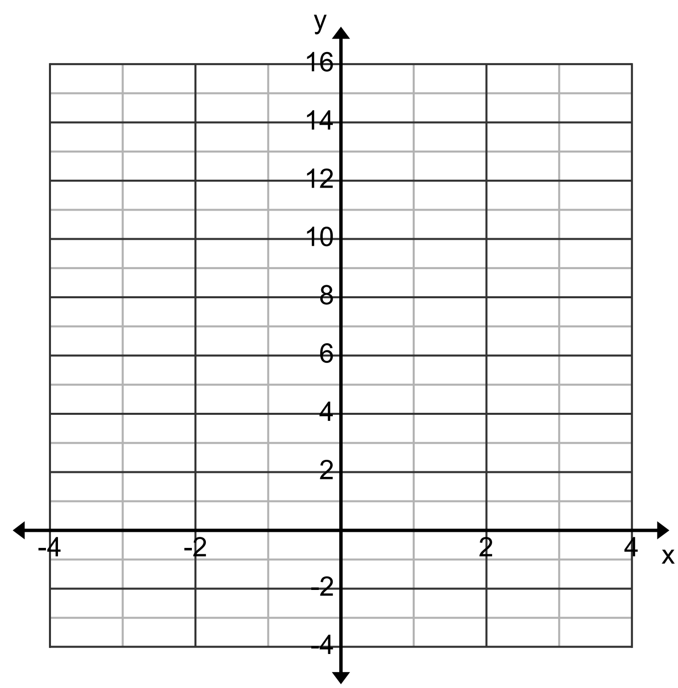
Is the vertex a Max or Min?

y-intercept:

Axis of symmetry is *x* = ____________

Vertex: (___________, __________)

Domain: Range:



5. $f(x) = 2(x+1)^2 + 1$

a = *b* = *c* =

Opens Up or Down?

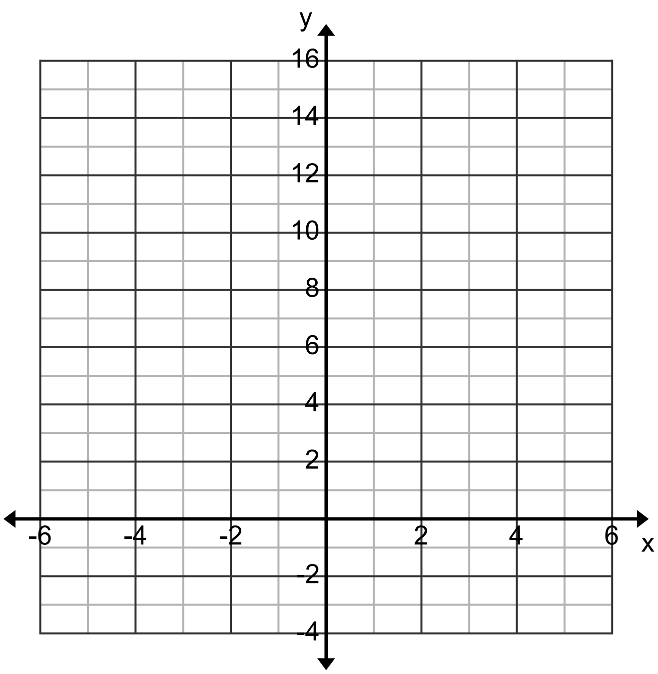
Is the vertex a Max or Min?

y-intercept:

Axis of symmetry is *x* = ____________

Vertex: (___________, __________)

Domain: Range:



6. $f(x) = -3x^2 - 12x + 1$

a = *b* = *c* =

Opens Up or Down?

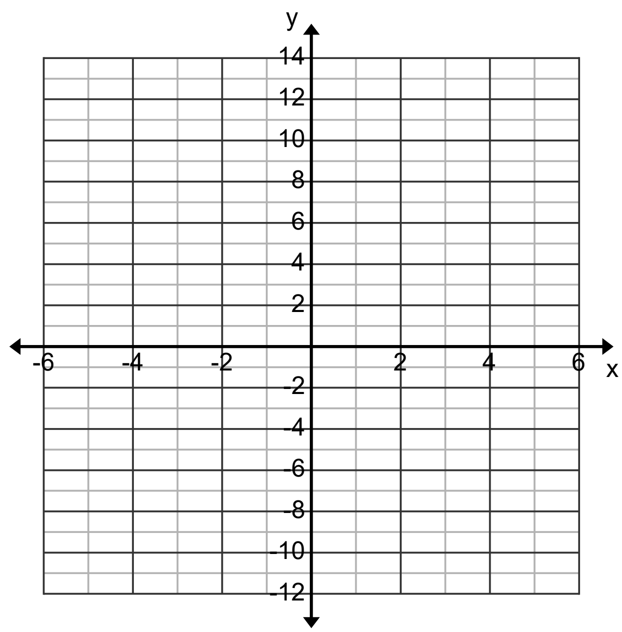
Is the vertex a Max or Min?

y-intercept:

Axis of symmetry is *x* = ____________

Vertex: (___________, __________)

Domain: Range:



Vocabulary

Term	Definition	Notation	Diagram/Visual
y-intercept			
zero			
symmetric about the y-axis			
symmetric about the origin			
even function			
odd function			

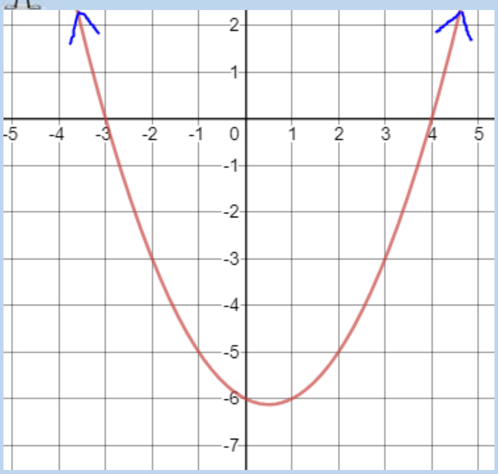
Key Ideas

Term	
other names for zeros	

Graph Characteristics, Part 2

There are additional characteristics about a graph that can be determined by observation. In this lesson, we will learn how to identify a y-intercept and zeros of a quadratic function from its graph. We will use the symmetry of a graph to determine whether the function is even, odd, or neither.

Example!



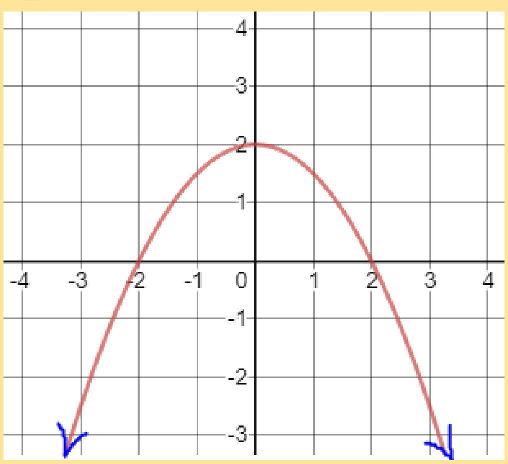
The y-intercept is the point where the graph crosses the y-axis. We observe that the graph crosses the y-axis at -6, so the y-intercept is $(0, -6)$.

A zero is a point where the graph crosses the x-axis. We observe the graph crossing the x-axis twice in this example, at -3 and 4. So, the zeros are $(-3, 0)$ and $(4, 0)$.

The graph is not symmetric about the y-axis. It is not symmetric about the origin either. So, we say the graph is neither even nor odd.

Questions To Ponder How many zeros are possible for a quadratic function? How many y-intercepts are possible?

Example!



The graph crosses the y-axis at _____, so the y-intercept is __________.

The graph crosses the x-axis at __________________. So, the zero(s) is/are _________________.

The graph is NOT symmetric about the __________. It IS symmetric about the __________. So, we say the graph is __________.

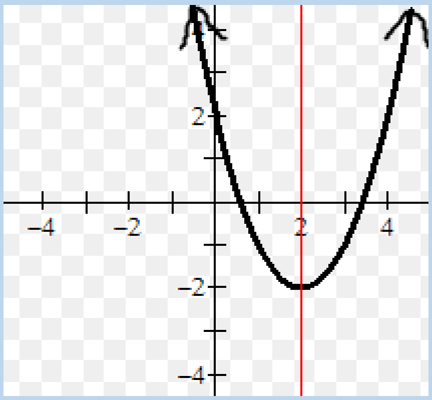
Questions To Ponder What is the same about all zeros? What is the same about all y-intercepts?

What do you know about the zeros of a quadratic function that is even?

Symmetry can be misunderstood, so we will make some explicit observations about some examples and non-examples next.

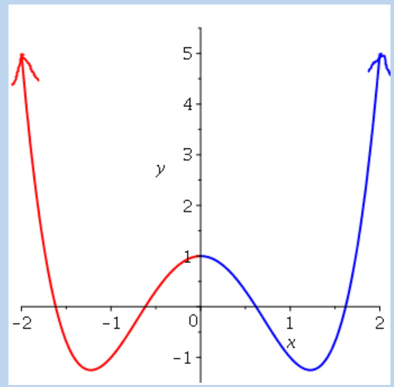
Example!

This function is <u>neither even nor odd</u>.



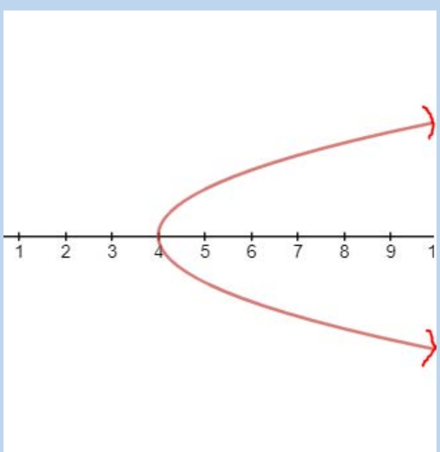
The quadratic function shown above has an axis of symmetry at x = 2, but the graph is not even because the axis of symmetry is not on the y-axis.

This function is <u>even</u>.



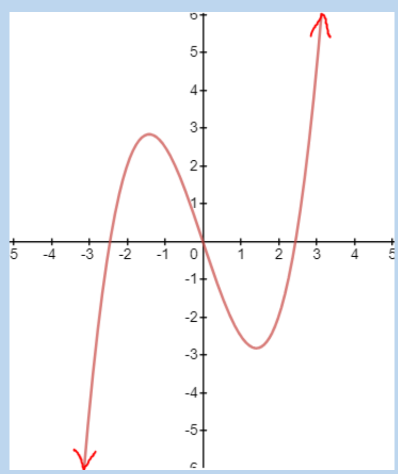
The function shown above is the same if it is reflected over the y-axis. It is a fourth degree polynomial (not quadratic).

This graph is <u>neither even nor odd</u>.



The graph shown above is symmetric about the x-axis. This kind of symmetry results in a relation that is not a function.

This graph is <u>odd</u>.

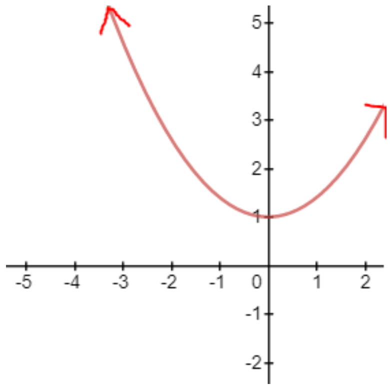


The graph shown above is symmetric about the origin. This can be described in two ways: 180^0 rotation or reflected over both the x-axis and the y-axis.

Questions To Ponder

Can a quadratic function be odd?

1.

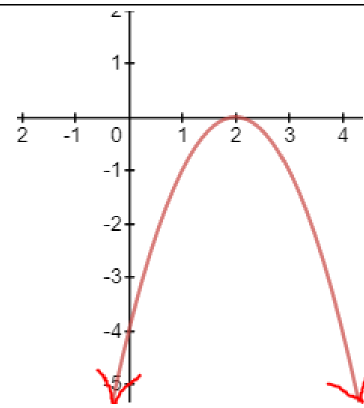


y-intercept:

zero(s):

even, odd, or neither:

2.

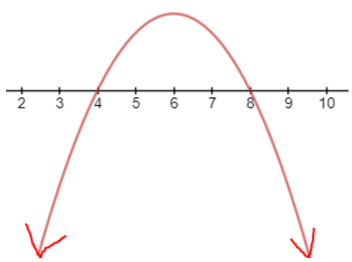


y-intercept:

zero(s):

even, odd, or neither:

3.



Which of the following could be the y-intercept of this quadratic function?

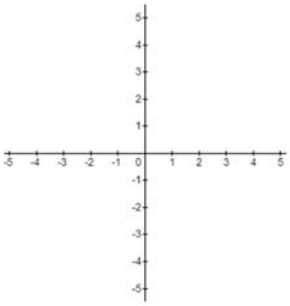
a. (0, 12) **b. (12, 0)** **c. (-12, 0)**

d. (0, -12) **e. there is not a y-intercept**

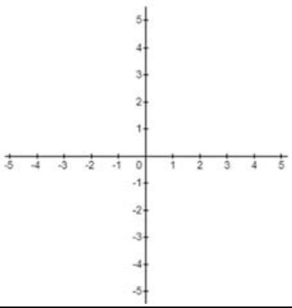
4. Sketch two possible graphs of quadratic functions with the given characteristics.

y-intercept: (0, 4)
even function

graph 1

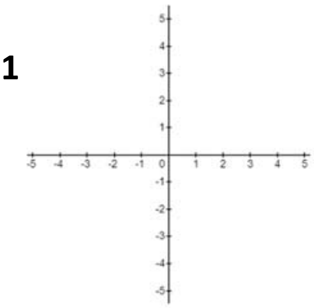


graph 2

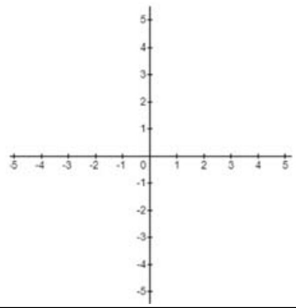


y-intercept: (0, -1)
zero: (2, 0)

graph 1



graph 2



QUADRATIC GRAPHING TASK

a. Graph these equations on your graphing calculator at the same time. What happens? Why?

$$y_1 = (x - 3)(x + 1)$$

$$y_2 = x^2 - 2x - 3$$

$$y_3 = (x - 1)^2 - 4$$

$$y_4 = x^2 - 2x + 1$$

b. Below are the first three equations from the previous problem.

$$y_1 = (x - 3)(x + 1)$$

$$y_2 = x^2 - 2x - 3$$

$$y_3 = (x - 1)^2 - 4$$

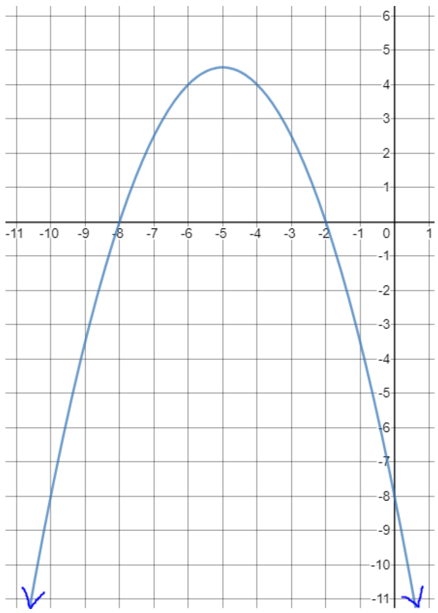
These three equations all describe the same function. What are the coordinates of the following points on the graph of the function? From which equation is each point most easily determined? Explain.

i. vertex: _____
ii. *y*-intercept: _____
iii. *x*-intercept(s): _____

c. Make up an equation for a quadratic function whose graph satisfies the given condition. Use whatever form is most convenient.

ii.Has a vertex at (−2,−5).
iii.Has a y-intercept at (0,6)
iv.Has x-intercepts at (3,0) and (5,0)
v.Has x-intercepts at the origin and (4,0)
vi.Goes through the points (4,2) and (1,2)

1. Identify the characteristics of each graph. For any points that do not have integer coordinates, estimate the values to the nearest half.



Domain:

Range:

End behavior:

Turning point:

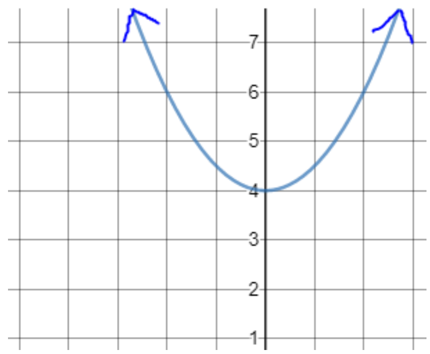
Interval of increase:

Interval of decrease:

y-intercept:

zero(s):

even, odd, or neither:



Domain:

Range:

End behavior:

Turning point:

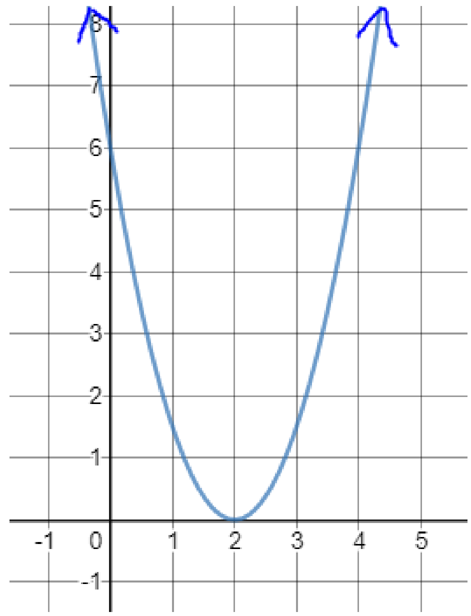
Interval of increase:

Interval of decrease:

y-intercept:

zero(s):

even, odd, or neither:



Domain:

Range:

End behavior:

Turning point:

Interval of increase:

Interval of decrease:

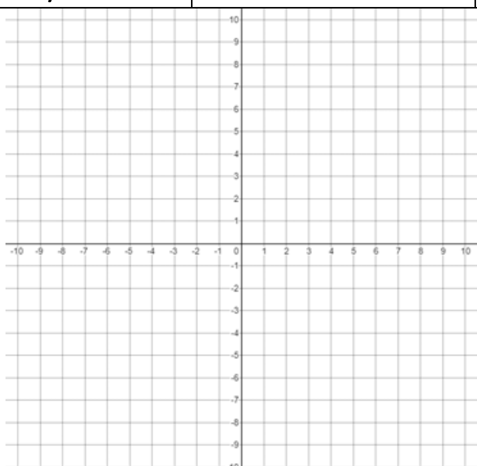
y-intercept:

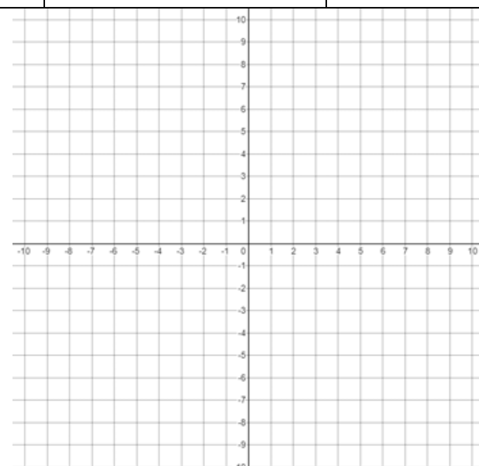
zero(s):

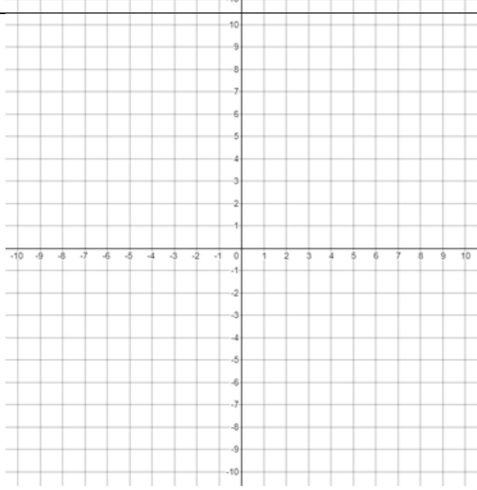
even, odd, or neither:

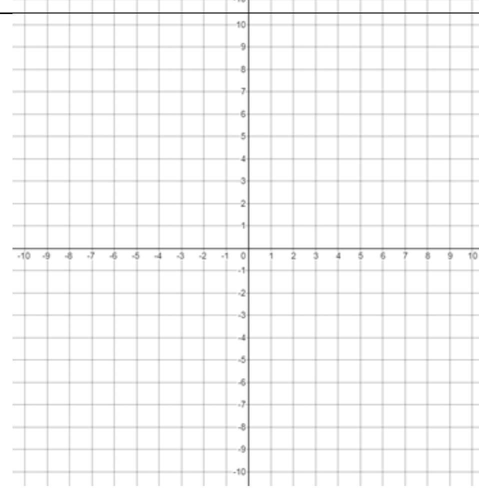
2. Given the selected characteristics of a graph of a quadratic function, list the missing characteristics. You may sketch a graph. If you do not know the exact value of a number, you may provide a realistic value.

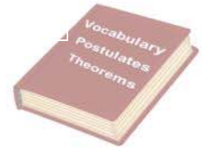
	Function 1	Function 2	Function 3	Function 4
Domain	$(-\infty, +\infty)$			
Range		$[-2, +\infty)$		$(-\infty, 7]$
End behavior	As $x \to -\infty$, $f(x) \to -\infty$.			
	As $x \to +\infty$, $f(x) \to -\infty$.			
Turning point	(3, 6)		(-8, -5)	
Int of increase				$(-\infty, 0)$
Int of decrease				
Y-intercept	(0, 0)		(0, 10)	
Zero(s)		(5, 0) and (-5, 0)		
Even/Odd/Neither				Even











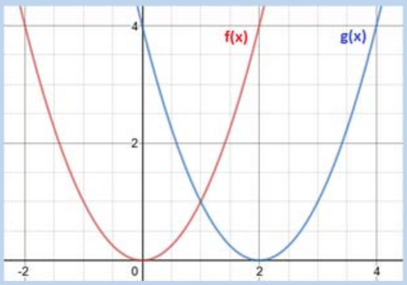
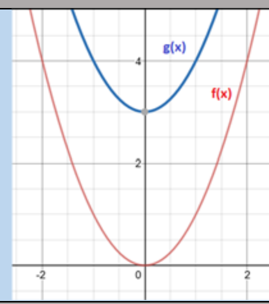
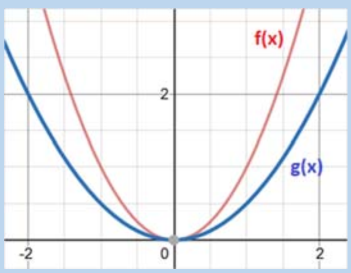
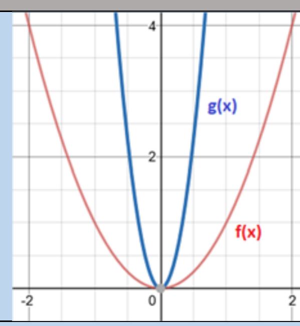
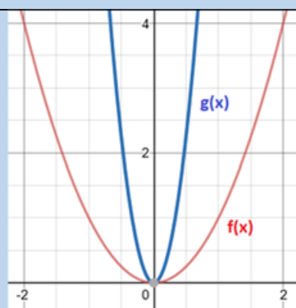
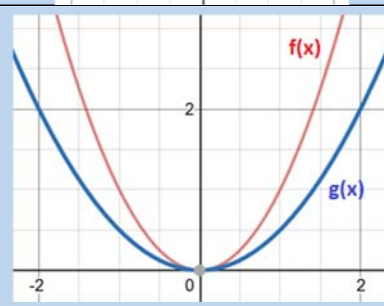
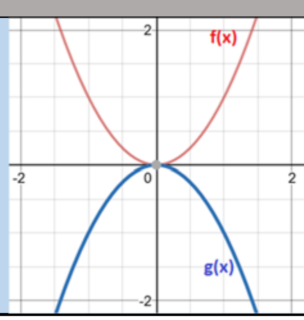
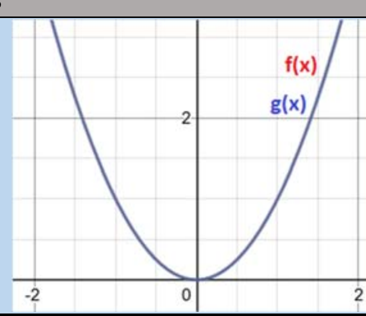
Vocabulary

	Term	Definition	Diagram/Visual
Translations	**Horizontal Shift**		
	Vertical Shift		
Dilations	**Horizontal Compression**		
	Horizontal Stretch		
	Vertical Compression		
	Vertical Stretch		
Reflections	**Reflection over X-axis**		
	Reflection over Y-axis		

Transformations of Graphs

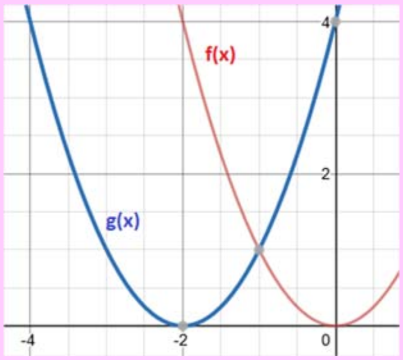
In this lesson, we will discover how the parent function of quadratics can be translated, dilated and reflected into a new graph.

The following examples are transformations of the parent function f(x) and the new graph g(x). Keep in mind **only one** transformation has occurred in each graph.

Translations			
Horizontal Shift			Vertical Shift
Dilations			
Horizontal Stretch			Vertical Stretch
Horizontal Compression			Vertical Compression
Reflections			
Reflection over the x-axis			Reflection over the y-axis

Which graphs of the transformations match? Explain your reasoning.

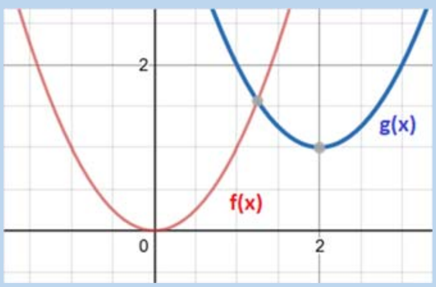
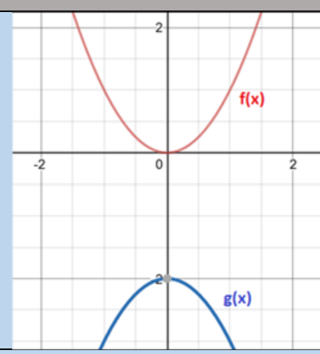
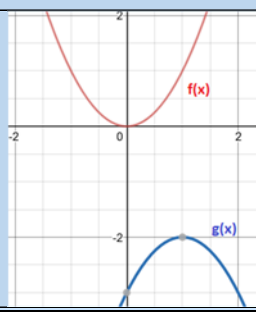
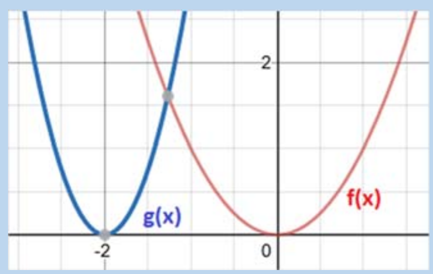
SELF CHECK



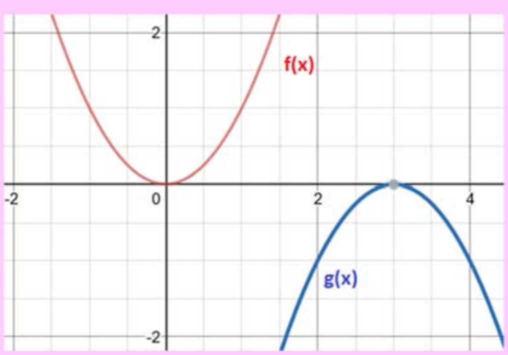
Which of the following transformations have occurred? Select all that apply.

_____ horizontal shift
_____ vertical shift
_____ horizontal stretch
_____ horizontal compression
_____ vertical stretch
_____ vertical compression
_____ reflection over x – axis
_____ reflection over y – axis

Example! The following examples are transformations of the parent function f(x) and the new graph g(x). Keep in mind **multiple** transformations have occurred in each graph.

Dilations			
Right 2 and up 1			Reflection over the x – axis and down 2
Reflection over the x axis, right 1 and down 2			Left 2 and vertical stretch OR Left 2 and horizontal compression

SELF CHECK



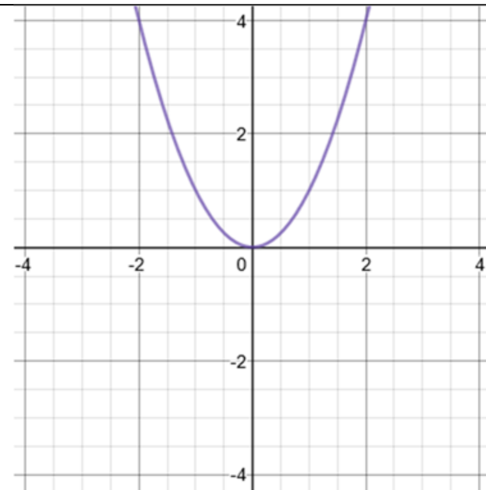
Which of the following transformations have occurred? Select all that apply.

_____ horizontal shift
_____ vertical shift
_____ horizontal stretch
_____ horizontal compression
_____ vertical stretch
_____ vertical compression
_____ reflection over x – axis
_____ reflection over y – axis

Now, we will learn how to sketch a graph given the parent function and the transformations.

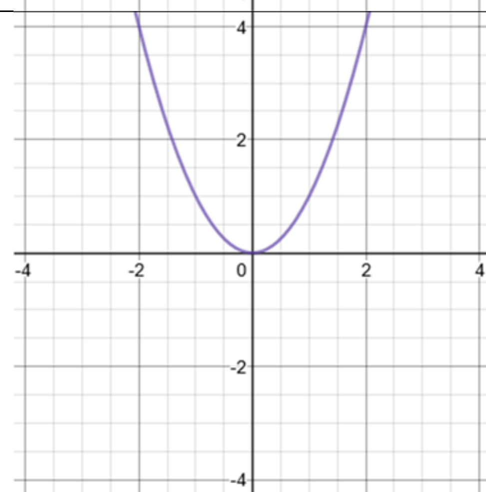
Transformations:

- **Right 3**
- **Down 2**



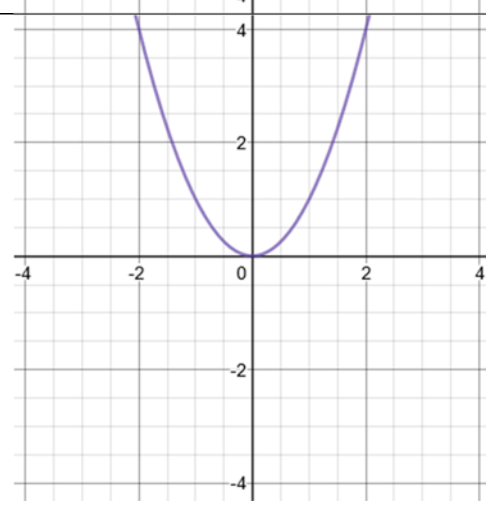
Transformations:

- **Reflect over the x – axis**
- **Left 1**
- **Up 2**

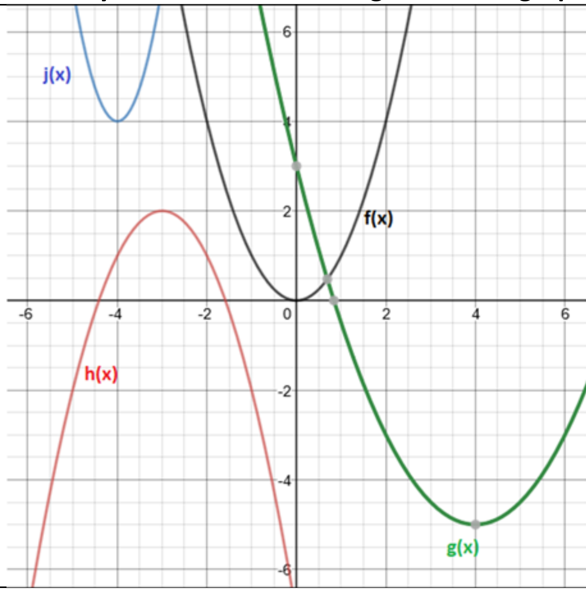


Transformations:

- **Right 3**
- **Horizontal Compression**



#1-3. Identify the transformations given in the graph.



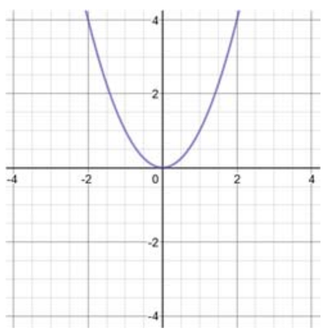
1. g(x):

2. h(x):

3. j(x):

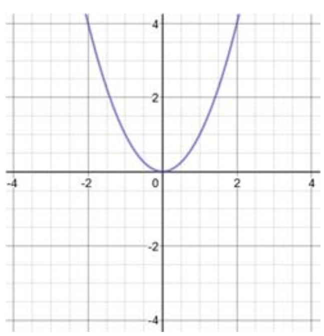
#4-6. Graph the new function given the transformation. Then identify key characteristics.

- Reflect x-axis
- Down 1
- Right 2



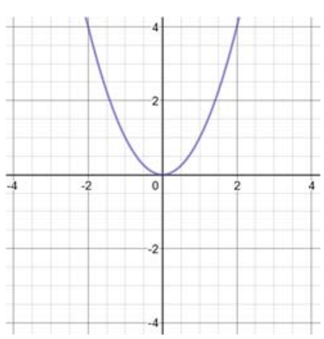
Domain: **Range:**
End Behavior: **y-intercept:**
Turning Point:
Interval of Increase: **Interval of Decrease:**
Even, odd, or neither: **Zero(s):**

- Left 3
- Up 2
- Reflect over y-axis
- Horizontal stretch



Domain: **Range:**
End Behavior: **y-intercept:**
Turning Point:
Interval of Increase: **Interval of Decrease:**
Even, odd, or neither: **Zero(s):**

- Vertical Shrink
- Right 2
- Down 1



Domain: **Range:**
End Behavior: **y-intercept:**
Turning Point:
Interval of Increase: **Interval of Decrease:**
Even, odd, or neither: **Zero(s):**

Quadratics Investigation: Transforming Parabolas

Directions: For this task, you should work in groups of about 3 students. ALL group members should do all steps on the calculator, and then you should discuss what you are seeing together and agree on your conjectures. Be prepared to report out to the class with your conjectures and what observations and thinking lead to them.

The parabola described by the equation is called the "parent function" for all quadratic functions. It is given this name because it is the most basic parabola: its vertex is at (0,0). It has not been stretched or shrunk (dilated). It has not been moved to the left, right, up, or down (translated). It has not been turned upside down (reflected).

A parabola that has been dilated, translated, or reflected is said to have undergone a transformation. We have already been studying transformations without really talking about them! Our investigation today will help us formalize this study.

1. Graph the parabola in y1 on your graphing calculator. Then, graph the each function given below in y2. Tell whether the function in y2 has been dilated, translated, or reflected with respect to the parent function in y1.

a.

b.

c.

d.

e.

Make a conjecture about the type of transformation a parabola described by the equation has undergone. Be as specific as possible.

2. Again, graph the parabola under y1 on your graphing calculator. Then, graph the function given in y2. Tell whether the function in y2 has been dilated, translated, or reflected with respect to the parent function in y1.

a.

b.

c.

d.

e.

Make a conjecture about the type of transformation a parabola described by the equation has undergone. Be as specific as possible.

3. Once more, graph the parabola under y1 on your graphing calculator. Then, graph the function given in y2. Tell whether the function in y2 has been dilated, translated, or reflected with respect to the parent function in y1.

a.

b.

c.

d.

e.

f.

Make a conjecture about the type of transformation a parabola described by the equation has undergone. Be as specific as possible.

4. Let's combine the transformations. Try to list all the transformations that the graph of each quadratic function below has undergone with respect to the parent function, . Check your lists using the graphing calculator.

a.

b.

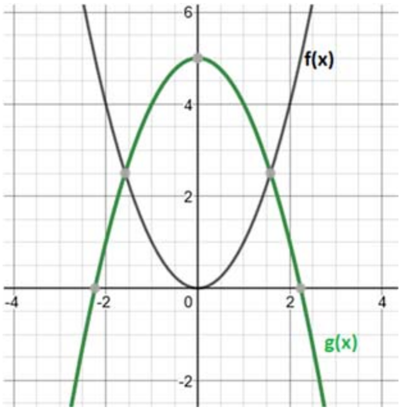
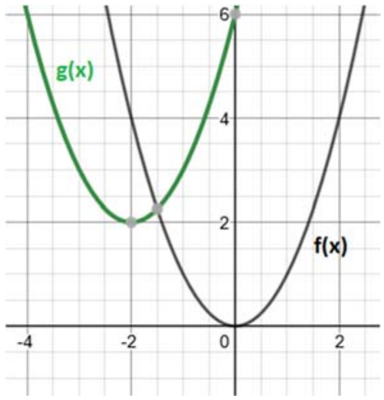
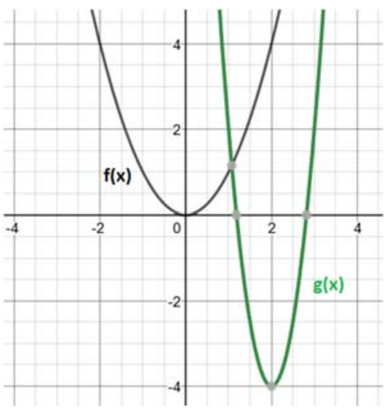
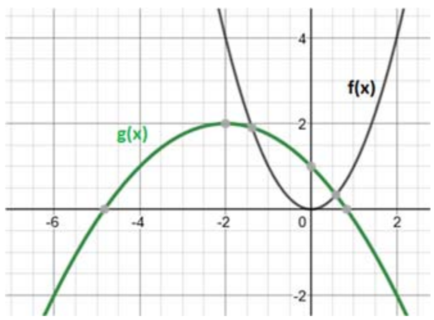
c.

5. Last step: try to write the vertex form equation for a parabola that has undergone each of the following transformations with respect to the parent function, :

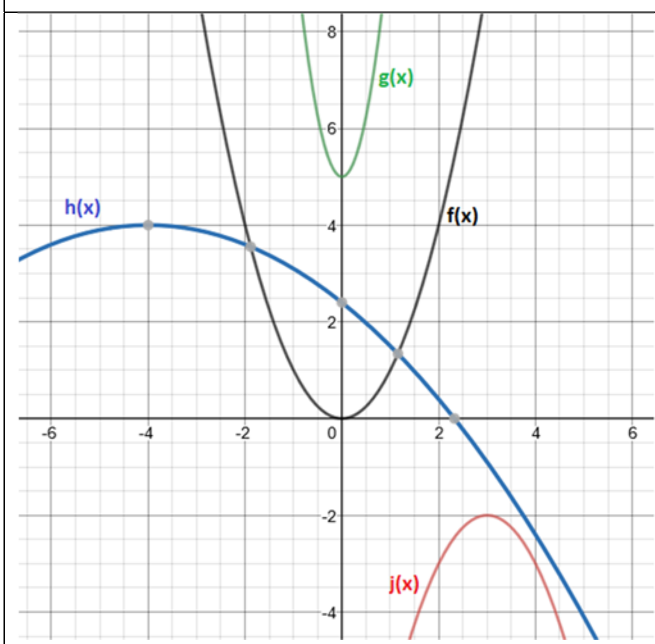
a. reflected over the x-axis, translated 2 units left and 1 unit up.

b. shrunk by a factor of .62, translated 4 units up.

c. stretched by a factor of 4, reflected over the x-axis, translated 3 units down and 4 units right.

Match the following graphs with the appropriate transformations.			
1. 	**2.** 	**A)** Right 2, down 4, and vertical stretch	**B)** Reflection across x – axis and up 5
3. 	**4.** 	**C)** Left 2 and up 2	**D)** left 2, reflection over x-axis, up 2, and horizontal stretch

Identify the following characteristics and their transformation for the below graphs.



g(x)

Transformations:

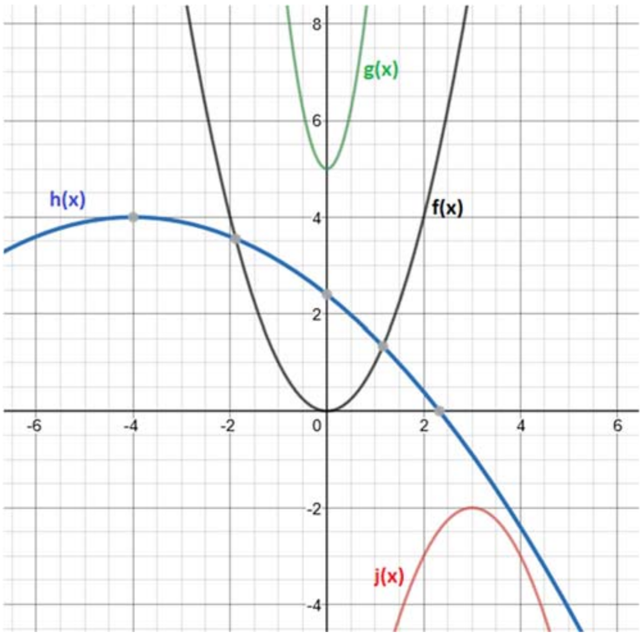
Domain: **Range:**

End Behavior: **y-intercept:**

Turning Point:

Interval of Increase: **Interval of Decrease:**

Even, odd, or neither: **Zero(s):**



h(x)

Transformations:

Domain:	**Range:**
End Behavior:	**y-intercept:**
	Turning Point:
Interval of Increase:	**Interval of Decrease:**
Even, odd, or neither:	**Zero(s):**

j(x)

Transformations:

Domain:	**Range:**
End Behavior:	**y-intercept:**
	Turning Point:
Interval of Increase:	**Interval of Decrease:**
Even, odd, or neither:	**Zero(s):**

Algebra 2 Unit #1 Concept #1

Lesson Name: Vertex and Standard Forms of Quadratics

Term	Definition	Diagram/Visual
Vertex Form		
Standard Form		
Vertex		
Axis of Symmetry		
Leading Coefficient		
Constant		

Vertex and Standard Forms of Quadratics

Now that you know how to identify key characteristics from a graph, we are going to look at the **vertex form** and **standard form** to algebraically identify characteristics. You can transform a graph in various way.

GUIDED NOTES

VERTEX FORM

- The axis of symmetry for a quadratic equation in vertex form is identified by the equation ________.
- The vertex of a quadratic equation in vertex form is given by the coordinates ___________.

$$f(x) = a(x - h)^2 + k$$

STANDARD FORM

- The axis of symmetry for a quadratic equation in standard for is given by the equation,___________.
- The vertex of a quadratic equation in standard form is given by the coordinates $\left(\_\_\_\_\_, f\left(-\frac{b}{2a}\right)\right)$.
- The y-intercept of a quadratic equation in standard form is identified by looking at the constant, _________.

$$f(x) = ax^2 + bx + c$$

Example!

Identify the axis of symmetry and vertex for the following quadratic equations.

1. $f(x) = (x-5)^2 + 2$ **Axis of Symmetry**: x = 5 **Vertex:** (5, 2)	2. . $f(x) = -2(x+7)^2 - 4$ **Axis of Symmetry**: x = -7 **Vertex:** (-7, -4)

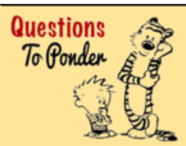
Identify the axis of symmetry, vertex, and y-intercept for the following quadratic equations.

3. $f(x) = x^2 - 2x - 5$ **Axis of Symmetry:** $-\frac{b}{2a} = -\frac{(-2)}{2(1)} = -\frac{-2}{2} = 1$ **Vertex:** $f\left(-\frac{b}{2a}\right) = f(1) = (1)^2 - 2(1) - 5 = -6$ Therefore, vertex is $(1, -6)$ **Y-Intercept:** $(0, -5)$	4. $f(x) = 4x^2 + 4x + 3$ **Axis of Symmetry:** $-\frac{b}{2a} = -\frac{(4)}{2(4)} = -\frac{4}{8} = -\frac{1}{2}$ **Vertex:** $f\left(-\frac{b}{2a}\right) = f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right) + 3 = 2$ Therefore, vertex is $\left(-\frac{1}{2}, 2\right)$ **Y-Intercept:** $(0, 3)$

SELF CHECK

What characteristics can you identify from each of the equations?

$f(x) = \frac{1}{2}(x+4)^2 - 2$	$f(x) = -x^2 + 2x + 4$
________ Axis of Symmetry ________ Vertex ________ Y-intercept	________ Axis of Symmetry ________ Vertex ________ Y-intercept



Axis of symmetry, vertex, and y-intercept are not the only characteristics of quadratic functions. What other previously learned characteristics are **needed** to graph?

Vertex Form: $f(x) = a(x - h)^2 + k$	**Standard Form:** $f(x) = ax^2 + bx + c$
#1-6. Identify the axis of symmetry, vertex, and y-intercept (if possible) for the given quadratic equation.	
1. $f(x) = 3x^2 - 2x - 2$	2. $f(x) = x^2 + 3x + 6$
3. $f(x) = 2(x-1)^2 - 4$	4. $f(x) = (x+3)^2 + 10$
5. $f(x) = -(x-3)^2 + 1$	6. $f(x) = 2x^2 - x + 4$

QUADRATIC FORMS – VERTEX, GRAPHED, AND STANDARD

Quadratic Functions: Putting it all Together

We've learned about three forms for quadratic functions, and we've covered all of the skills needed to convert between the forms (distributing, factoring, and completing the square). You can now find enough information about a quadratic function (vertex, zeros, *y*-intercept) to be able to sketch the graph without a calculator.

Three forms for quadratic functions

Here are three general forms that can be used for writing formulas for quadratic functions. Each has its advantages.

form	function formula	main advantage
standard	$f(x) = ax^2 + bx + c$	ready for using the Quadratic Formula
factored	$f(x) = (px + q)(rx + s)$ OR $f(x) = a(x - x_1)(x - x_2)$	Find zeros by solving the equations $px + q = 0$ and $rx + s = 0$. OR x_1 and x_2 are the zeros
vertex	$f(x) = a(x - h)^2 + k$	The vertex is (h, k).

Equation Solving Methods

- Factoring
- Quadratic Formula
- Completing the Square
- Square Roots

Converting between the forms

You've already learned all of the skills needed to change a quadratic function from any of the forms to another form. Specifically here's what's needed in each case:

conversion	how to do it
standard to factored	factoring, and maybe some extra steps
standard to vertex	completing the square OR find the vertex
factored to standard	distributing (mult. table or "FOIL") and simplifying (combine like terms)
vertex to standard	distributing (mult. table or "FOIL") and simplifying (combine like terms)

To get back and forth between factored and vertex forms, use standard form as an in-between step.

Example: Convert $f(x) = 3(x - 6)(x - 2)$ into vertex form.

Steps to get from factored to standard form

First multiply $(x - 6)(x - 2)$: $f(x) = 3(x^2 - 8x + 12)$

Distribute the 3: $f(x) = 3x^2 - 24x + 36$

Steps to get from standard to vertex form

Factor out 3 from first two terms: $f(x) = 3(x^2 - 8x \qquad) + 36$

Calculate: $-8/2 = -4$, $(-4)^2 = 16$

Complete the square: $f(x) = 3(x^2 - 8x + 16) + 36 - 48$

Write perfect square and simplify: $f(x) = 3\,(x - 4)^2 \qquad - 12$

Problems: Converting

1. Change each function into the form specified. If you're not sure what to do, see the chart on page 1.

a. $f(x) = x^2 - 4x - 96$ into factored form.

b. $f(x) = 4x^2 - 4x - 3$ into factored form.

c. $f(x) = 3(x - 4)(x + 2)$ into standard form.

Hint: First do $(x - 4)(x + 2)$ then use the 3.

d. $f(x) = -2\ (x + 5)^2 + 6$ into standard form.

Hint: First do $(x + 5)^2$, then use the –2, then the 6.

e. $f(x) = 2x^2 + 16x + 28$ into vertex form.

f. $f(x) = (x + 3)(x - 5)$ into vertex form.

Hint: First distribute, then completing-the-square.

g. $f(x) = (x - 1)^2 - 1$ into factored form.

Hint: Distribute, then simplify, then factor.

h. $f(x) = 3(x - 4)(x + 2)$ into vertex form.

i. $f(x) = 2(x - 3)^2 - 8$ into factored form.

j. $f(x) = -(x - 5)^2 - 3$ into standard form.

k. $f(x) = -2x^2 - 12x - 18$ into factored form.

l. $f(x) = -2x^2 - 12x - 18$ into vertex form.

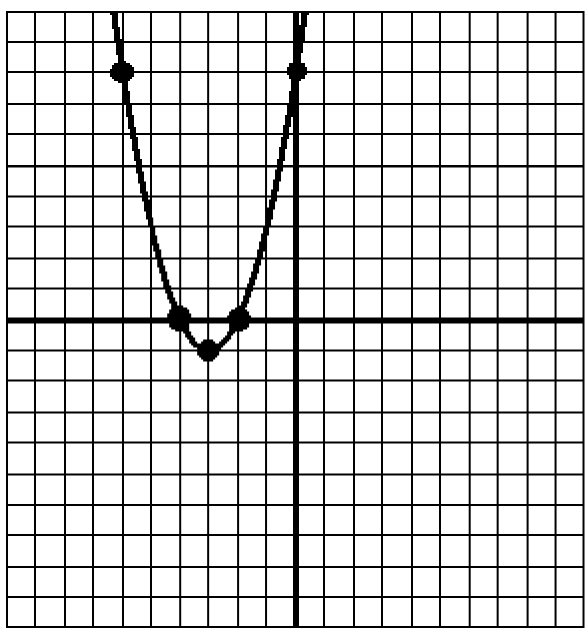
Making Graphs by Hand

If you're asked to graph of a quadratic function by hand, here's a set of steps that will produce a reasonably accurate graph:

- Use the *a* value (from any of the three forms) to get the shape: $\cup$ if $a > 0$, $\cap$ if $a < 0$.
- Find the zeros using the easiest method (factoring, square roots, completing the square, quadratic formula). Provided that there are two zeros, this will give you two points to draw.
- Find the vertex (using vertex form **or** $-b/2a$ **or** averaging the zeros). This usually gives a third point to draw.
- Find the *y*-intercept (evaluate $f(0)$, or just take the *c* value from $ax^2 + bx + c$ form). This usually gives a fourth point to draw.
- Draw a dotted line for the axis of symmetry (vertical line through the vertex). Then you can often get a fifth point by drawing the reflected image of the *y*-intercept point.
- Make sure that all the points you've found fit with the shape that you anticipated ($\cup$ or $\cap$). If everything looks OK, draw a parabola shape passing through the points you have.

Example: Make a graph of $f(x) = x^2 + 6x + 8$.

- The *a* value is 1, so expect the $\cup$ shape.
- Factoring gives the zeros: $f(x) = (x + 2)(x + 4)$ so there are points $(-2, 0)$ and $(-4, 0)$.
- Completing-the-square leads to vertex form: $f(x) = (x + 3)^2 - 1$ so the vertex is $(-3, -1)$.
- $f(0) = 8$ so the *y*-intercept is at $(0, 8)$.
- The axis of symmetry $x = -3$ shows that there's a reflected image point at $(-6, 8)$.
- Draw a graph using the five known points:



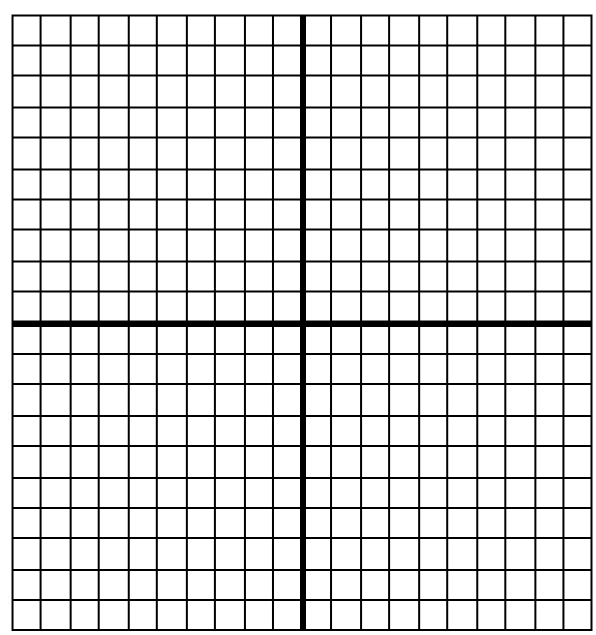
2. Do all of the following for the function $f(x) = 2x^2 - 8x + 6$, **<u>without using a calculator</u>**.

a. Using factoring, find the zeros.

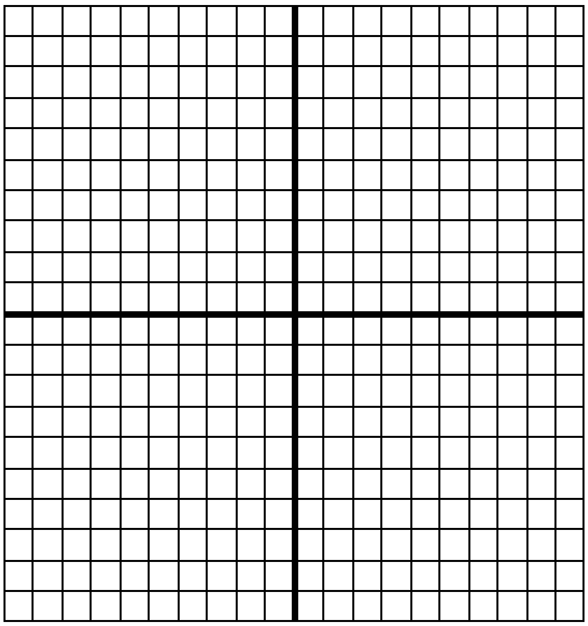
b. Using completing-the-square, find the vertex.

c. Find the y-intercept.

d. Draw the 4 points found so far, use the axis of symmetry to find a 5th point, then sketch the graph of $f(x)$.



3. Using the same sequence of steps as in problem **2**, find five points then draw the graph for $f(x) = -x^2 + 6x - 8$, **without using a calculator**.



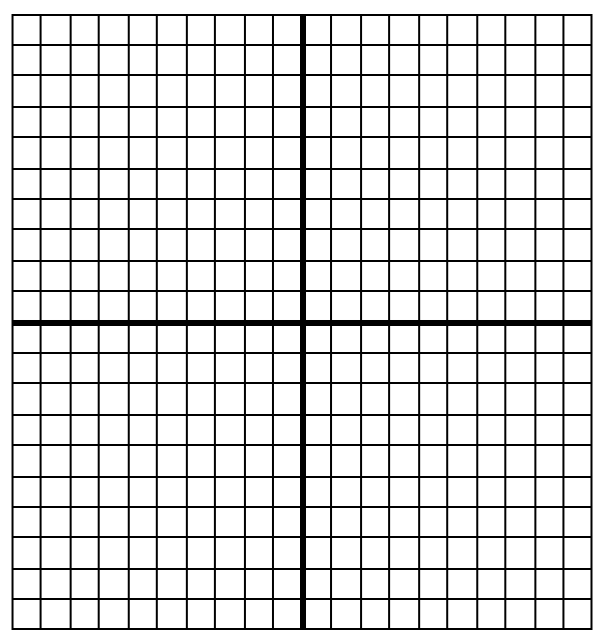
4. Do all of the following for the function $f(x) = \frac{1}{2}x^2 + 2x - 4$, **without using a calculator**.

a. Using the Quadratic Formula, find the zeros.

b. Using a formula for the *x*-coordinate of the vertex, find the vertex.

c. Find the *y*-intercept.

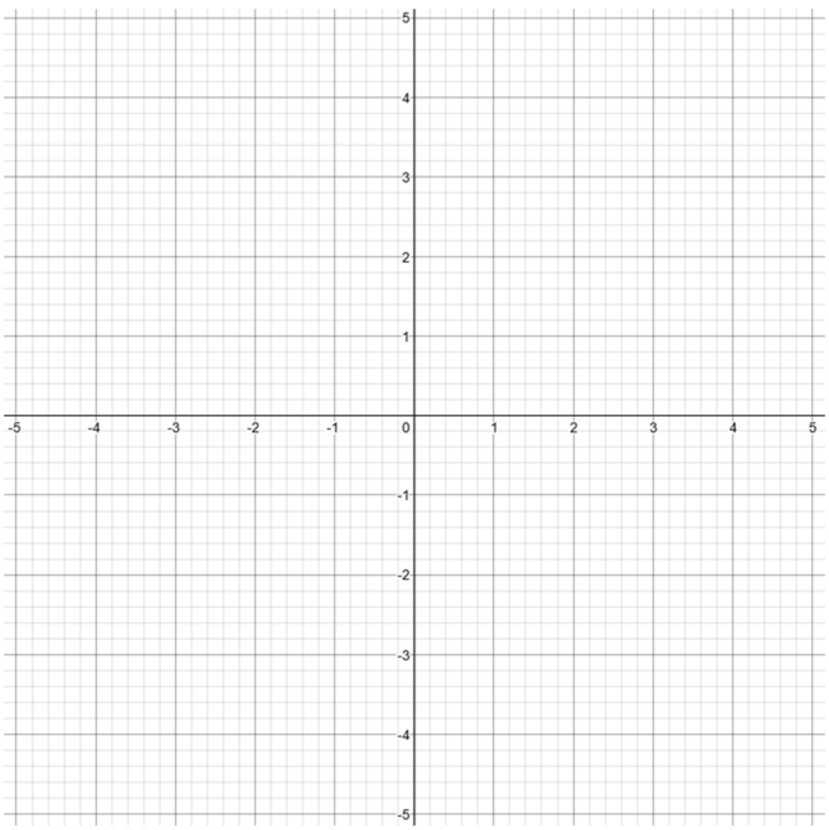
d. Draw the 4 points found so far (some of them will involve non-whole numbers but put them in approximately the correct place). Use the axis of symmetry to find a 5th point, then sketch the graph of $f(x)$.



Write the vertex form for quadratics:	**Write the standard form for quadratics:**

Complete the following table given the information provided. If you do not know the exact characteristic, provide a realistic one.			
Function	**Axis of Symmetry**	**Vertex**	**Y-Intercept**
$f(x) = x^2 - 3x - 2$			
		(2,3)	N/A
$f(x) = -2x^2 +$ ____$x +$____	x = 4		$(0, -10)$

Given the vertex and the y-intercept, graph the given function and identify other key characteristics.



Axis of Symmetry:

Vertex: (2,0)

Y-Intercept: $(0, -3)$

Domain:

Range:

End Behavior:

Intervals of Increasing:

Intervals of Decreasing:

Algebra 2 Unit #1 Concept #1

Lesson Name: Transformations of Quadratic Equations

Match the following key vocabulary terms from previous lessons with the appropriate definition or image.

__________ Vertex

__________ Horizontal Shift

__________ Reflection over X-axis

__________ Axis of Symmetry

__________ Vertical Shift

__________ Horizontal Compression

__________ Vertical Compression

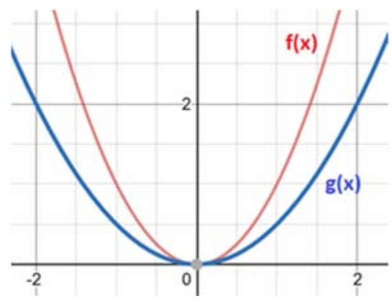
a. A transformation which moves the graph right or left

b. A vertical line that goes through the vertex of a graph.

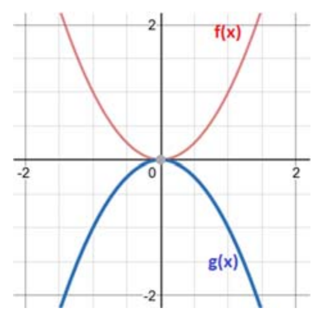
c. The turning point of a quadratic function

d. A transformation which moves the graph up or down.

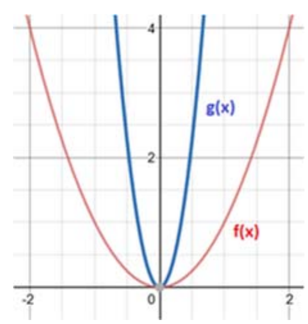
e.



f.



g.



Vertex and Standard Forms of Quadratics

We can use the **vertex form** of a quadratic function to identify and graph a transformed function.

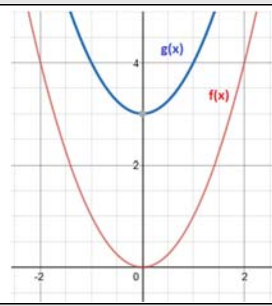
GUIDED NOTES

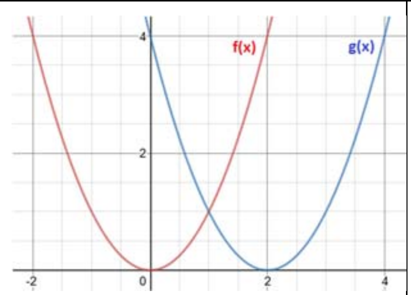
VERTEX FORM

$$f(x) = a(bx - h)^2 + k$$

Translations

Vertical Shift
"k"
If *k* is positive, the graph will shift ______.
If *k* is negative, the graph will shift ______.

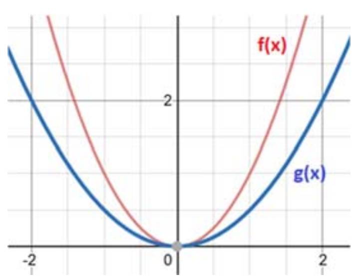


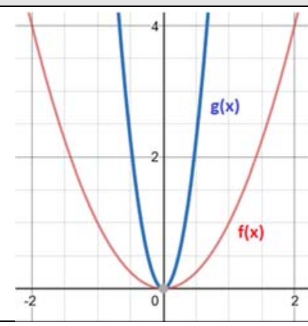


Horizontal Shift
"h"
If *h* is positive, the graph will shift ______.
If *h* is negative, the graph will shift ______.

Dilations

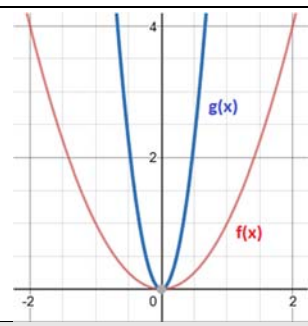
Horizontal Stretch
"b"
If 0 < b < 1 , the graph will be ______.

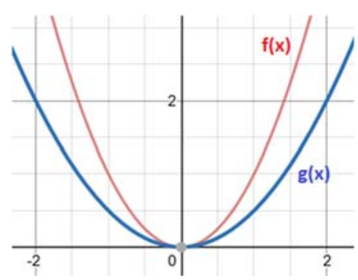




Vertical Stretch
"a"
If a > 1, the graph will be ______.

Horizontal Compression
"b"
If b > 1, the graph will be ______.



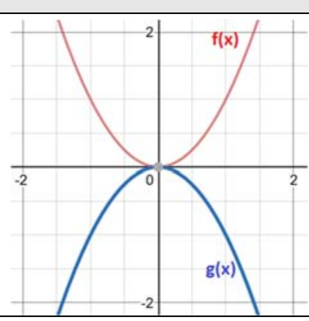


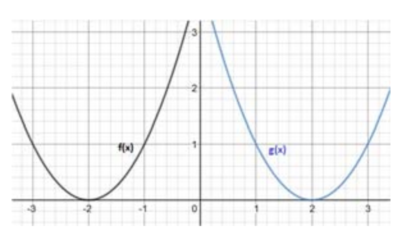
Vertical Compression
"a"
If 0 < a < 1 , the graph will be ______.

Reflections

Reflection over the x- axis

If a < _______, the graph will open down.





Reflection over the y- axis

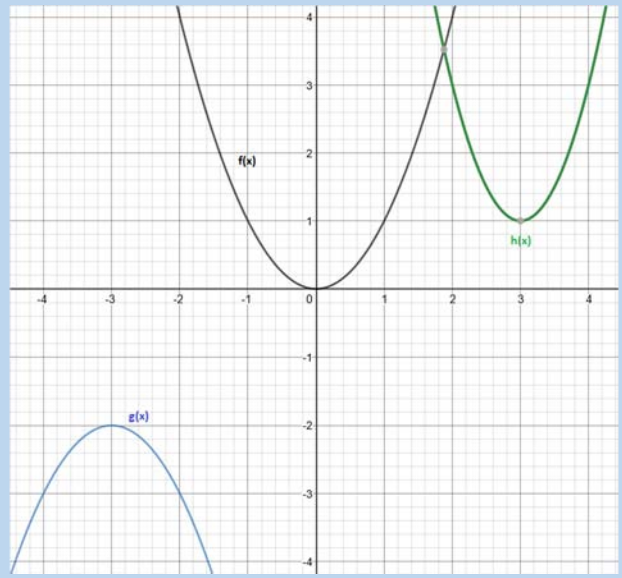
b < ______

Questions To Ponder

In the previous lesson, the vertex form did not include *b,* the coefficient of *x.* What is the importance of this variable?

Example!

Graph the following transformed quadratic equations of the parent function of f(x) = x^2.



1. $g(x) = -(x+3)^2 - 2$

Transformations:
- **Reflect over the x-axis,**
- **down two units,**
- **left 3 units**

2. $h(x) = 2(x-3)^2 + 1$

Transformations:
- **vertical stretch by a factor of two (multiply the current y value by 2)**
- **right 3 units,**
- **up one unit**

SELF CHECK

Identify the transformations of the given quadratic, $g(x) = -3(x-4)^2 + 7$.

1. ______________________________________
2. ______________________________________
3. ______________________________________
4. ______________________________________

Questions To Ponder

The effects of a horizontal compression and a vertical stretch appear to be the same. Explain how they are different algebraically.

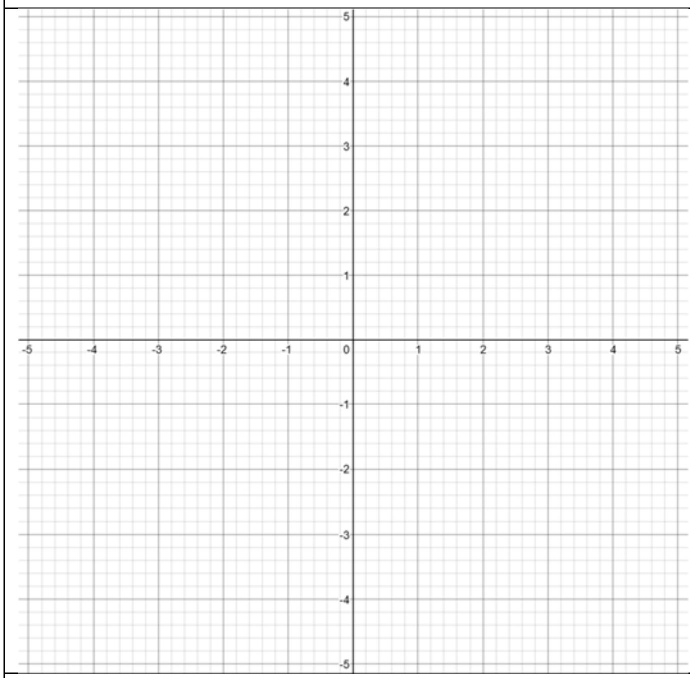
Algebra 2 Unit #1 Concept #1

Lesson Name: Transformations of Quadratic Equations

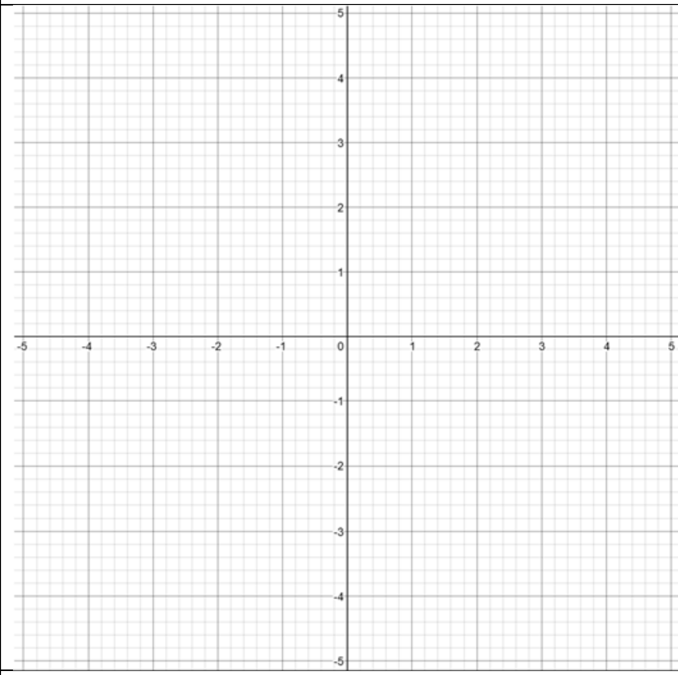
Independent Practice

Identify the vertex and transformations of the following parent function, $f(x) = x^2$. Then graph.

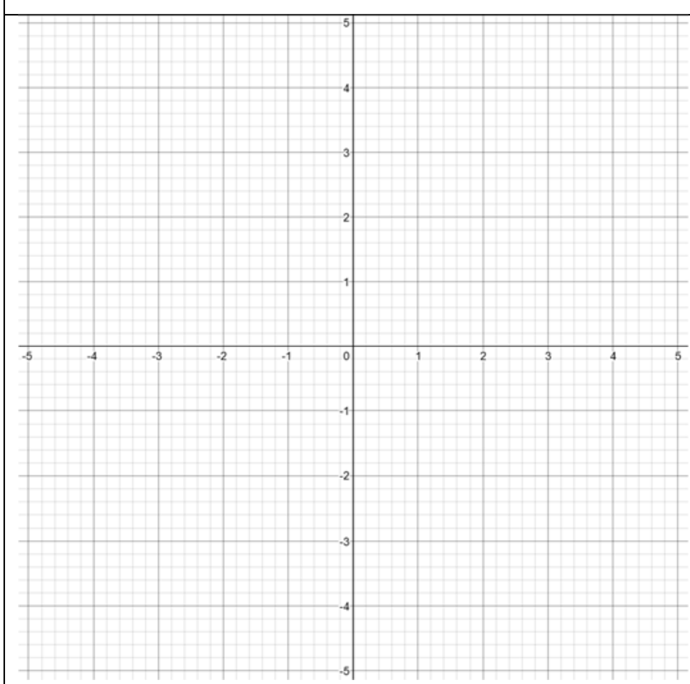
1. $g(x) = -\frac{1}{2}(x-4)^2 + 2$



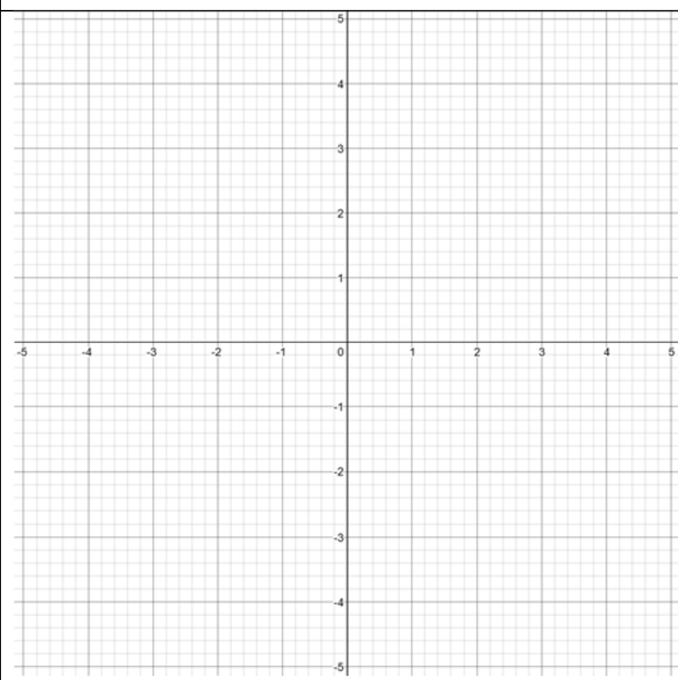
2. $g(x) = 3(-2x+1)^2$



3. $g(x) = -(x+2)^2 - 2$



4. $g(x) = -2\left(\frac{1}{3}x + 4\right)^2 + 5$



QUADRATICS INVESTIGATION: TRANSFORMING PARABOLAS 2

Transformations of the Quadratic Parent Function, $f(x)=x^2$

Classwork

Example 1: Quadratic Expression Representing a Function

a. A quadratic function is defined by $g(x) = 2x^2 + 12x + 1$. Write this in the completed-square (vertex) form and show all the steps.

b. Where is the vertex of the graph of this function located?

c. Look at the completed-square form of the function. Can you name the parent function? How do you know?

d. What transformations have been applied to the parent function to arrive at function g? Be specific.

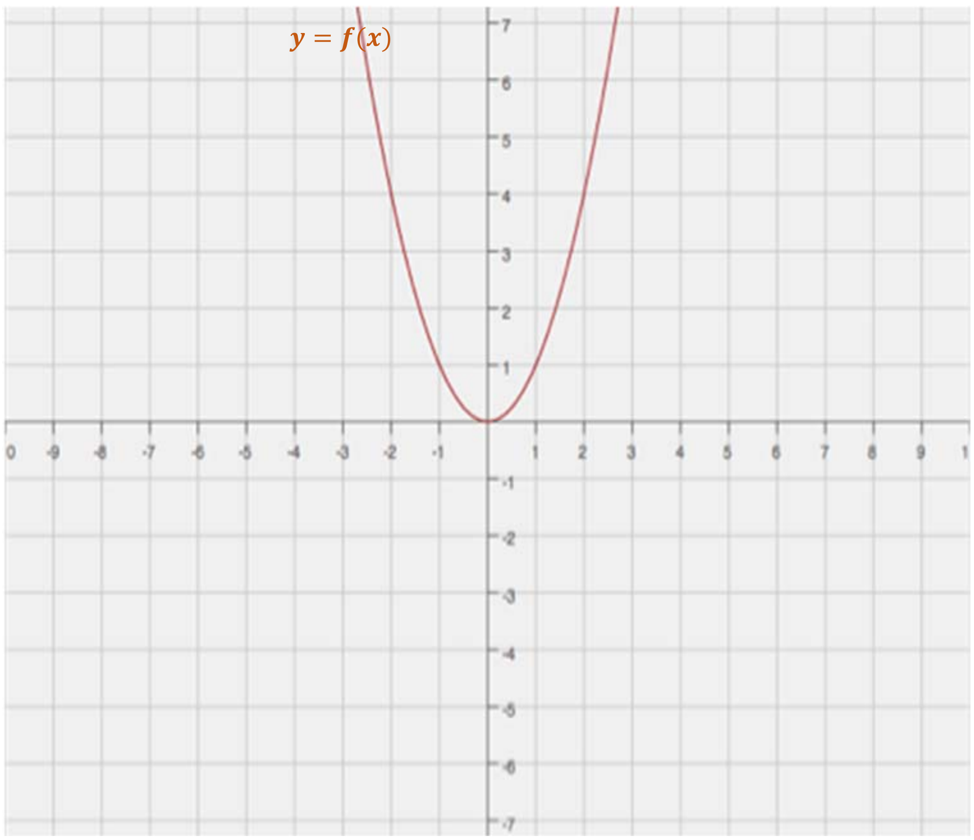
e. How does the completed-square form relate to the quadratic parent function $f(x) = x^2$?

Example 2

The graph of a quadratic function $f(x) = x^2$ has been translated 3 units to the right, vertically stretched by a factor of 4, and moved 2 units up. Write the formula for the function that defines the transformed graph.

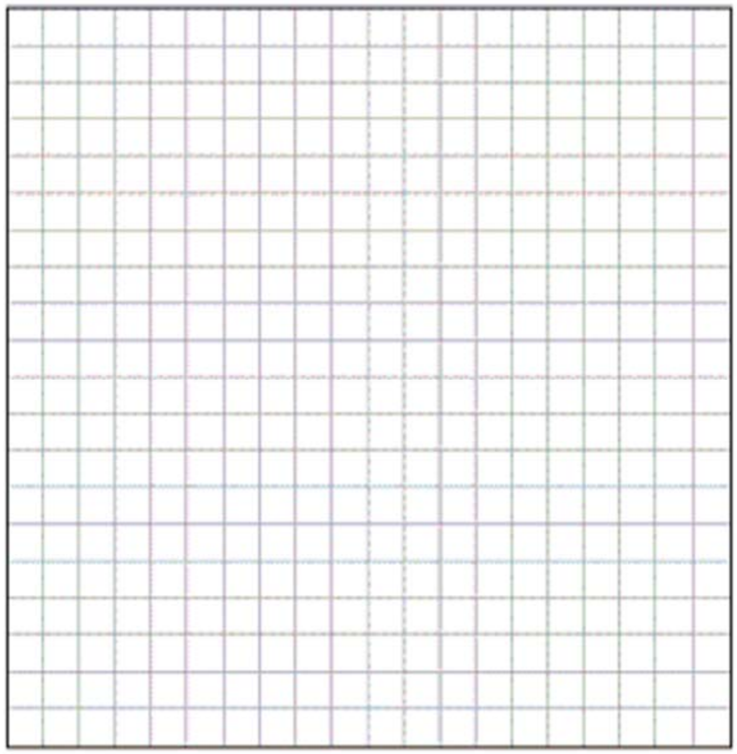
Exercises

1. Without using a graphing calculator, sketch the graph of the following quadratic functions on the same coordinate plane, using transformations of the graph of the parent function $f(x) = x^2$.
   a. $g(x) = -2(x-3)^2 + 4$
   b. $h(x) = -3(x+5)^2 + 1$
   c. $k(x) = 2(x+4)^2 - 3$
   d. $p(x) = x^2 - 2x$
   e. $t(x) = x^2 - 2x + 3$



2. Write a formula for the function that defines the described transformation of the graph of the quadratic parent function $f(x) = x^2$.
   a. 3 units shift to the right
   b. Vertical shrink by a factor of 0.5
   c. Reflection across the x-axis
   d. 4 units shift up

   Then, graph both the parent and the transformed functions on the same coordinate plane.



3. Describe the transformation of the quadratic parent function $f(x) = x^2$ that results in the quadratic function $g(x) = 2x^2 + 4x + 1$.

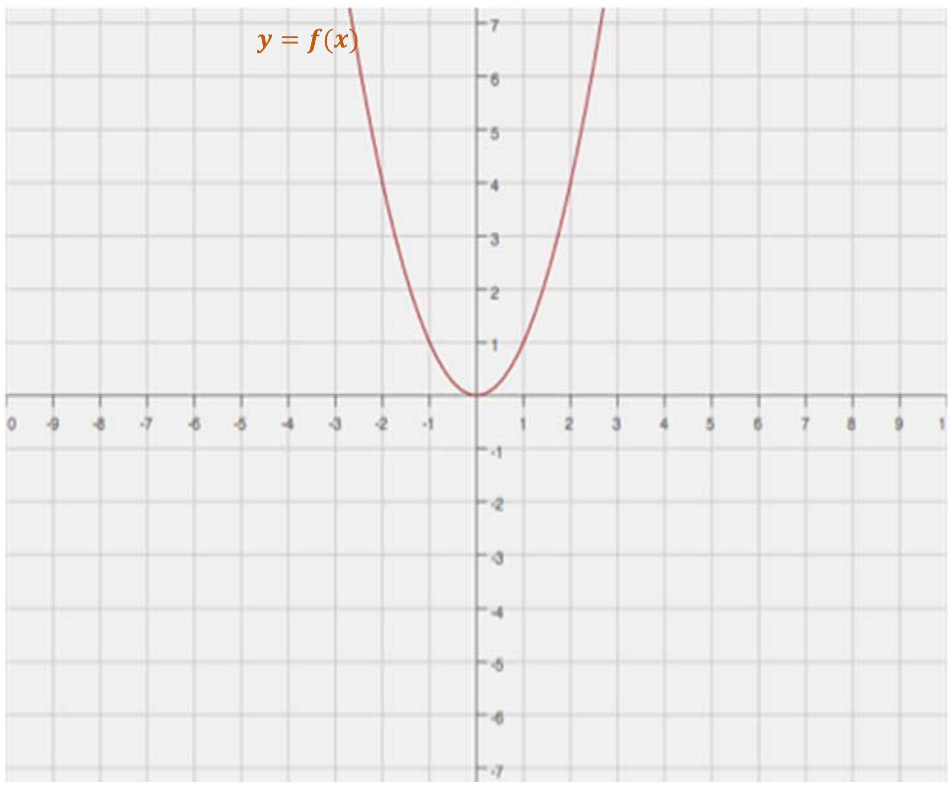
4. Sketch the graphs of the following functions based on the graph of the function $f(x) = x^2$. If necessary, rewrite some of the functions in the vertex (completed-square) form. Label your graphs.

   a. $g(x) = -(x-4)^2 + 3$

   b. $h(x) = 3(x-2)^2 - 1$

   c. $k(x) = 2x^2 + 8x$

   d. $p(x) = x^2 + 6x + 5$



Lesson Summary

Transformations of the quadratic parent function, $f(x) = x^2$, can be rewritten in form $g(x) = a(x - h)^2 + k$, where (h, k) is the vertex of the translated and scaled graph of f, with the scale factor of a, the leading coefficient. We can then quickly and efficiently (without the use of technology) sketch the graph of any quadratic function in the form $f(x) = a(x - h)^2 + k$ using transformations of the graph of the quadratic parent function, $f(x) = x^2$.

Problem Set

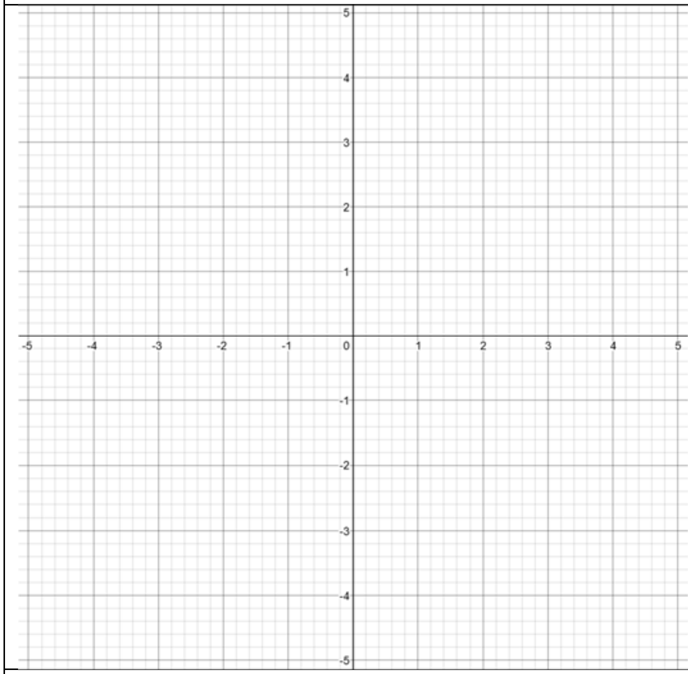
1. Write the function $g(x) = -2x^2 - 20x - 53$ in completed-square form. Describe the transformations of the graph of the parent function $f(x) = x^2$ that result in the graph of g.

2. Write the formula for the function whose graph is the graph of $f(x) = x^2$ translated 6.25 units to the right, vertically stretched by a factor of 8, and translated 2.5 units up.

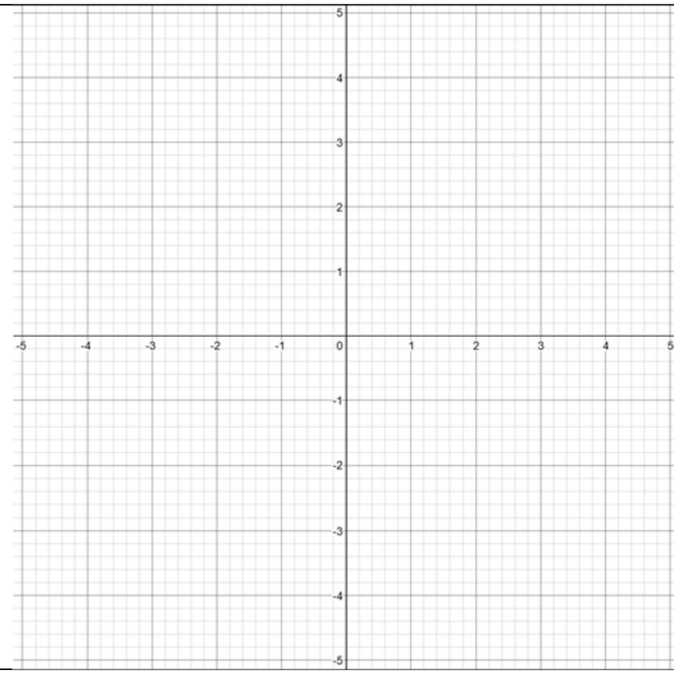
3. Without using a graphing calculator, sketch the graphs of the functions below based on transformations of the graph of the parent function $f(x) = x^2$. Use your own graph paper, and label your graphs.
   a. $g(x) = (x + 2)^2 - 4$
   b. $h(x) = -(x - 4)^2 + 2$
   c. $k(x) = 2x^2 - 12x + 19$
   d. $p(x) = -2x^2 - 4x - 5$
   e. $q(x) = 3x^2 + 6x$

Identify the vertex and transformations of the following parent function, $f(x) = x^2$. Then graph.

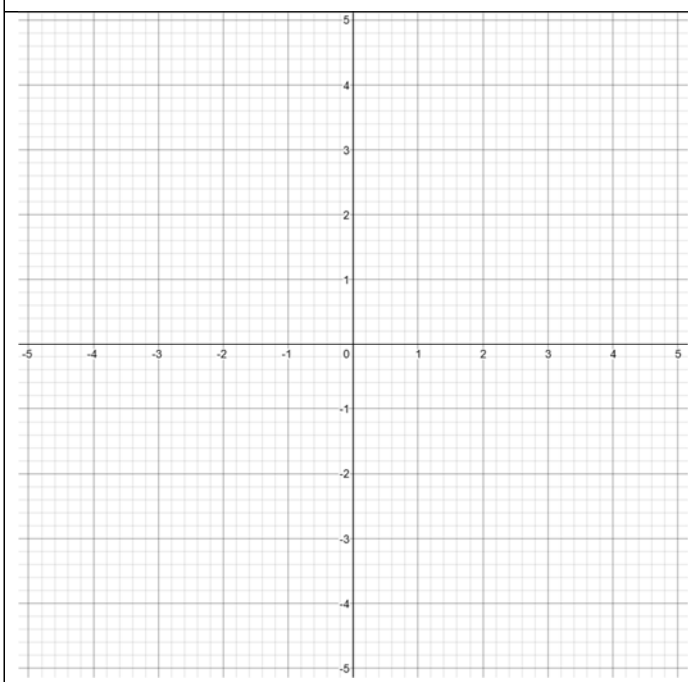
1. $g(x) = (-x+4)^2 - 2$



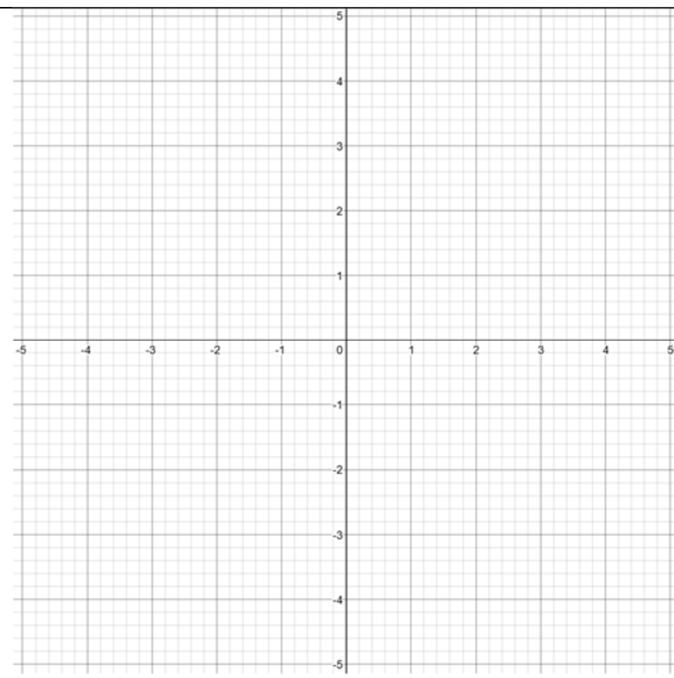
2. $g(x) = (x+2)^2 - 3$



3. $g(x) = -(x-3)^2 + 2$



4. $g(x) = -2x^2 + 1$



Algebra 2 Unit #1 Concept #1
Lesson Name: Solving Quadratics by Factoring

Vocabulary

Term	Definition	Notation	Diagram/Visual
Factor	______ ______ ______		
Zeros	______ ______ ______		
Standard form	______ ______ ______		
Factored form	______ ______ ______		
Like terms	______ ______ ______		
Distributive Property	______ ______ ______		

Key Ideas

Term	
Zero Product Property	

Solving Quadratics by Factoring

In the previous lessons, you discovered which characteristics of quadratic functions can be found in the vertex forms and standard forms of the equation. You also recognized that the zeros of a quadratic function cannot be found in either vertex or standard forms. In this lesson, you will learn to find the factored form of a quadratic function and use it to find the zeros of the quadratic function.

GUIDED NOTES

When you know the ____________________ of a quadratic equation, you can apply the Zero Product Property to find the zeros of the function.

Zero Product Property: If ________ $= 0$, then _____ $= 0$ or _____ $= 0$.

Note: We will wait until the next page to learn how to convert to this form.

Example! Use the Zero Product Property to solve $(2x+5)(x-4)=0$.

The factors of the quadratic equation are $2x+5$ and $x-4$. When they are multiplied together, their product is zero. So, we are ready to apply to Zero Product Property to solve the equation.

$(2x+5)(x-4)=0$			Given
$2x+5=0$	or	$x-4=0$	Zero Product Property
$x=-\frac{5}{2}$	or	$x=4$	Solve each linear equation (show the steps if you need to)

Questions To Ponder The order of operations tells us that we should divide before we subtract. Why is the first equation solved by subtracting then dividing?

Example! Use the Zero Product Property to solve the following equations.

$(x+1)(x+9)=0$ $\qquad$ $(x+2)(3x-1)=0$ $\qquad$ $x(x-6)=0$

Questions To Ponder Do you know how to write the factored form of a quadratic equation with the zeros 5 and 12?

We are ready to learn to find the factored form of a quadratic expression. This process is called ________________. We will start with the opposite process, which is multiplying.

Let's review the Distributive Property.

Multiply and simplify $(2x+5)(x-4)$.

$2x(\qquad\qquad) + 5(\qquad\qquad)$

$2x \cdot \_\_\_ + 2x \cdot \_\_\_ + 5 \cdot \_\_\_ + 5 \cdot \_\_\_$

$\_\_\_\_x^2 + \_\_\_\_x + \_\_\_\_x + \_\_\_\_$

Now, combine the like terms. $\_\_\_\_x^2 + \_\_\_\_x + \_\_\_\_$

Let's make some observations about the standard form and the factored form.

1. The factors of the leading coefficient in standard form are seen in the linear terms of the factored form.
2. One pair of factors of the constant in standard form are seen in the constant term of the factored form.

We will use these observations to guide us in factoring. Then, we will use the linear term to check each guess.

Example! Write the factored form of the quadratic expression

$2b^2 + 17b + 21$

Factors of $2b^2$: 2b, b

Factor pairs of 21: 1, 21
3, 7

Factors of $2b^2$ — Factor pairs of 21

(__2b__ + __7__) (__b__ + __3__)

Use any combination of pairs to make your first guess

7b

Find the inside product

6b

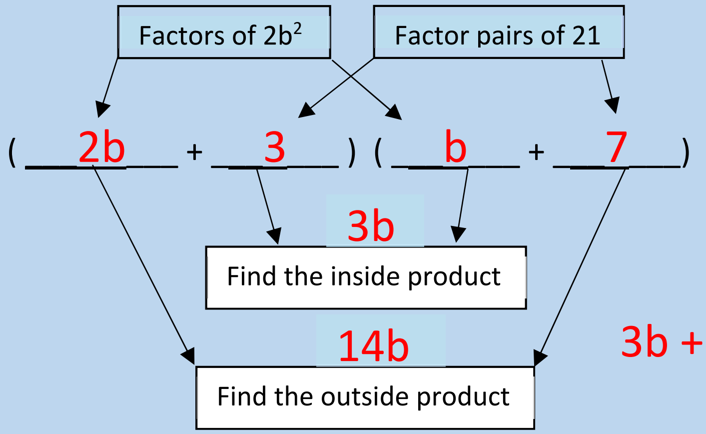
Find the outside product

$7b + 6b = 13b \neq 17b$

Since the sum of the inside and outside products is not the linear term of the quadratic expression, we chose the wrong pairs or the wrong order of factors.

Next, we will try another combination of factors.

$$2b^2 + 17b + 21$$



Factors of 2b^2: 2b, b

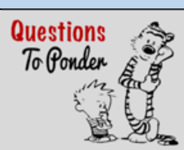
Factor pairs of 21: 1, 21
3, 7

Use any combination of pairs to make your first guess

3b + 14b = 17b

Since the sum of the inside and outside products IS the linear term of the quadratic expression, we have the correct pairs of factors. The factored form of the quadratic expression is $(2b + 3)(b + 7)$.

Tips for factoring:
1. Check the inside and outside products mentally so you do not get frustrated by a lot of writing.
2. For expressions with a negative constant, be sure to list factor pairs with both +/- and -/+ combinations.
3. If the sign of the linear term is wrong but the number is correct, switch the signs of the factors.
4. Do not erase your incorrect combinations of factors because it is easy to write the same thing again without making progress. Simply, draw a line through the combinations that do not work.
5. More tips from class discussion: ____
____.
6. ____
____.
7. ____
____.



What skills do you need from elementary and middle school to help you factor quadratic expressions?

What is the linear term of the expression $4x^2 - 9$?

Example! We are ready to factor a quadratic expression with a negative constant.
Factor $5p^2 - p - 18$.

Factors of $5p^2$ | Factor pairs of -18

(_______ + ________) (________ + ________)

Find the inside product

Find the outside product

Factors of $5p^2$:

Factor pairs of -18:

Use any combination of pairs to make your first guess

______ + ______ = ______

Is this equal to the linear term?
If not, try another combination.

(_______ + ________) (________ + ________)

Find the inside product

Find the outside product

Use any combination of pairs to make your next guess

______ + ______ = ______

Is this equal to the linear term?
If not, try another combination.

(_______ + ________) (________ + ________)

Find the inside product

Find the outside product

Use any combination of pairs to make your next guess

______ + ______ = ______

Is this equal to the linear term?
If not, try another combination.

(_______ + ________) (________ + ________)

Find the inside product

Find the outside product

Use any combination of pairs to make your next guess

______ + ______ = ______

Is this equal to the linear term?

Questions To Ponder In the previous examples, the leading coefficient was prime. What changes when the leading coefficient is composite?

You are ready to combine factoring with the Zero Product Property to find zeros of quadratic functions.

Example! Find the zeros of the quadratic functions.

$y = 3x^2 - 5x + 2$	Given
$0 = 3x^2 - 5x + 2$	Substitute 0 for y to find where the graph crosses the x-axis.
$0 = (3x - 2)(x - 1)$	Factor
$0 = 3x - 2$ or $0 = x - 1$	Zero Product Property
$x = \frac{2}{3}$ or $x = 1$	Solve each linear equation
$\left(\frac{2}{3}, 0\right)$ and $(1,0)$ are the zeros	Write the points for each zero

Questions To Ponder Can you graph the function in the previous example? What are the easiest characteristics to identify? Which characteristics are the minimum you must find to be able to sketch the graph?

Example! Now we can solve a quadratic equation.

$7x^2 = 32x + 60$	Given
______ + ______ + ______ = 0	Put the equation in standard form
	List the factors of the quadratic term:
	List the factors of the constant:
(____ + ____) (____ + ____) = 0	Make your first guess for the factors.
_____ + _____ = _____	Are the factors correct? If not, try another one.
(____ + ____) (____ + ____) = 0	Make your next guess for the factors.
_____ + _____ = _____	Are the factors correct? If not, try another one.
(____ + ____) (____ + ____) = 0	Make your next guess for the factors.
_____ + _____ = _____	Are the factors correct? If not, try another one.

The factored form is
(7x + ____)(____ - 6) = 0

_________ = 0 or _________ = 0	Zero Product Property
x = _____ or x = _____	Solve each linear equation

SELF CHECK Since you can use the Distributive Property to check factoring, you are able to check your work for problems in this lesson. Try the following:

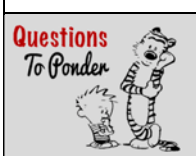
Factor the expressions.

1. $2x^2 + 11x + 12$

2. $10x^2 - 31x + 15$

3. Find the zeros of the function. $16x^2 - 25 = 0$

4. Solve the quadratic equation. $2 = 4x^2 - 7x$



$f(x) = x^2 + 3x + 3$ is a quadratic function that does not factor. What ideas do you have for finding its zeros?

Factor the expressions.

$2x^2 - 21x - 11$

$6x^2 + 11x + 4$

Find the zeros of the functions.

$f(x) = x^2 - 5x + 6$

$g(x) = 2x^2 - 18$

Solve the quadratic equation.

$5x^2 - 10x = 2x^2 + 4x - 5$

$3x^2 + 20x - 1 = -6x^2 + 8x - 5$

Quadratics Factor to Solve– *The Protein Bar Toss Learning Task*

Blake and Zoe were hiking in a wilderness area. They came up to a scenic view at the edge of a cliff. As they stood enjoying the view, Zoe asked Blake if he still had some protein bars left, and, if so, could she have one. Blake said, "Here's one; catch!" As he said this, he pulled a protein bar out of his backpack and threw it up to toss it to Zoe. But the bar slipped out of his hand sooner than he intended, and the bar went straight up in the air with his arm out over the edge of the cliff. The protein bar left Blake's hand moving straight up at a speed of 24 feet per second. If we let t represent the number of seconds since the protein bar left the Blake's hand and let $h(t)$ denote the height of the bar, in feet above the ground at the base of the cliff, then, assuming that we can ignore the air resistance, we have the following formula expressing $h(t)$ as a function of t,

$$h(t) = -16t^2 + 24t + 160.$$

In this formula, the coefficient on the t^2-term is due to the effect of gravity and the coefficient on the t-term is due to the initial speed of the protein bar caused by Blake's throw. In this task, you will explore, among many things, the source of the constant term.

1. In Algebra 1, you considered a formula for the distance fallen by an object dropped from a high place. List some ways in which this situation with Blake and the protein bar differs from the situation previously studied.

2. **Use technology to graph the equation** $y = -16t^2 + 24t + 160$. Find a viewing window that includes the part of this graph that corresponds to the situation with Blake and his toss of the protein bar. What viewing window did you select?

3. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

4. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what happened and position himself to catch it? Justify your answer graphically and algebraically.

5. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer graphically. Then write a quadratic equation that you would need to solve to justify the answer algebraically.

The equation from item 5 can be solved by factoring, but it requires factoring a quadratic polynomial where the coefficient of the x^2-term is not 1. Our next goal is to learn about factoring this type of polynomial. We start by examining products that lead to such quadratic polynomials.

6. For each of the following, perform the indicated multiplication and use a rectangular model to show a geometric interpretation of the product as area for positive values of *x*.
   a. $(2x+3)(3x+4)$
   b. $(x+2)(4x+11)$
   c. $(2x+1)(5x+4)$

7. For each of the following, perform the indicated multiplication.
   a. $(2x-3)(9x+2)$
   b. $(3x-1)(x-4)$
   c. $(4x-7)(2x+9)$

The method for factoring general quadratic polynomial of the form ax^2+bx+c, with *a, b,* and *c* all non-zero integers, is similar to the method learned in Mathematics I for factoring quadratics of this form but with the value of *a* restricted to *a* = 1. The next item guides you through an example of this method.

8. Factor the quadratic polynomial $6x^2+7x-20$ using the following steps.
   a. Think of the polynomial as fitting the form ax^2+bx+c.

      What is *a?* ____ What is *c*? ____ What is the product *ac*? ____

   b. List all possible pairs of integers such that their product is equal to the number *ac*. It may be helpful to organize your list in a table. Make sure that your integers are chosen so that their product has the same sign, positive or negative, as the number *ac* from above, and make sure that you list all of the possibilities.

   c. What is *b* in the quadratic polynomial given? _____ Add the integers from each pair listed in part b. Which pair adds to the value of *b* from your quadratic polynomial? We'll refer to the integers from this pair as *m* and *n*.

d. Rewrite the polynomial replacing *bx* with $mx + nx$. [*Note either m or n could be negative; the expression indicates to add the terms mx and nx including the correct sign.*]

e. Factor the polynomial from part d by grouping.

f. Check your answer by performing the indicated multiplication in your factored polynomial. Did you get the original polynomial back?

9. Use the method outlined in the steps of item 8 to factor each of the following quadratic polynomials. Is it necessary to always list all of the integer pairs whose product is *ac*? Explain your answer.

a. $2x^2 + 3x - 54$

b. $4w^2 - 11w + 6$

c. $3t^2 - 13t - 10$

d. $8x^2 + 5x - 3$

e. $18z^2 + 17z + 4$

f. $6p^2 - 49p + 8$

10. If you are reading this, then you should have factored all of the quadratic polynomials listed in item 9 in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers.
    a. Compare your answers with other students, or other groups of students. Did everyone in the class write their answers in the same way? Explain how answers can look different but be equivalent.

    b. Factor $24q^2-4q-8$ completely.

Show that $24q^2-4q-8$ can be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers, using three different pairs of factors.

    c. How should answers to quadratic factoring questions be expressed so that everyone who works the problem correctly lists the same factors, just maybe not in the same order?

11. If a quadratic polynomial can be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers, the method you have been using will lead to the answer, specifically called the correct ***factorization***. As you continue your study of mathematics, you will learn ways to factor quadratic polynomials using numbers other than integers. For right now, however, we are interested in factors that use integer coefficients. Show that each of the quadratic polynomials below <u>cannot</u> be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers.
    a. $4z^2+z-6$
    b. t^2+2t+8
    c. $3x^2+15x-12$

Factors		Product	Difference
1	24	24	23
2	12	24	10
3	8	24	5
4	6	24	2

Factors		Product	Difference
1	36	36	35
2	18	36	16
3	12	36	9
4	9	36	5
6	6	36	0

12. Now we return to our goal of solving the equation from item 5. Recall that you solved quadratic equations of the form $ax^2 + bx + c = 0$, with *a* = 1, in Mathematics I. The method required factoring the quadratic polynomial and using the Zero Factor Property. The same method still applies when $a \neq 1$, its just that the factoring is more involved, as we have seen above. Use your factorizations from items 9 and 10 as you solve the quadratic equations below.
    a. $2x^2 + 3x - 54 = 0$
    b. $4w^2 + 6 = 11w$
    c. $3t^2 - 13t = 10$
    d. $2x(4x+3) = 3 + x$
    e. $18z^2 + 21z = 4(z-1)$
    f. $8 - 13p = 6p(6-p)$
    g. $24q^2 = 4q + 8$

13. Solve the quadratic equation from item 5. Explain how the solution gives an algebraic justification for your answer to the question.

14. Suppose the cliff had been 56 feet higher. Answer the following questions for this higher cliff.
    a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?
    b. What is the formula for the height function in this situation?
    c. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what has happened and position himself to catch it? Justify your answer algebraically.
    d. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

15. Suppose the cliff had been 88 feet lower. Answer the following questions for this lower cliff.

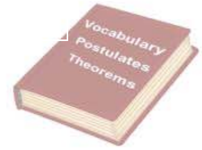
   a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

   b. What is the formula for the height function in this situation?

   c. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what has happened and position himself to catch it? Justify your answer algebraically.

If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

Your teacher will assess your class to determine what homework is best for you. Factoring is an important concept in Algebra 2, so your fluency with the skill is imperative. Please do your best!



Vocabulary

Term	Definition	Notation	Diagram/Visual
Discriminant			
Real number			
Non-real number			
Rational number			
Irrational number			

Key Ideas

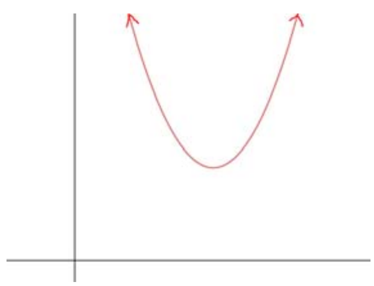
Term	
Simplest form of a square root	

Solving Quadratics by the Quadratic Formula

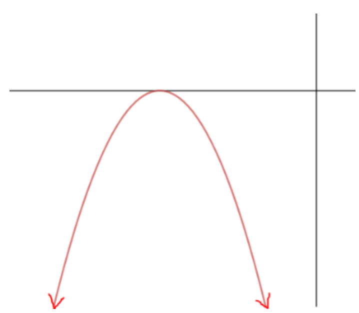
We are continuing our study of characteristics quadratic functions. In the last lesson, we focused on the zeros of a function, and we reviewed factoring. It is important that the student knows that not all quadratic expressions can be factored. In this lesson, we will study a strategy for solving quadratics that can be used with all expressions. We will observe graphs and equations that have non-real solutions as well as graphs and equations that have irrational solutions.

The number of real zeros of a quadratic function can be observed on a graph. The x-axis is a real number line, so each point where the graph crosses the x-axis is a real zero. There are three cases:

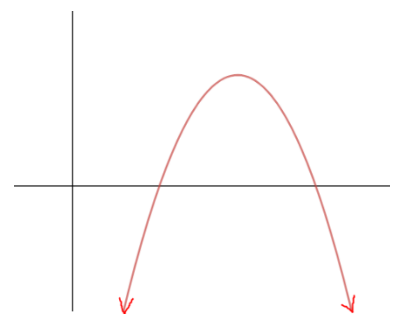
no real zeros



one real zero



two real zeros



When a quadratic function cannot be factored, we can use the ________________ __________________ to find the zeros.

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

You should pay attention to the form of the equation in the hypothesis. It is a quadratic equation in ________________ form, and it is equal to _____________. Before you can apply the Quadratic Formula, the equation must be in this form.

In the conclusion, there are _____ zeros represented by the formula. One is found by adding, and one is found by subtracting.

The expression $b^2 - 4ac$ is called the ____________________. The value of the discriminant will reveal important information about the zeros of the function.

Value of discriminant	Types of zeros	Representation on a graph
$b^2 - 4ac < 0$	No real zeros	
	One real zero	
$b^2 - 4ac > 0$, and it is a perfect square		
	Two real, irrational zeros	

Example! Find the zeros of the function.

$$f(x) = 3x^2 + 11x + 5$$

All zeros have y-coordinates of 0, so substitute for $f(x)$.
Check that the function does not factor. This is relatively simple since 3 and 5 are prime.
Since the function is in standard form, we can use the Quadratic Formula to solve the equation.

$0 = 3x^2 + 11x + 5$ — We see that $a = 3, b = 11, c = 5$.

$$x = \frac{-(11) \pm \sqrt{(11)^2 - 4(3)(5)}}{2(3)}$$ Substitute into the Quadratic Formula.

$$x = \frac{-11 \pm \sqrt{121 - 60}}{6}$$ Simplify exponents and perform multiplication.

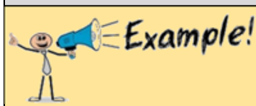
$$x = \frac{-11 \pm \sqrt{61}}{6}$$ Simplify the radical.

The discriminant is 61, which is greater than zero but it is not a perfect square. To graph the points, we will need decimal approximations of these two irrational numbers.

$$x = \frac{-11+\sqrt{61}}{6} \approx -0.53 \text{ and } x = \frac{-11-\sqrt{61}}{6} \approx -3.14$$

Therefore, the zeros of the function are (−0.53, 0) and (−3.14, 0).

Are the zeros real? Are the zeros irrational?



Find the zeros of the function.

$$f(x) = -2x^2 - 6x - 7$$

All zeros have y-coordinates of ______, so substitute for $f(x)$. Check that the function does not factor. Since the function is in standard form, we can use the ________________ ________________ to solve the equation.

$0 =$ ________________________ — We see that $a =$ _____, $b =$ _____, $c =$ _____.

$$x = \frac{-(\quad) \pm \sqrt{(\quad)^2 - 4(\quad)(\quad)}}{2(\quad)}$$ Substitute into the Quadratic Formula.

$$x = \frac{\quad \pm \sqrt{\quad - \quad}}{\quad}$$ Simplify exponents and perform multiplication.

$$x = \frac{\quad \pm \sqrt{\quad}}{\quad}$$ Simplify the radical.

The discriminant is ______ which is less than zero. So, we cannot graph the points. The graph does not cross the x-axis.

Questions To Ponder

How do you simplify the radical if the discriminant is 20? 24? 75? 90?

Example!

Solve the equation.

$$4x^2 = 9$$

First, get the equation in standard form with 0 on the other side. The equation does factor, but we will continue with the Quadratic Formula for this lesson.

$$4x^2 - 9 = 0$$

We see that $a =$ _____, $b =$ _____, $c =$ _____.

$$x = \frac{-(\quad) \pm \sqrt{(\quad)^2 - 4(\quad)(\quad)}}{2(\quad)}$$

Substitute into the Quadratic Formula.

$$x = \frac{\quad \pm \sqrt{\quad - \quad}}{\quad}$$

Simplify exponents and perform multiplication.

$$x = \frac{\quad \pm \sqrt{\quad}}{\quad}$$

Simplify the radical.

$x =$ ______ and $x =$ ______

List the solutions separately.

Questions To Ponder

Which types of discriminants indicate that the quadratic function could have been factored?

Use the Quadratic Formula to find the real zeros of each function. If the zeros are non-real, you do not need to simplify the expression. If the zeros are real, tell whether they are irrational or rational.

$f(x) = x^2 + 6x - 9$

The zeros are ____________________.

$g(x) = -5x^2 - 20$

The zeros are ____________________.

Use the Quadratic Formula to solve the quadratic equations.

$8x^2 - 20x = 8 + 9x - 4x^2$

$5x + 10 - 3x^2 = x^2$

Quadratic formula

1. Start with $ax^2 + bx + c = 0$, and derive the quadratic formula by completing the square.

2. Verify that it is true.

3. Explore what happens for the cases $b = 0$, $c \neq 0$, $b \neq 0$, $c = 0$, and $b = c = 0$.

4. The part under the square root symbol is called the discriminant.
   a. What happens when the discriminant is equal to zero?

   b. What happens when the discriminant is greater than zero?

   c. What happens when the discriminant is less than zero?

   d. What happens when the discriminant is a perfect square?

Use the Quadratic Formula to find the real zeros of each function. If the zeros are non-real, you do not need to simplify the expression. If the zeros are real, tell whether they are irrational or rational.

$f(x) = 3x^2 + 7x + 2$

$g(x) = 6x^2 - x + 10$

The zeros are ___________________.

The zeros are ___________________.

- -

Use the Quadratic Formula to solve the quadratic equations.

$2x - 5 - x^2 = -3x^2 + 11x$

$0 = -x^2 - 4x + 8$

Key Ideas

Term	
Order of operations	
$\sqrt{x^2} = \pm x$	

Solving Quadratics by Inverse Operations

As we continue our study of characteristics of quadratic functions, we recall that we have learned how to find zeros of quadratic functions when they are given in standard form. We can factor or use the Quadratic Formula. In this lesson, we will learn how to find the zeros of a quadratic function when it is given in vertex form.

Example! When a function is given in vertex form, we can find the zeros by solving for x using inverse operations.

$f(x) = -3(x-4)^2 + 12$	Given
$0 = -3(x-4)^2 + 12$	Substitute 0 for the y-value because the zeros have y-coordinates of 0.
$-12 = -3(x-4)^2$	Subtract 12.
$4 = (x-4)^2$	Divide by -3.
$\pm 2 = x - 4$	Take the square root.
$2 = x - 4$ or $-2 = x - 4$	Separate into two equations.
$6 = x$ or $2 = x$	Solve each linear equation.
(6, 0) or (2, 0)	Write the zeros as points on a graph.

Questions To Ponder Is $g(x) = 10x^2 - 40$ in standard form or vertex form? Which strategy would you use to find its zeros?

Example! Use inverse operations to solve the equation.

$7 = 2(x+5)^2 + 1$	Given
_____$= 2(x+5)^2$	__________ 1.
_____ $= (x$_____$)^2$	_________ by _____.
$\pm$ _____ $=$ ______	Take the square root.
____________ or ____________	Separate into two equations.
___________ or ___________	Solve each linear equation.

Walk-Around Task: Using the Square Root Property

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Table of Contents

Instructions

This activity is designed to help students with solving quadratic equations using the square root property. This is a beginning to level 1 activity, which means all the square roots are perfect squares.

This activity also gets students up and about. Place the 10 cards on the wall around your room. Students pick any card to begin with. They should work the problem on the page, then students look around the room for a graph that matches theirs. They should continue working until they return to the card they started with.

Included are 2 versions of this activity.

- One with 10 cards for use as a walk-around activity (and an absent student size)
- A shorter version with only 5 cards great for table work, bell work or use as a review.

To help with grading, I have included 2 student answer sheets: One that includes room to show work, and another with just boxes.

Hints and suggestions:

When making copies of the shorter version, I use different colored paper. This allows for easier identification when students misplace a piece.

Possible Uses

- Mid-Lesson or End of Lesson Check for understanding
- Math Station for students that have finished work early
- Test Review
- Homework Alternative
- Bell work

1

Solve for m

$$m^2 - 12 = 109$$

Answer from another card → **4 or -6**

2

Solve for c

$$3c^2 - 17 = 10$$

Answer from another card → **3.5**

3

Solve for x

$$\frac{1}{2}x^2 = 18$$

Answer from another card

11 or 5

4

Solve for v

$$(v+5)^2 = 4$$

Answer from another card → **11 or -11**

5

Solve for y

$$(y+1)^2 = 25$$

Answer from another card

-1 or 13

6

Solve for x

$$(x-6)^2 = 49$$

Answer from another card → **-4 or 14**

7

Solve for x

$$(2x-7)^2 = 0$$

Answer from another card → **-7 or -3**

8

Solve for y

$$(y-4)^2 = 64$$

Answer from another card → **6 or - 6**

9

Solve for w

$$(w-5)^2 = 81$$

Answer from another card → **12 or -4**

10

Solve for x

$$3(x-8)^2 = 27$$

Answer from another card → **3 or - 3**

Name: ____________________

1	**2**	**3**
$m^2 - 12 = 109$	$3c^2 - 17 = 10$	$\frac{1}{2}x^2 = 18$
________	________	________
Answer from another card → **11 or 5**	Answer from another card → **3.5**	Answer from another card → **-1 or 13**
4	**5**	
$(v+5)^2 = 4$	$(y+1)^2 = 25$	
________	________	
Answer from another card → **11 or -11**	Answer from another card → **11 or 5**	

6

$(x-6)^2 = 49$

______

Answer from another card → **-4 or 14**

7

$(2x-7)^2 = 0$

______

Answer from another card → **-7 or -3**

8

$(y-4)^2 = 64$

______

Answer from another card → **6 or -6**

9

$(w-5)^2 = 81$

______

Answer from another card → **12 or -4**

10

$3(x-8)^2 = 27$

______

Answer from another card → **3 or -3**

Student Response Page – Version A1

Name ____________________

Period _____

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

Show work below

Card ____	Card ____	Card ____
Card ____	Card ____	Card ____
Card ____	Card ____	Card ____
Card ____	Card ____	Card ____

Student Response Page – Version A2

Name ____________________

Period _____

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

Student Response Page

Name ____________________

Period _____

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

Student Response Page

Name ____________________

Period _____

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

1

Solve for m

$$m^2 - 12 = 109$$

Answer from another card

11 or 5

2

Solve for c

$$3c^2 - 17 = 10$$

Answer from another card → **3.5**

3

Solve for x

$$(2x-7)^2 = 0$$

Answer from another card → **-7 or -3**

4

Solve for v

$$(v+5)^2 = 4$$

Answer from another card → **11 or -11**

5

Solve for x

$$3(x-8)^2 = 27$$

Answer from another card → **3 or - 3**

Name: ____________________

1

$m^2 - 12 = 109$

____

Answer from another card → 11 or 5

2

$3c^2 - 17 = 10$

____

Answer from another card → 3.5

3

$(2x - 7)^2 = 0$

____

Answer from another card → -7 or -3

4

$(v + 5)^2 = 4$

____

Answer from another card → 11 or -11

5

$3(x - 8)^2 = 27$

____

Answer from another card → 3 or -3

Student Response Page – Version B1

Name ____________________

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

Show work below

Card ____ →	Card ____ →	Card ____
Card ____ →	Card ____ →	Card ____

Student Response Page

Name ____________________

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

Show work below

Card ____ →	Card ____ →	Card ____
Card ____ →	Card ____ →	Card ____

Student Response Page – Version B2

Name ____________________
Period _____

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

Student Response Page

Name ____________________
Period _____

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

Student Response Page

Name ____________________
Period _____

Pick a card to start with. Write the number in the 1st box. Solve the quadratic equation using the square root property, then find the solution on another card. Write the card number in the 2nd box, and then continue until you reach your first card again.

KEYS

Version A - Set of 10 Cards

1	4	7	2	10	3	8	9	6	5

Note: Students may start with any cards; shift the answer to match the starting point.

Version B: Set of 5 Cards

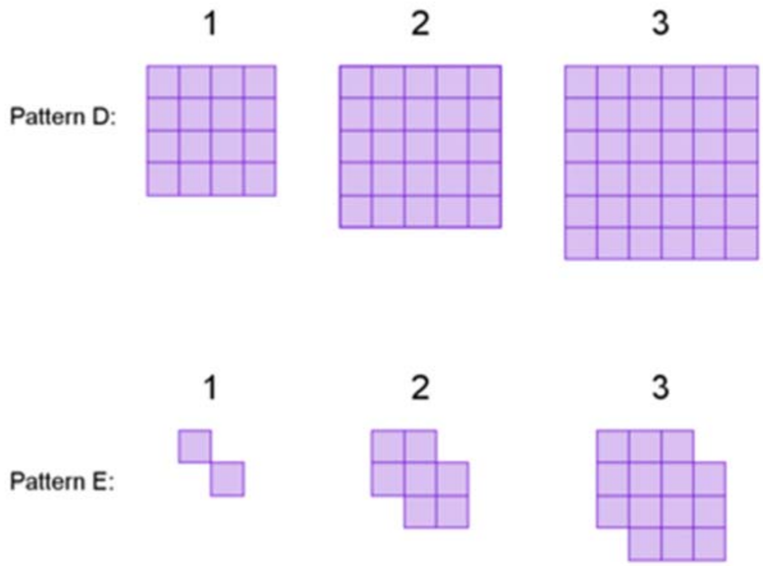
1	4	3	2	5

1. $m^2 - 12 = 109$ $m^2 = 121$ $\sqrt{(m)^2} = \pm\sqrt{121}$ $m = \pm 11$	2. $3c^2 - 17 = 10$ $3c^2 = 27$ $c^2 = 9$ $\sqrt{(c)^2} = \pm\sqrt{9}$ $c = \pm 3$	3. $\frac{1}{2}x^2 = 18$ $x^2 = 36$ $\sqrt{(x)^2} = \pm\sqrt{6}$ $x = \pm 6$
4. $(v + 5)^2 = 4$ $\sqrt{(v+5)^2} = \pm\sqrt{4}$ $v + 5 = \pm 2$ $v = -5 \pm 2$ $v = -7 \; or -3$	5. $(y + 1)^2 = 25$ $\sqrt{(y+1)^2} = \pm\sqrt{25}$ $y + 1 = \pm 5$ $y = -1 \pm 5$ $y = -6 \; or \; 4$	6. $(x - 6)^2 = 49$ $\sqrt{(x-6)^2} = \pm\sqrt{49}$ $x - 6 = \pm 7$ $y = 6 \pm 7$ $y = -1 \; or \; 13$
7. $(2x - 7)^2 = 0$ $\sqrt{(2x-7)^2} = \pm\sqrt{0}$ $2x - 7 = 0$ $2x = 7$ $v = \frac{7}{2}$	8. $(y - 4)^2 = 64$ $\sqrt{(y-4)^2} = \pm\sqrt{64}$ $y - 4 = \pm 8$ $y = 4 \pm 8$ $y = -4 \; or \; 12$	9. $(w - 5)^2 = 81$ $\sqrt{(w-5)^2} = \pm\sqrt{81}$ $w - 5 = \pm 9$ $w = 5 \pm 9$ $w = -4 \; or \; 14$
10. $3(x - 8)^2 = 27$ $(x-8)^2 = 9$ $\sqrt{(x-8)^2} = \pm\sqrt{9}$ $x - 8 = \pm 3$ $x = 8 \pm 3$ $x = 5 \; or \; 11$		

SOLVE BY SQUARE ROOTING

Task

The first three steps of two visual patterns are shown below.



The number of tiles in step *n* of Pattern D is defined by $d(n) = (n + \mathbf{3})^2$. The number of tiles in step *n* of Pattern E is defined by $e(n) = (n + \mathbf{1})^2 - \mathbf{2}$.

a. For each pattern, decide whether there is a step with 167 tiles in it. If so, which step is it? If not, explain how you know.

b. For each pattern, decide whether there is a step with 169 tiles in it. If so, which step is it? If not, explain how you know.

c. Describe and justify the steps for solving the following equation for *x*:

$$a(x - h)^2 = k$$

Find the zeros of the functions. Write your answers as points on a graph.

1. $g(x) = -2(x+1)^2 + 50$

2. $h(x) = 7(x-3)^2$

Solve each equation.

3. $26 = 6(x-2)^2 - 4$

4. $19 = 16x^2 - 1$

5. A ball is dropped from a height of 64 feet. Its height, in feet, can be modeled by the function $h(t) = -16t^2 + 64$, where *t* is the time in seconds since the ball was dropped. Use inverse operations to determine the zeros of the function. Which of the two zeros is the time it takes the ball to hit the ground?

6. The equation below is true for all real numbers *x* and only one real number *b*.

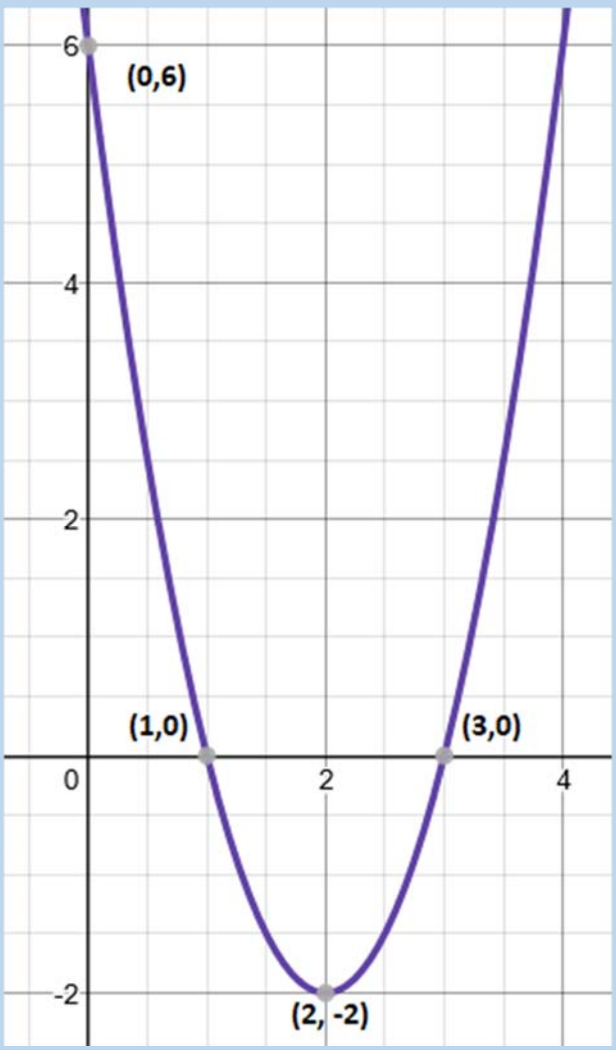
$$x^2 + 4x + 9 = (x+b)^2 + 5$$

Determine the value of *b*. Show your work.

Writing Functions Given a Graph

If given the graph of a quadratic equation, you are able to develop an equation in either vertex, factored, or standard form.

Example! Depending on the information given, you will choose which method to use. We will use the same graph to demonstrate the two methods.



Method One: Given the vertex and another point using vertex form.

Vertex: $(\mathbf{2}, -\mathbf{2})$ **Point:** $(\mathbf{1}, \mathbf{0})$

Step 1: Substitute what is given and solve for the unknown.

$$f(x) = a(x-h)^2 + k$$
$$0 = a(1-2)^2 - 2$$
$$a = 2$$

Step 2: Substitute *a, h,* and *k* into vertex form.

$$f(x) = a(x-h)^2 + k$$
$$f(x) = 2(x-2)^2 - 2$$

Converting Vertex to Standard form:

$$f(x) = a(x-h)^2 + k \quad \rightarrow \quad f(x) = ax^2 + bx + c$$

$f(x) = 2(x-2)^2 - 2$	* Expand quadratic and use distributive property
$f(x) = 2[(x-2)(x-2)] - 2$	
$f(x) = 2(x^2 - 4x + 4) - 2$	* Combine like terms
$f(x) = 2x^2 - 8x + 8 - 2$	
$\boldsymbol{f(x) = 2x^2 - 8x + 6}$	***Standard form**

Method Two: Given the zeros and another point using factored form.

Zeros: $(\mathbf{1}, \mathbf{0})$ **&** $(\mathbf{3}, \mathbf{0})$ **Additional Point:** $(\mathbf{2}, -\mathbf{2})$

Step 1: Substitute what is given and solve for the unknown.

$$f(x) = a(x - zero_1)(x - zero_2)$$
$$-2 = a(2-1)(2-3)$$
$$a = 2$$

Step 2: Substitute *a* and zeros into factored form.

$$f(x) = a(x - zero_1)(x - zero_2)$$
$$f(x) = 2(x-1)(x-3)$$

Converting Factored to Standard Form:

$$f(x) = a(x - zero_1)(x - zero_2) \rightarrow f(x) = ax^2 + bx + c$$

$f(x) = 2(x-1)(x-3)$	*Use distributive property
$f(x) = 2(x^2 - 4x + 3)$	
$\boldsymbol{f(x) = 2x^2 - 8x + 6}$	***Standard form**

Questions To Ponder

In both methods, *a* had a value of 2. Explain why this will always be true.

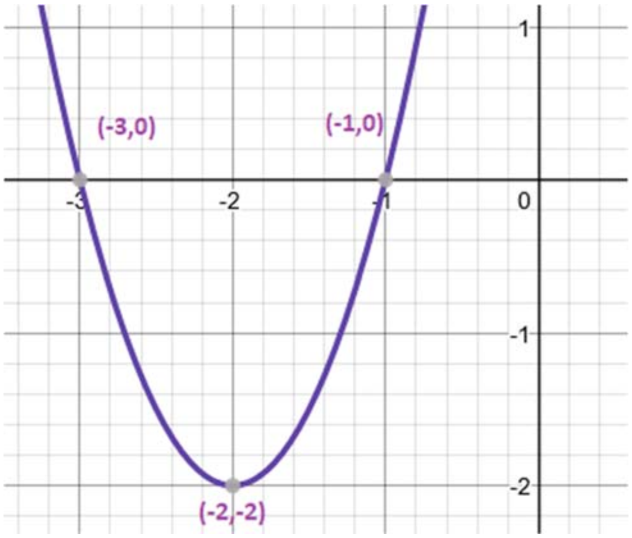
SELF CHECK

Which form would you use for each of the scenarios?

Given: Zeros: $(-\mathbf{2}, \mathbf{0})$ **&** $(-\mathbf{4}, \mathbf{0})$ **Additional Point:** $(-\mathbf{3}, -\mathbf{1})$	**Given: Vertex:** $(\mathbf{3}, \mathbf{2})$ **Point:** $(-\mathbf{1}, \mathbf{10})$

Given the graph, write an equation for the quadratic function.

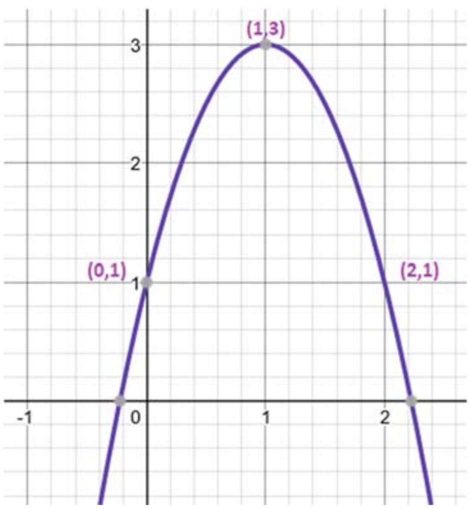
1.



Equation:

Standard Form:

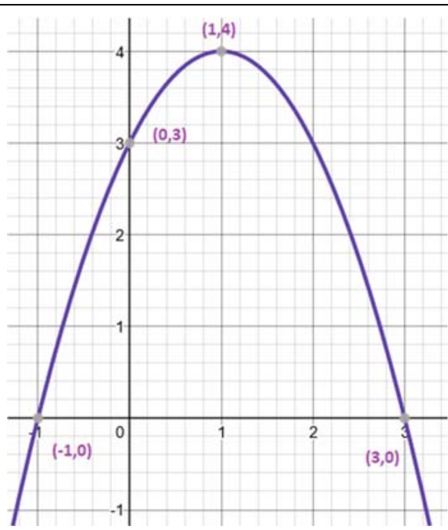
2.



Equation:

Standard Form:

3.



Equation:

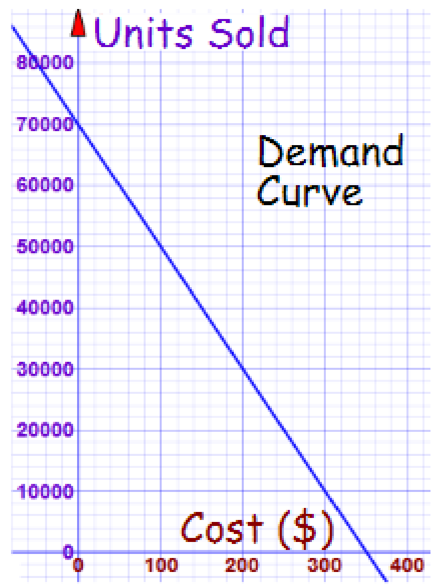
Standard Form:

WRITING A QUADRATIC FUNCTIONS FROM REAL SITUATIONS - NEW SPORTS BIKE

You have designed a new style of sports bicycle!
Now you want to make lots of them and sell them for profit.

You know that your **costs** are going to be:

- **$700,000 for manufacturing set-up costs, advertising, etc**
- **$110 to make each bike**



Based on similar bikes, you know that you can expect **sales** to follow this "Demand Curve":

- **Unit Sales:** $= \mathbf{70,000 - 200}p$

Where "p" is the price of one bike.

1. If you set the selling price, p:

a. at $0, what happens? ___________________________

at $350, what happens? _________________________

c. at $300 , what happens? ________________________

____

2. Explain the unit sales demand curve in your own words and in context. _____________________________

____
____
____
____
____
____

In lower grades social studies, you learned about some economics principles. As a reminder for yourself, define the economics terms below, explain them, and define a general equation in terms of the variables given, the unit costs, fixed costs and unit sales

Economic Term	Explain what it means in words & How you should calculate it.	Equation/Expression for this
Selling Price	**3.**	p
Unit Cost	**4.**	$\$110.00$
Fixed Costs	**5.**	$\$700,000.00$
Unit Sales (y)	**6.**	$y = 70,000 - 200p$
Total Production Cost	**7.**	**8.**
Total Sales, in Dollars	**9.**	**10.**
Total Costs	**11.**	**12.**
Total Profit (A)	**13.**	**14.**

15. What type of equation is the "total profit" equation?

16. Based on this equation type, what should its shape be?

17. Graph this equation on your own paper or using technolology.

18. Solve for where the profit equation $A = 0$ and find its solutions. Then interpret these solutions in this context.

19. But we want to know the MAXIMUM profit, don't we? Where would the best (maximum) profit be on your graph, relative to your solutions in #18?

Based on the maximum profit from #19, you can expect:

20. Unit Sales = ____________________________

21. Sales in Dollars = _____________________________

22. Costs = ______________________________________

23. Profit = ____

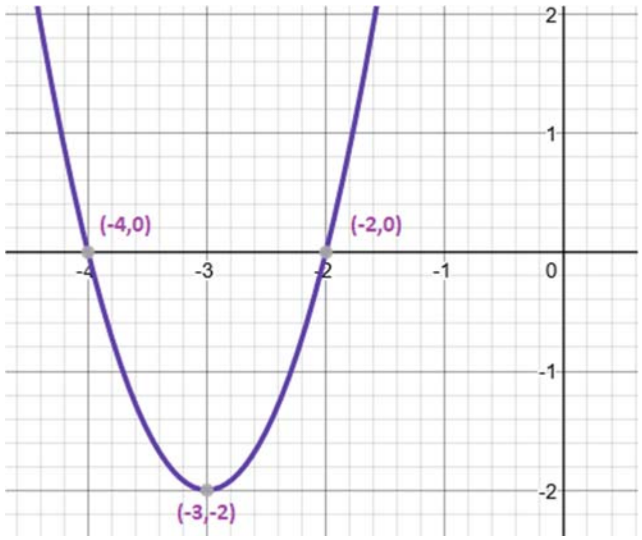
24. Was this a profitable business if you sold for maximum profit? ____________________________________

Algebra 2 Unit #1 Concept #1 HOMEWORK

Lesson Name:

Given the graph, write an equation for the quadratic function.

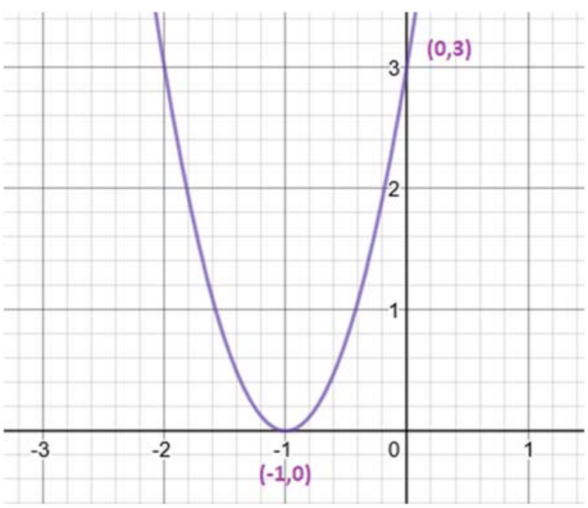
1.



Equation:

Standard Form:

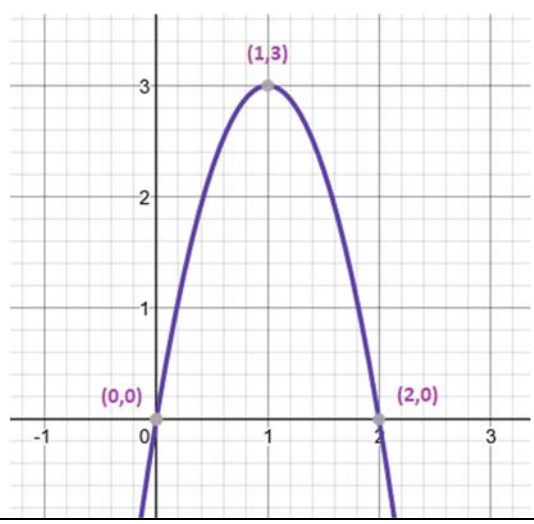
2.



Equation:

Standard Form:

3.



Equation:

Standard Form:

Vocabulary

Term	Definition	Notation	Diagram/Visual
Rate of change	________________ ________________ ________________		

Key Ideas

Term	
Slope	

Finding Average Rate of Change from Functions, Tables, and Graphs

In Algebra 1, you learned about slope of a line. For any two points on the graph, the slope of the line is always the same. For a quadratic function, however, the slope is not the same over the entire domain. So, we will learn to calculate and interpret the average rate of change of a non-linear function in this lesson.

GUIDED NOTES

Average rate of change is a measure of how much the output changes as the input changes. In other words, we ignore the behavior of the graph between two points, and determine the slope of the imaginary line that connects those points.

In every calculation of average rate of change, you will need _______ points. Then, you can calculate the slope with

$$\text{Average rate of change} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

What formula do you remember for slope from Algebra 1? Compare and contrast it with the formula above for average rate of change?

A ball is launched upward at a rate of 48 ft/s from a platform 100 ft above the ground. The function that represents its height in feet after t seconds is $f(t) = -16t^2 + 48t + 100$. What is the meaning of the average rate of change from 2 seconds to 3 seconds after launch?

First, find the points of interest on the graph.

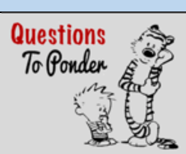
$f(2) = -16(2)^2 + 48(2) + 100 = 132$
$f(3) = -16(3)^2 + 48(3) + 100 = 100$
So, $(2, 132)$ and (3, 100).

Second, substitute the points into the formula.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{100-132}{3-2} = \frac{-32}{1} = -32$$

Third, use the average rate of change and the units in the problem to interpret the meaning of the number you found.

The ball falls an average of 32 feet per second in the interval from 2 seconds to 3 seconds.



How do we know the ball "falls" in the example above?

Example! A battery factory did a study to determine what profit they earn for making various numbers of AAA batteries. They summarized their data in a table, shown below.

Thousands of batteries manufactured	Profit earned in dollars
0	−12,000
20	8,000
40	12,000
60	0
80	−28,000

What is the meaning of the average rate of change between 20,000 batteries and 40,000 batteries manufactured?

First, find the points of interest on the table.

$f($_______$) =$ ________
$f($_______$) =$ ________
So, (_______, _______) and (_______, _______).

Second, _____________ the points into the formula.

$\frac{f(x_2)-f(x_1)}{x_2-x_1} =$ ___________ $=$ ______ $=$

Third, use the __________________ and the _______ in the problem to interpret the meaning of the number you found.

The profit increases an average of ________ per **additional battery** manufactured between _________ and _________ batteries.

Questions To Ponder

1. Use your work above to complete the answer a different way:

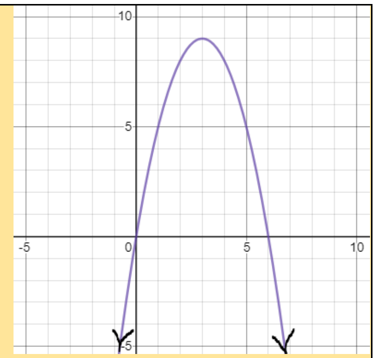
   The profit increases and average of ________ per **additional thousand batteries** manufactured between _________ and __________ batteries.

2. What do the negative numbers mean in the profit earned column in the example above?

3. Assume that the behavior of the profit function is quadratic. Approximate the zeros of the function? What do those represent in this context?

Example!

The bunny population on an island is shown in the graph to the right. The number of bunnies, in hundreds, is a function of the time in years since 2000.



What is the meaning of the average rate of change from x = 1 to x = 5?

First, find the points of interest on the graph.

$f($_______$) =$ ________
$f($_______$) =$ ________
So, (______, ______) and (______, ______).

Second, ____________ the points into the formula.

$\frac{f(x_2)-f(x_1)}{x_2-x_1} =$ __________ $=$ _____ $=$

Third, use the _________________ and the ______ in the problem to interpret the meaning of the number you found.

The average rate of change from the year _____ to the year ______ is ____. This means that the number of bunnies __________________.

Questions To Ponder

Is the number of bunnies on the island constant from 2001 to 2005?

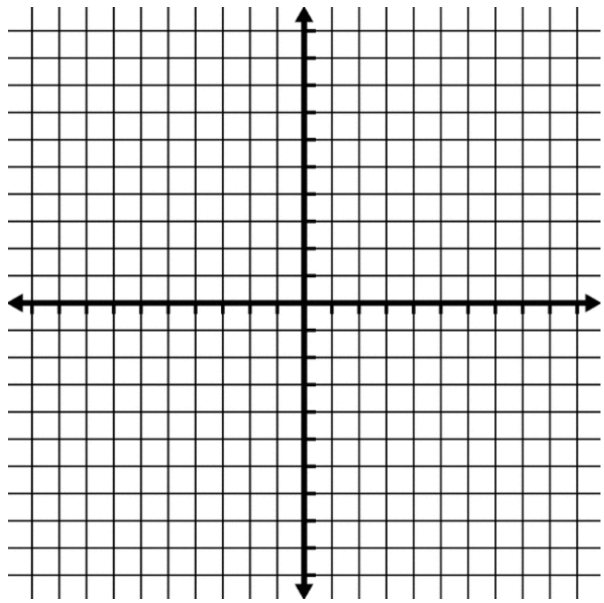
The graph shown in the above example includes more than we need to represent the bunny population. What interval should be used to represent the bunnies?

1. The table represents plant height measured in inches over a six-week period. What is the meaning of the average rate of change over the interval [0, 6]?

x	0	1	2	3	4	5	6
f(x)	0.9	1.5	2.5	3.9	5.7	7.9	10.5

2. The weight of the wood that is used to build a deck is a function of the length of the deck, measured in feet. The function $w(f) = f(f + 4)$ is used to calculate the weight of the deck in pounds. What is the meaning of the average rate of change from a length of 6 feet to 12 feet?

3. A certain quadratic function has an average rate of change of 2 apples per minute over the interval [3, 6]. Sketch the graph of one quadratic function that satisfies these specifications. Label the axes appropriately.



List the following characteristics for your graph:

Domain: Range: End behavior:

Vertex: Interval of increase: Interval of decrease:

y-intercept: zeros:

Transformations from the parent function:

Average Rate of Change of a Function

DESMOS:

Teachers – Create class code from this link:

https://teacher.desmos.com/activitybuilder/custom/599ece21cc74fa1a46f0c427

Tell students to go to student.desmos.com and enter your class code.

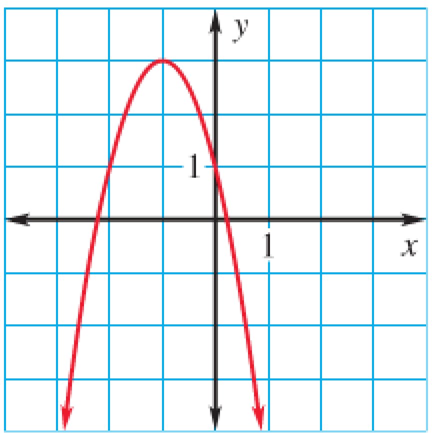
You can find your teacher guide at this link:

https://teacher.desmos.com/activitybuilder/teacherguide/599ece21cc74fa1a46f0c427

Characteristics of Quadratic Functions Homework

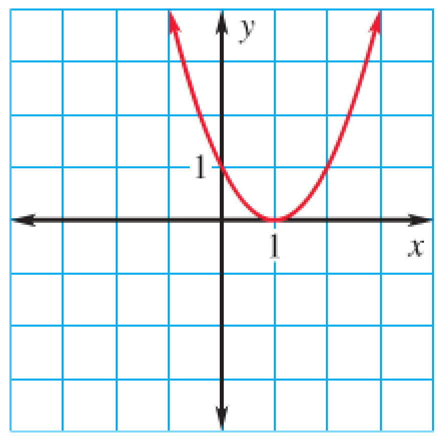
Review, plus average rates of change: Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, end behavior and rates of change.

1.



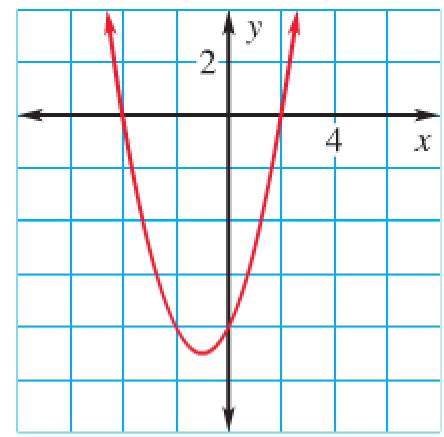
domain: ___________
range: ___________
vertex: ___________
asymptote: ___________
zero(s): ___________
interval of increase: ___________
interval of decrease: ___________
extrema: ___________
x-intercept(s): ___________
y-intercept: ___________
average rate of change on the interval:
$\underline{-1} \le x \le \underline{0}$: ___________
End Behavior:_____________

2.



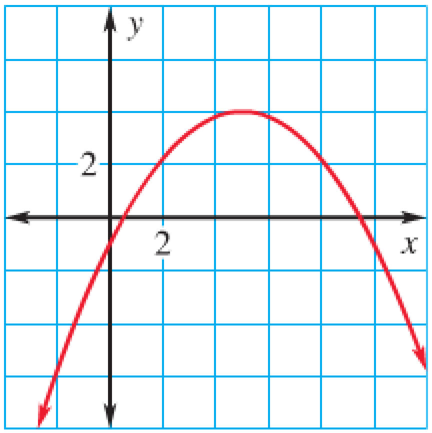
domain: ___________
range: ___________
vertex: ___________
asymptote: ___________
zero(s): ___________
interval of increase: ___________
interval of decrease: ___________
extrema: ___________
x-intercept(s): ___________
y-intercept: ___________
average rate of change on the interval:
$\underline{0} \le x \le \underline{1}$: ___________
End Behavior:_____________

3.



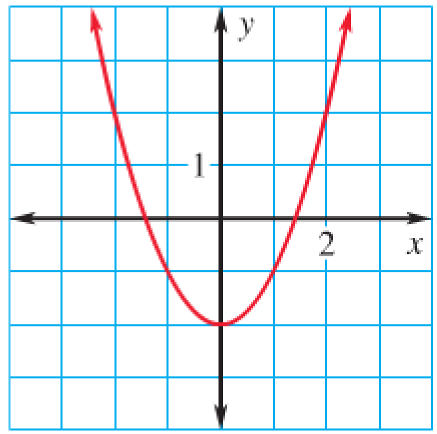
domain: ___________
range: ___________
vertex: ___________
asymptote: ___________
zero(s): ___________
interval of increase: ___________
interval of decrease: ___________
extrema: ___________
x-intercept(s): ___________
y-intercept: ___________
average rate of change on the interval:
$\underline{-2} \le x \le \underline{-1}$: ___________
End Behavior:_____________

4.



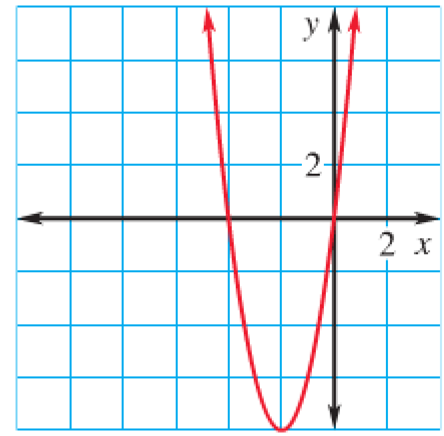
domain: ___________
range: ___________
vertex: ___________
asymptote: ___________
zero(s): ___________
interval of increase: ___________
interval of decrease: ___________
extrema: ___________
x-intercept(s): ___________
y-intercept: ___________
average rate of change on the interval:
$\underline{-1} \le x \le \underline{1}$: ___________
End Behavior:_____________

5.



domain: ___________
range: ___________
vertex: ___________
asymptote: ___________
zero(s): ___________
interval of increase: ___________
interval of decrease: ___________
extrema: ___________
x-intercept(s): ___________
y-intercept: ___________
average rate of change on the interval:
$\underline{0} \le x \le \underline{2}$: ___________
End Behavior:_____________

6.



domain: ___________
range: ___________
vertex: ___________
asymptote: ___________
zero(s): ___________
interval of increase: ___________
interval of decrease: ___________
extrema: ___________
x-intercept(s): ___________
y-intercept: ___________
average rate of change on the interval:
$\underline{-2} \le x \le \underline{-1}$: ___________
End Behavior:_____________

Algebra 2 Unit #1 Concept #2

Lesson Name: The Meaning of i^u

Term	Definition	Example
i		
Complex Number		
Standard Form		

The Meaning of i^u

In previous lessons, negative numbers appeared under a radical, and we noticed that this resulted in a non-real number. Today we will learn that these numbers are called imaginary numbers.

GUIDED NOTES

What is i?

- An imaginary number is any number that can be written in the form _______, where b is a real number and i is imaginary.

Example: $\sqrt{-16} = \sqrt{16} \cdot \sqrt{-1} = 4\sqrt{-1} = 4i$

- Therefore, $i^2 =$ ______.
- Any number that can be written in the form ___________, where a and b are real numbers, is called a complex number. We refer to the form $a + bi$ as the standard form of a complex number and call the a the real part and bi the imaginary part.

Example: $-12 + \sqrt{-100} = -12 + 10i$

Real ↙ ↘ Imaginary

- Higher powers of i:

$i^1 = i$	$i^5 = i$	$i^9 =$ _____	$i^{13} =$ _____
$i^2 = -1$	$i^6 = -1$	$i^{10} =$ _____	$i^{14} =$ _____
$i^3 = -i$	$i^7 = -i$	$i^{11} =$ _____	$i^{15} =$ _____
$i^4 = 1$	$i^8 = 1$	$i^{12} =$ _____	$i^{16} =$ _____

- To calculate the simplified form of a higher power of i, we can ____.

Example! **Write each of the following imaginary numbers in standard form.**

1. $\sqrt{-49} = \sqrt{49} \cdot \sqrt{-1} = 7i$
2. $-\sqrt{-18} = -\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{-1} = -3\sqrt{2}i$
3. $i^{43} = i^3 = -i$
4. $i^{100} = i^4 = 1$

SELF CHECK **Write each of the following imaginary numbers in standard form.**

1. $-\sqrt{-7}$
2. $\sqrt{-24}$
3. $2\sqrt{-9} + 6$

I. Identify the real part of each complex number.

a. $2+5i$ b. $-4+\sqrt{3}-6i$ c. $-\frac{1}{8}-\frac{5}{8}i$ d. $-10i$

II. Identify the imaginary part of each complex number.

a. $2+5i$ b. $-4+(\sqrt{3}-6)i$ c. $-\frac{1}{8}-\frac{5}{8}i$ d. 12

III. Simplify each expression, and write each complex number in standard form.

a. $(3-4)+(1+6)i$

b. $3+\sqrt{-16}$

c. $\frac{6+2i}{5}$

d. $\frac{-2+\sqrt{-14}}{3}$

IV. Consider the following questions, and write your answers on the page.

a. Can we write the addition of the same radicand as one term?

$6\sqrt{3} + 7\sqrt{3}$

b. Can we write the addition of different radicands as one term?

$2\sqrt{5} + \sqrt{11}$

c. Can we write the multiplication of different radicands as one radicand?

$\sqrt{14} \cdot \sqrt{15}$

Imagining a New Number

Part 1

Prior to the 16th century, the square root of a negative number was generally accepted as a mathematical impossibility. It wasn't until 1545, when the Italian mathematician, Gerolamo (or Girolamo) Cardano, known in English as Jerome Cardan, posed the problem of dividing ten into two parts whose product is 40. In his solution to this problem, Cardano stated that these solutions were "manifestly impossible", but plunged ahead by saying "nevertheless, we will operate." He "operated" by treating these expressions as numbers that follow standard rules of algebra and checked that they satisfied his original problem. So, Cardano was the first to imagine that there might be some numbers in addition to the real numbers that we represent as directed lengths. However, Cardano did not pursue this idea.

In 1637, Rene Descartes, the French mathematician who gave us the Cartesian coordinate system for plotting points, did not see a geometric interpretation for the square root of a negative number so in his book, *La Geometrie*, he called such a number "imaginary". This term stuck so that we still refer to the square root of a negative number as "imaginary". By the way, Descartes is also the one who coined the term "real" for the real numbers.

We now turn to the mathematics of these "imaginary" numbers.

In 1748, Leonard Euler (pronounced 'oiler'), one of the greatest mathematicians of all times, started the use of the notation "*i*" to represent the square root of – 1, that is, $i = \sqrt{-1}$. Thus, $i^2 = \sqrt{-1} \cdot \sqrt{-1} = -1$, since *i* represents the number whose square is – 1.

This definition preserves the idea that the square of a square root returns us to the original number, but also reminds us that for any real numbers *a* and *b*,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

is true only when both *a* and *b* are greater than or equal to zero.

1. As we have seen, the number *i* is not a real number; it is a new number. We want to use it to expand from the set of real numbers to a larger system of numbers.

a. What meaning could we give to each of the following?
1. $2i$
2. $5i$
3. $7i$

b. Find the square of each of the following:
1. $2i$
2. $5i$
3. $7i$

c. Use *i* to write an expression for each of the following.
1. $\sqrt{-9}$
2. $\sqrt{-16}$
3. $\sqrt{-64}$

d. Write an expression involving *i* for each of the following.
1. $-\sqrt{-4}$
2. $-\sqrt{-25}$
3. $-\sqrt{-49}$

2. We are now ready to be explicit about imaginary numbers. An ***imaginary number*** is any number that can be written in the form bi, where *b* is a real number and $i=\sqrt{-1}$. Imaginary numbers are also sometimes called ***pure imaginary numbers***. Write each of the following imaginary numbers in the standard form *bi*:

a. $\sqrt{-\frac{1}{36}}$ b. $\sqrt{-16}$ c. $\sqrt{-\frac{5}{64}}$

d. $-\sqrt{-7}$ e. $-\sqrt{-18}$

3. Any number that can be written in the form $a+bi$, where *a* and *b* are real numbers, is called a ***complex number***. We refer to the form $a+bi$ as the ***standard form*** of a complex number and call *a* the ***real part*** and *bi* the ***imaginary part***. Write each of the following as a complex number in standard form and state its real part and its imaginary part.

a. $6-\sqrt{-1}$ b. $-12+\sqrt{-100}$ c. $31-\sqrt{-20}$

We say that two complex numbers are ***equal*** if their real parts are equal and their imaginary parts are equal, that is,

if $a+bi$ and $c+di$ are complex numbers,

then $a+bi=c+di$ if and only if *a* = *c* and *b* = *d*.

Name________________________________

Simplifying Radicals/Imaginary Numbers Worksheet

Date________________ Period____

Simplify.

1) $\sqrt{343}$

2) $\sqrt{112}$

3) $\sqrt{54}$

4) $\sqrt{216}$

5) $\sqrt{-147}$

6) $\sqrt{-36}$

7) $\sqrt{-72}$

8) $\sqrt{-12}$

9) $\sqrt{8}$

10) $\sqrt{50}$

11) $\sqrt{-128}$

12) $\sqrt{-512}$

13) $(-6-7i)+(2+6i)$

14) $(6i)+(6i)-(7-3i)$

15) $(4-8i)+(-2-6i)$

16) $(-5-4i)+(8-7i)$

17) $2-(-4+6i)+3$

18) $(2i)-8-(-2+i)$

19) $(-3-8i)-(6i)-(3i)$

20) $(-4-5i)-(4+6i)$

Answers to Simplifying Radicals/Imaginary Numbers Worksheet

1) $7\sqrt{7}$
3) $3\sqrt{6}$
5) $7i\sqrt{3}$
7) $6i\sqrt{2}$
9) $2\sqrt{2}$
11) $8i\sqrt{2}$
13) $-4 - i$
15) $2 - 14i$
17) $9 - 6i$
19) $-3 - 17i$

Adding, Subtracting, and Multiplying Complex Numbers

Now that we know what complex numbers are, we will combine them using addition, subtraction, and multiplication. We will save division for the next lesson.

Example! Add, subtract, and multiply the numbers $3 - 6i$ and $-4 - i$.

To add complex numbers, add the real parts and the imaginary parts separately. Always write the answer in standard form.

$$(3 - 6i) + (-4 - i) = (3 + -4) + (-6 + -1)i$$
$$= -1 + -7 \quad i$$
$$= -1 - 7i$$

To subtract complex numbers, subtract the real parts and the imaginary parts separately, being careful of the order presented. Always write the answer in standard form.

$$(3 - 6i) - (-4 - i) = (3 - -4) + (-6 - -1)i$$
$$= 7 + -5 \quad i$$
$$= 7 - 5i$$

To multiply complex numbers, use the Distributive Property. If both numbers have an imaginary part, then simplify $i^2 = -1$ and combine like terms to write the answer in standard form.

$$(3 - 6i) \cdot (-4 - i) = -12 - 3i + 24i + 6i^2$$
$$= -12 - 3i + 24i + 6(-1)$$
$$= -12 - 3i + 24i - 6$$
$$= -18 + 21i$$

Questions To Ponder Is i a variable? How do you know?

Example! Sometimes, the complex numbers are not simplified before we perform operations on them. So, we will simplify them and then add, subtract, and multiply.

Add the numbers $\sqrt{-18}+\sqrt{20}$ and $\sqrt{125}+\sqrt{-45}$.

First, simplify each number. $\sqrt{-18}+\sqrt{20}$ $\sqrt{125}+\sqrt{-45}$

$=\sqrt{9}\sqrt{\quad}\sqrt{\quad}+\sqrt{4}\sqrt{\quad}$ $=\sqrt{5}\sqrt{\quad}+\sqrt{\quad}\sqrt{\quad}\sqrt{\quad}$

$=$ _______ + _______i $=$ _______ + _______i

To add complex numbers, add the real parts and the imaginary parts separately. Always write the answer in standard form.

$(\qquad\qquad) + (\qquad\qquad)i$

$= \qquad\qquad + \qquad\qquad i$

$=$

Subtract the numbers $3i^6-2i^3$ and $\frac{-5-\sqrt{-16}}{2}$.

First, simplify each number. $3i^6-2i^3$ $\frac{-5-\sqrt{-16}}{2}$

$=$ _______ + _______i $=$ _______ + _______i

To subtract complex numbers, subtract the real parts and the imaginary parts separately, being careful of the order presented. Always write the answer in standard form.

$(\qquad\qquad) + (\qquad\qquad)i$

$= \qquad\qquad + \qquad\qquad i$

$=$

Multiply the numbers $12i^{48}+\sqrt{-200}$ and $2-\sqrt{-32}$.

First, simplify each number. $12i^{48}+\sqrt{-200}$ $2-\sqrt{-32}$

$=$ _______ + _______i $=$ _______ + _______i

To multiply complex numbers, use the Distributive Property. If both numbers have an imaginary part, then simplify $i^2=-1$ and combine like terms to write the answer in standard form.

$(\qquad\qquad)\cdot(\qquad\qquad)$

$=$

$=$

$=$

$=$

Are these equivalent? $\sqrt{(-3)(-12)}$ and $\sqrt{-3}\cdot\sqrt{-12}$

What ideas do you have for simplifying $\frac{5-3i}{1+2i}$? Is it different from simplifying $\frac{-4+i}{6i}$?

Add or subtract the complex numbers.

1. $(2 + 11i) + (-10 - 4i)$

2. $(6 - 29i) - (17 + 8i)$

3. $(16 - 5i) + (9i)$

Multiply the complex numbers.

4. $(-8 + 12i)(10 - i)$

5. $(7 - 5i)(1 + 9i)$

6. $(-1 + i)(5 + 2i)$

7. $(1 - 3i)^2$

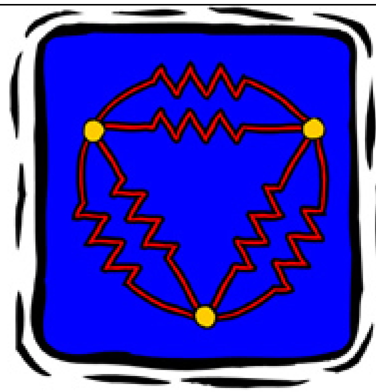
8. $(2 + 3i)(2 - 3i)$

9. $\left(\frac{2}{3} + \frac{4}{3}i\right)(3 - 6i)$

COMPLEX NUMBERS AND ELECTRICITY

*It is the practice in electrical engineering to represent the imaginary unit, **i**, by the letter "**j**", to avoid confusion with the symbol for electric current which is **I**. This page will, however, continue to use the letter **i** for the imaginary unit. Be sure to read the passage following question #5 before attempting the remaining questions.*

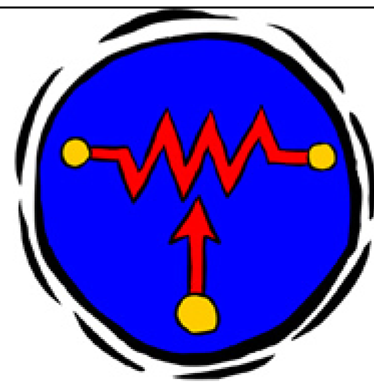
1. The relationship between voltage, *E (volts)*, current, *I (amps)*, and impedance, *Z (ohms)*, in an alternating circuit, is given by the formula $E = I \bullet Z$. If the circuit has a current $I = 3 + 2i$and an impedance $Z = 2 - i$, what is the voltage of this circuit?



Choose:

- $8 + i$
- $8 + 7i$
- $4 + i$
- $4 - i$

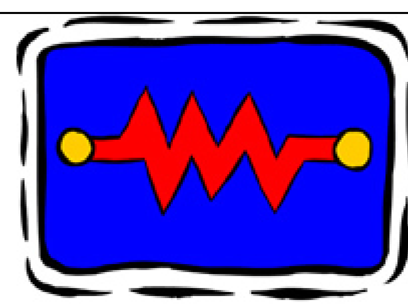
2. Impedance measures the opposition of an electrical circuit to the flow of electricity. The total impedance in a parallel circuit is given by the formula $Z_T = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$. What is the total impedance of a circuit, Z_T, if $Z_1 = 1 + 2i$ and $Z_2 = 1 - 2i$?



Choose:

- 1
- 0
- 5/2
- -3/2

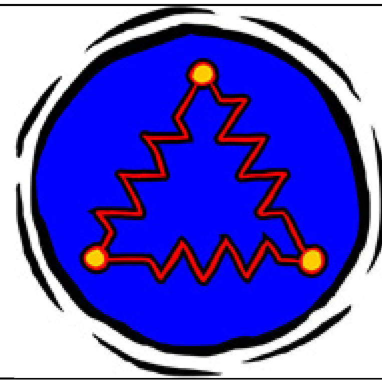
3. The impedance in one component of a series circuit is $2 + 6i$ ohms, and the impedance in another component of the circuit is $5 - 3i$ ohms. The "impedance" for a series circuit is the sum of the impedances for its individual components. Find the impedance in this circuit.



Choose:

- $7 - 3i$
- $7 + 3i$
- $10i$
- $4i$

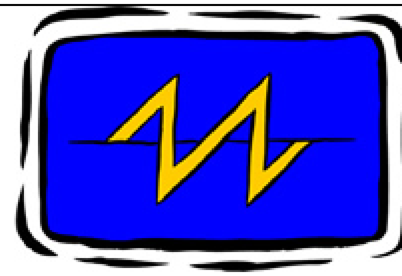
4. The current in a circuit is 4 + 3*i* amps and the impedance is 6 - 2*i* ohms. By using the formula *E* = *I* • *Z* described in question 1, find the voltage.



Choose:

- ☐ 30 + 10*i*
- ☐ 5 - *i*
- ☐ 15 + 10*i*
- ☐ 7 + 2*i*

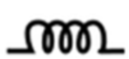
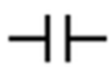
5. The voltage in a circuit is 35 + 12*i* volts and the impedance is 4 + 5*i* ohms. By using the formula *E* = *I* • *Z* described in question 1, find the current.



Choose:

- ☐ $\frac{200}{41}+\frac{127}{41}i$
- ☐ $\frac{200}{41}-\frac{127}{41}i$
- ☐ 35 + 12*i*
- ☐ $\frac{80}{41}-\frac{127}{41}i$

AC circuit components such as resistors, inductors and capacitors all oppose the flow of current. The opposition to current is referred to as *resistance* for resistors and *reactance* for inductors and capacitors. The total opposition to current flow in a circuit is called *impedance*, *Z*, measured in ohms, Ω . (Note in the table below that *impedance* referred to as *resistance* is represented with a Real Number while *impedance* referred to as *reactance* is represented with an Imaginary Number. Impedance from Inductors is positive, and from Capacitors is negative.)

Circuit Component	Symbol	Impedance (Z)
Resistor	 (5 Ω)	Z = 5
Inductor	 (6 Ω)	Z = 6i
Capacitor	 (7 Ω)	Z = -7i

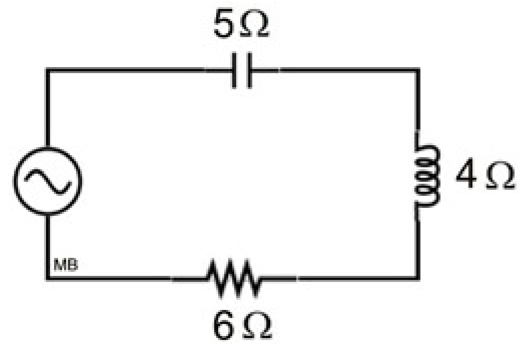
 Alternating Current (Voltage) Source

In a series circuit, the impedance is the sum of the impedances for the individual components.
In a parallel circuit, there is more than one pathway through which the current can flow. To find the total impedance, Z_T, first calculate the i of the pathways separately, by treating each pathway as a series circuit. Then apply the formula

$$Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

.

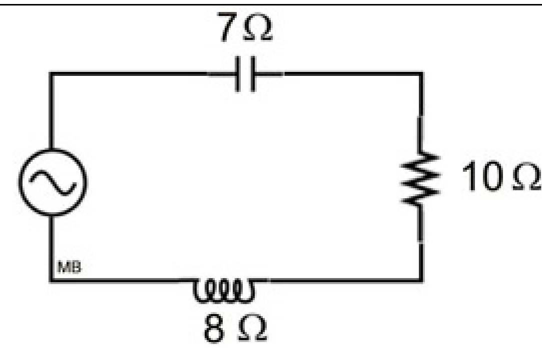
6. A series circuit is shown at the right, where electricity flows in only one direction. Find the impedance, *Z*, of this circuit, in ohms.



Choose:

- ☐ 15 Ω
- ☐ 6 + 9*i* Ω
- ☐ 6 - *i* Ω
- ☐ 5 - 2*i* Ω

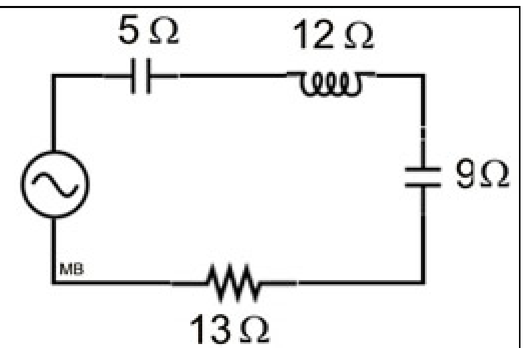
7. A series circuit is shown at the right. Find the impedance, *Z*, of the circuit.



Choose:

- ☐ 10 - 15*i* Ω
- ☐ 10 + 15*i* Ω
- ☐ 10 - *i* Ω
- ☐ 10 + *i* Ω

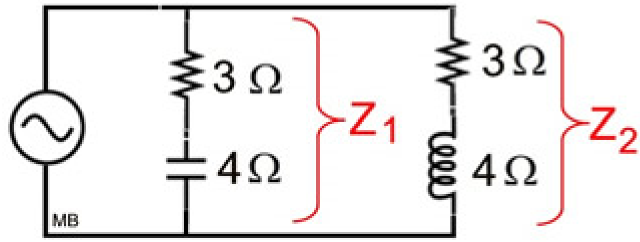
8. A series circuit is shown at the right. Find the impedance, *Z*, of the circuit.



Choose:

- ☐ 12 - *i* Ω
- ☐ 13 -2*i* Ω
- ☐ 13 + 2*i* Ω
- ☐ 13 - 28*i* Ω

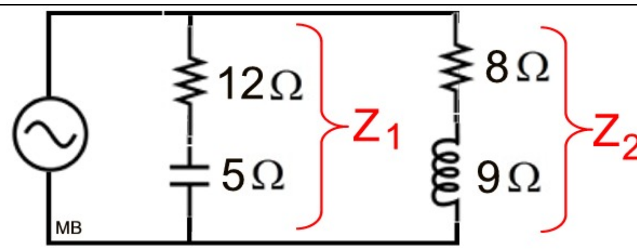
9. A parallel circuit is shown at the right. Find the impedance, *Z* of the circuit.



Choose:

- ☐ -7/6 Ω
- ☐ 7/6 Ω
- ☐ 25/6 Ω
- ☐ -25/6 Ω

10. A parallel circuit is shown at the right. Find the impedance, *Z*, of the circuit.



Choose:

- ☐ $20 + 4i$ Ω
- ☐ $\frac{96}{13} + \frac{87}{52}i$ Ω
- ☐ $\frac{192}{52} + \frac{87}{52}i$ Ω
- ☐ $\frac{773}{104} + \frac{199}{104}i$ Ω

Vocabulary

Term	Definition	Notation	Diagram/Visual
Complex conjugate	____________________ ____________________ ____________________		

Dividing Complex Numbers

After adding, subtracting, and multiplying complex numbers in the previous lesson, the next operation we will learn is dividing complex numbers. We will use a new technique to write the quotient in standard form.

Questions To Ponder

Is it true? $\frac{3}{2-5i} = \frac{3}{2} - \frac{3}{5}i$

Hint: Think about $\frac{4}{1+6} = \frac{4}{1} - \frac{4}{6}$

Example!

When complex conjugates are multiplied together, the product is a real number. We will use this fact to simplify the quotient of complex numbers.

Divide $2+3i$ by $-5+i$.

$\frac{2+3i}{-5+i}$	Given
$\frac{2+3i}{-5+i} \cdot \frac{-5-i}{-5-i}$	Multiply the numerator and denominator by the conjugate of the denominator. Recall that $\frac{-5-i}{-5-i} = 1$, so the value of the fraction is the same.
$\frac{-10-2i-15i-3i^2}{25+5i-5i-i^2}$	Multiply across.
$\frac{-10-17i-3(-1)}{25+0i-(-1)}$	Simplify the numerator and denominator separately.
$\frac{-7-17i}{25+1}$	Observe that the imaginary parts of the denominator add to zero.
$\frac{-7-17i}{26}$	
$\frac{-7}{26} - \frac{17}{26}i$	Write the complex number in standard form.

Questions To Ponder

Explain the process of simplifying a quotient of complex numbers in your own words.

Example! Divide -10 by $6i$.

$\frac{-10}{6i}$	Given
$\frac{-10}{6i} \cdot \frac{\quad}{\quad}$	Multiply the numerator and denominator by the conjugate of the denominator. Recall that $\frac{\quad}{\quad} = 1$, so the value of the fraction is the same.
$\frac{\quad}{\quad}$	Multiply across.
$\frac{\quad}{\quad}$	Simplify the numerator and denominator separately.
$\frac{\quad}{\quad}$	Observe that the imaginary parts of the denominator add to zero.
$\frac{\quad}{\quad} - \frac{\quad}{\quad}i$	Write the complex number in standard form.

Questions To Ponder Do you get the same result if you multiply by the denominator itself when it is imaginary?

Divide the complex numbers. Write your answers in standard form.

1. $\frac{3+2i}{4-i}$

2. $\frac{4-i}{3+2i}$

3. $\frac{4}{-1-i}$

4. $\frac{-8-5i}{9}$

5. $\frac{-8-5i}{9i}$

Find the sum, difference, product, and quotient of the pairs of complex numbers. Write all of your answers in standard form.

A. $12 + 5i$ and $-6i + 1$

1. sum
2. difference
3. product
4. quotient

B. $\sqrt{75} + \sqrt{27} - \sqrt{-175}$ and $4i - 3i^2 - 2i^3 + i^4$

1. sum
2. difference
3. product
4. quotient

Irrational and Complex Solutions of Quadratics

In Lesson G, we learned that zeros of quadratic functions can be classified in three ways: non-real, irrational, and rational. We did not simplify expressions with negative discriminants, but we will do so in this lesson.

Example! Recall this example from Lesson G:

Find the zeros of the function $f(x) = -2x^2 - 6x - 7$.

All zeros have y-coordinates of 0, so substitute for $f(x)$. Check that the function does not factor. Since the function is in standard form, we can use the Quadratic Formula to solve the equation.

$$0 = -2x^2 - 6x - 7$$

We see that $a = -2, b = -6, c = -7$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(-2)(-7)}}{2(-2)}$$

Substitute into the Quadratic Formula.

$$x = \frac{6 \pm \sqrt{36 - 56}}{-4}$$

Simplify exponents and perform multiplication.

$$x = \frac{6 \pm \sqrt{-20}}{-4}$$

Simplify the radical.

We stopped here and said the discriminant is -20, which is less than zero. So, we cannot graph the points. The graph does not cross the x-axis.

Now, let's continue to simplify so we can make some observations about the complex zeros.

$$x = \frac{6 \pm \sqrt{4}\sqrt{5}\sqrt{-1}}{-4}$$

$$x = \frac{6 \pm 2\sqrt{5}\,i}{-4}$$

$$x = \frac{2(3 \pm \sqrt{5}\,i)}{2(-2)}$$

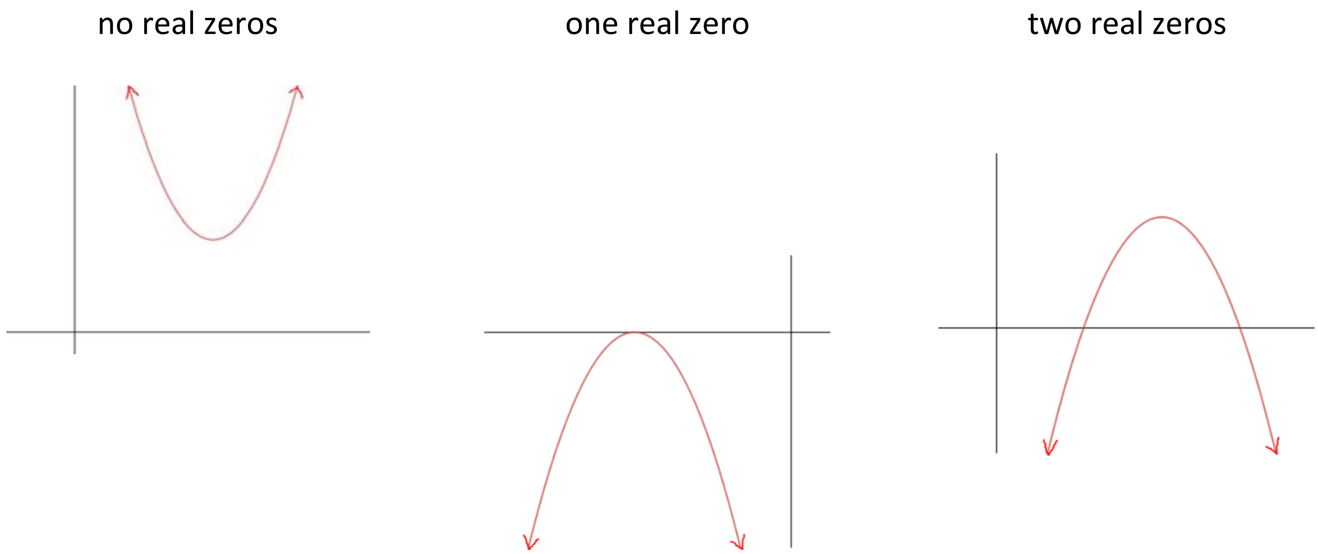
$$x = \frac{3 \pm \sqrt{5}\,i}{-2}$$

We have two complex zeros: $-\frac{3}{2} - \frac{\sqrt{5}}{2}i$ and $-\frac{3}{2} + \frac{\sqrt{5}}{2}i$. We can observe that the zeros are complex conjugates. We cannot graph them on the x-axis because it is a real number line, but we can now explain mathematically why the graph does not cross the x-axis.

GUIDED NOTES

Let's also revisit these notes:

The number of real zeros of a quadratic function can be observed on a graph. The x-axis is a real number line, so each point where the graph crosses the x-axis is a real zero. There are three cases:



When a quadratic function cannot be factored, we can use the ______________ ______________ to find the zeros.

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The expression $b^2 - 4ac$ is called the discriminant. The value of the discriminant will reveal important information about the zeros of the function.

Value of discriminant	Types of zeros	Representation on a graph
$b^2 - 4ac < 0$	Two complex zeros	
$b^2 - 4ac = 0$	One real zero	
$b^2 - 4ac > 0$, and it is a perfect square	Two rational zeros	
$b^2 - 4ac > 0$, and it is not a perfect square	Two real, irrational zeros	

Example!

ENRICHMENT

Now, we can revisit another example to learn more about irrational zeros. Find the zeros of the function $f(x) = 3x^2 + 11x + 5$.

All zeros have y-coordinates of 0, so substitute for $f(x)$.
Check that the function does not factor. Since the function is in standard form, we can use the Quadratic Formula to solve the equation.

$$0 = 3x^2 + 11x + 5$$ We see that $a = 3, b = 11, c = 5$.

$$x = \frac{-(11) \pm \sqrt{(11)^2 - 4(3)(5)}}{2(3)}$$ Substitute into the Quadratic Formula.

$$x = \frac{-11 \pm \sqrt{121 - 60}}{6}$$ Simplify exponents and perform multiplication.

$$x = \frac{-11 \pm \sqrt{61}}{6}$$ Simplify the radical.

We stopped here and used a calculator to approximate the zeros: (−0.53, 0) and (−3.14, 0).

Now, we can say more about these numbers. First, observe that irrational zeros come in conjugate pairs just like complex zeros do. They are $\frac{-11+\sqrt{61}}{6}$ and $\frac{-11-\sqrt{61}}{6}$. Since these numbers are irrational, they are real and can be located on the real number line. Let's use our knowledge of fractions and radicals to approximate their values without a calculator.

We will follow the order of operations.

$\frac{-11 + \sqrt{61}}{6}$	Too small	Too big
Approximate $\sqrt{61}$ $49 < 61 < 64$	7	8
Add -11	-4	-3
Divide by 6	$-\frac{2}{3}$	$-\frac{1}{2}$

$\frac{-11 - \sqrt{61}}{6}$	Too small	Too big
Approximate $-\sqrt{61}$ $-64 < -61 < -49$	-8	-7
Add -11	-19	-18
Divide by 6	$-3\frac{1}{6}$	-3

So, we have found approximations for the irrational zeros within 0.1 without using a calculator.

Vocabulary

Term	Definition	Notation	Diagram/Visual
Irrational zeros	______________ ______________ ______________		
Complex zeros	______________ ______________ ______________		

Key Ideas

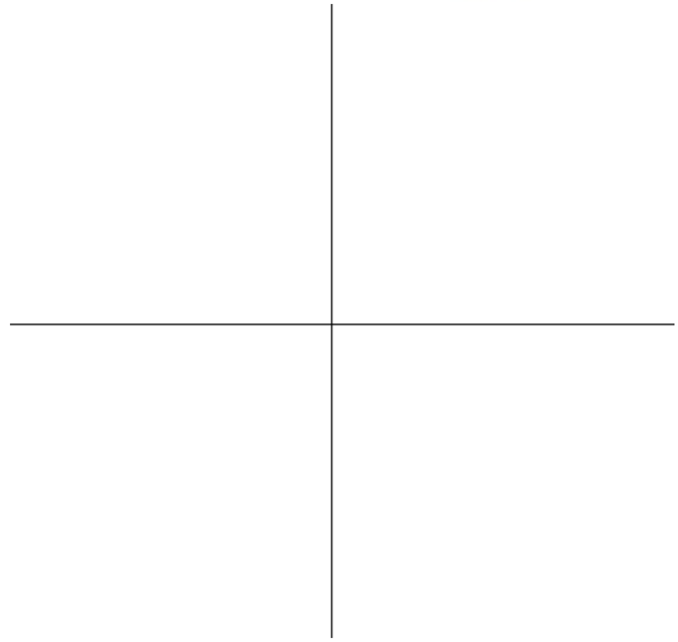
Term	
Real number line	

Find the zeros of each function, and list them separately. Verify the type of each zero by sketching a graph of the function.

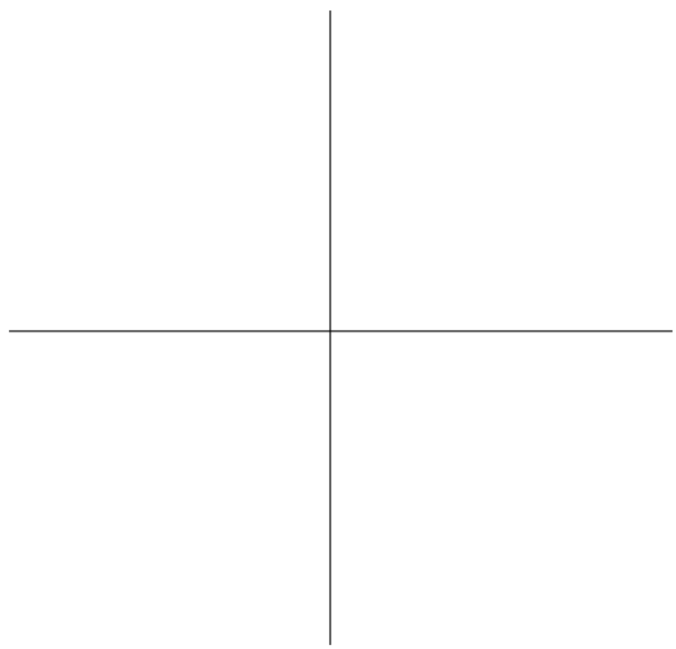
1. $p(x) = 3x^2 + 5x + 6$

2. $m(x) = 5x^2 + 2x - 5$

3. $h(x) = 12x^2 + 13x - 4$



4. $k(x) = 7x^2 - 2x + 3$



AVERAGE RATE OF CHANGE DESMOS TASK

Complex Numbers as Solutions to Equations

Classwork

Opening Exercises

1. The expression under the radical in the quadratic formula, $b^2 - 4ac$, is called the *discriminant.*
   Use the quadratic formula to solve the following quadratic equations. Calculate the discriminant for each equation.
   a. $x^2 - 9 = 0$

   b. $x^2 - 6x + 9 = 0$

   c. $x^2 + 9 = 0$

2. How does the value of the discriminant for each equation relate the number of solutions you found?

Example 1

Consider the equation $3x + x^2 = -7$.

What does the value of the discriminant tell us about number of solutions to this equation?

Solve the equation. Does the number of solutions match the information provided by the discriminant? Explain.

Exercise

Compute the value of the discriminant of the quadratic equation in each part. Use the value of the discriminant to predict the number and type of solutions. Find all real and complex solutions.

a. $x^2 + 2x + 1 = 0$

b. $x^2 + 4 = 0$

c. $9x^2 - 4x - 14 = 0$

d. $3x^2 + 4x + 2 = 0$

e. $x = 2x^2 + 5$

f. $8x^2 + 4x + 32 = 0$

Lesson Summary

- A quadratic equation with real coefficients may have real or complex solutions.
- Given a quadratic equation $ax^2 + bx + c = 0$, the discriminant $b^2 - 4ac$ indicates whether the equation has two distinct real solutions, one real solution, or two complex solutions.
  - If $b^2 - 4ac > 0$, there are two real solutions to $ax^2 + bx + c = 0$.
  - If $b^2 - 4ac = 0$, there is one real solution to $ax^2 + bx + c = 0$.
  - If $b^2 - 4ac < 0$, there are two complex solutions to $ax^2 + bx + c = 0$.

Homework – solving with real and complex solutions

Problem Set

1. Give an example of a quadratic equation in standard form that has …
   a. Exactly two distinct real solutions.
   b. Exactly one distinct real solution.
   c. Exactly two complex (non-real) solutions.

2. Suppose we have a quadratic equation $ax^2 + bx + c = 0$ so that $a + c = 0$. Does the quadratic equation have one solution or two distinct solutions? Are they real or complex? Explain how you know.

3. Solve the equation $5x^2 - 4x + 3 = 0$.

4. Solve the equation $2x^2 + 8x = -9$.

5. Solve the equation $9x - 9x^2 = 3 + x + x^2$.

6. Solve the equation $3x^2 - x + 1 = 0$.

7. Solve the equation $6x^4 + 4x^2 - 3x + 2 = 2x^2(3x^2 - 1)$.

8. Solve the equation $25x^2 + 100x + 200 = 0$.

9. Write a quadratic equation in standard form such that -5 is its only solution.

10. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all positive real numbers?

11. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all negative real numbers?

Extension:

12. Show that if $k > 3.2$, the solutions of $5x^2 - 8x + k = 0$ are not real numbers.

13. Let k be a real number, and consider the quadratic equation $(k + 1)x^2 + 4kx + 2 = 0$.
    a. Show that the discriminant of $(k + 1)x^2 + 4kx + 2 = 0$ defines a quadratic function of k.
    b. Find the zeros of the function in part (a), and make a sketch of its graph.
    c. For what value of k are there two distinct real solutions to the original quadratic equation?
    d. For what value of k are there two complex solutions to the given quadratic equation?
    e. For what value of k is there one solution to the given quadratic equation?

14. We can develop two formulas that can help us find errors in calculated solutions of quadratic equations.
    a. Find a formula for the sum S of the solutions of the quadratic equation $ax^2 + bx + c = 0$.
    b. Find a formula for the product R of the solutions of the quadratic equation $ax^2 + bx + c = 0$.

c. June calculated the solutions 7 and -1 to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solution is correct.

d. Paul calculated the solutions $3 - i\sqrt{2}$ and $3 + i\sqrt{2}$ to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in his solutions? If not, determine if his solutions are correct.

e. Joy calculated the solutions $3 - \sqrt{2}$ and $3 + \sqrt{2}$ to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solutions are correct.

f. If you find solutions to a quadratic equation that match the results from parts (a) and (b), does that mean your solutions are correct?

g. Summarize the results of this exercise.

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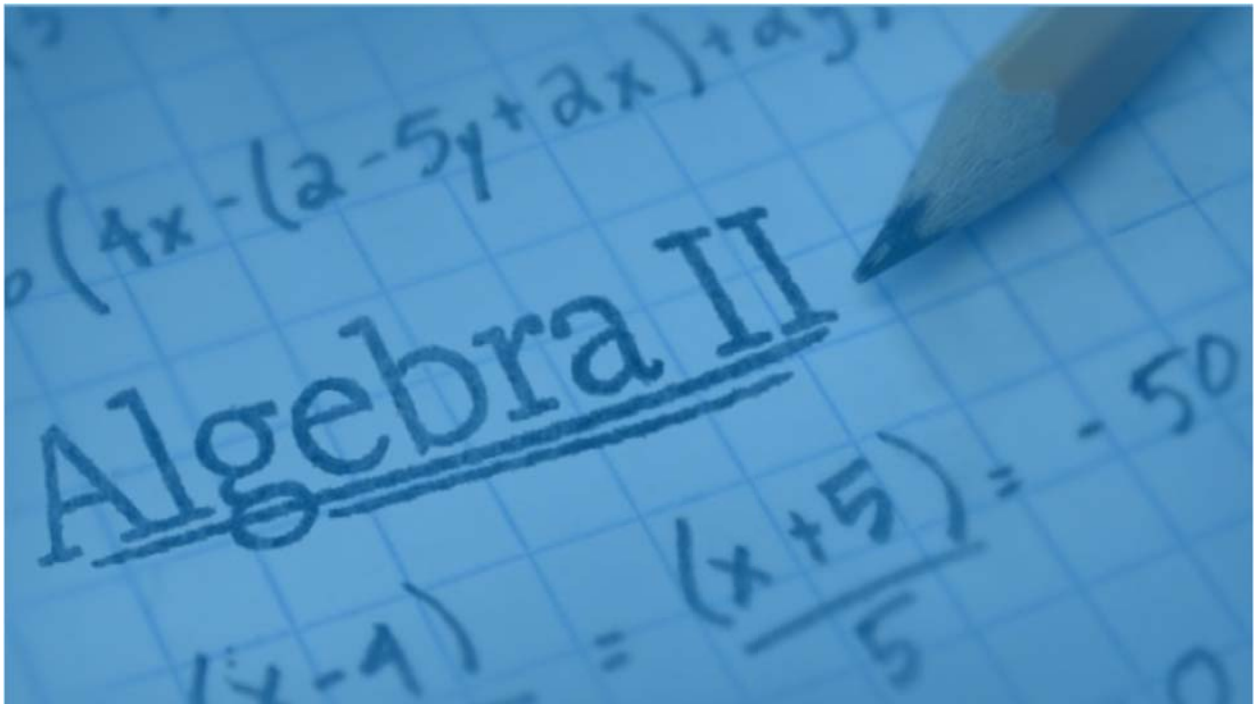
Science Technology Engineering Mathematics



Unit 2

Operations with Polynomials

Algebra 2

Unit 2: Operations with Polynomials

Concept 1: Polynomial Basics

Lesson A: Polynomial Basics	(A2.U2.C1.A.____.equations)

Concept 2: Linear Functions as Sequences

Lesson B: Adding and Subtracting Polynomial Basics	(A2.U2.C2.B.____.AddSubtPolynomials)
Lesson C: Multiplying Polynomials	(A2.U2.C2.C.____.MultPolynomials)
Lesson D: Dividing Polynomial Functions	(A2.U2.C2.D.____.DivPolynomials)
Lesson E: Remainder Theorem	(A2.U2.C2.E.____.RemainderThm)

Vocabulary

Term	Definition	Notation	Diagram/Visual
Term			
Degree			
Monomial			
Binomial			
Trinomial			
Polynomial			

Key Ideas

Term	
Standard Form	

Polynomial Basics

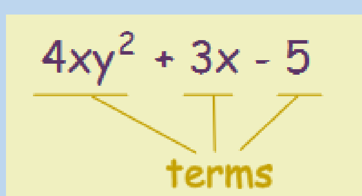
In this lesson, we will learn the definition of polynomial functions, how to write polynomials in standard form, and classify them by the number of terms and degree.

The degrees of a polynomial are the exponents. In naming polynomials, we use the name for the *highest* exponent.

Degree	Name Using Degree	Polynomial Example
0		6
1	linear	$x + 3$
2		
3	cubic	
4		$x^4 + 3x^2$
5	quintic	
6		

We also name the polynomials based on how many terms it contains.

Number of Terms	Name Using Number of Terms
	monomial
2	
3	trinomial
4	



In this example, the highest degree is 2 and there are 3 terms. Therefore, it is called a **quadratic trinomial.** Note that it is also written in standard form, which is when the terms are written in descending order, starting with the highest degree.

Questions To Ponder

What is the name of a polynomial with a degree of 4 and has 7 terms?

Example!

Name the following polynomials.

1. $3x^5 - 2x^4 + 9$
2. $21p^2$
3. 9x - $12x^3$
4. $6a^3 + 7a^2 - 11a + 5$
5. $10x^3$
6. $8x^2$+ $4x^4$- $5x^7$ + x^5

1.
2.
3.
4.
5.
6.

Questions To Ponder

Which of the examples listed above are in standard form? How do you know?

For those that are not in standard form, rewrite them so that they are in standard form.

Polynomials	**What is the degree?**	**How many terms?**	**What is the name of the polynomial?**	**Write the polynomial in standard form.**
$2x^4 + x^3$				
$4x^2$				
$10n - 5n^4 + 8$				
$8p^5 + 6p^4 - 2n^2 + 7n$				
$3x - 9x^2 + 4x^3 - 5x^4$				

Classifying Polynomials - Application

Classifying Polynomials (Task)

Mathematical Goals

· Understand the definition of a polynomial
· Classify polynomials by degree and number of terms

Georgia Standards of Excellence

Interpret the structure of expressions

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see x4 – y4 as (x2)2 - (y2)2, thus recognizing it as a difference of squares that can be factored as (x2 – y2) (x2 + y2).

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Look for and make use of structure.
6. Look for and express regularity in repeated reasoning.

Introduction

In this task, we are going to explore the definition and classification of polynomial functions. We will identify different parts of these expressions and explain their meaning within the context of problems

Materials

- Pencil
- Handout

Classifying Polynomials

Previously, you have learned about linear functions, which are first degree polynomial functions that can be written in the form $f(x) = a_1x^1 + a_0$ where a_1 is the slope of the line and a_0 is the y-intercept (Recall: $y = mx + b$, here m is replaced by a_1 and b is replaced by a_0.) Also, you have learned about quadratic functions, which are 2nd degree polynomial functions and can be expressed as $f(x) = a_2x^2 + a_1x^1 + a_0$.

These are just two examples of polynomial functions; there are countless others. A polynomial is a mathematical expression involving a sum of nonnegative integer powers in one or more variables multiplied by coefficients. A polynomial in one variable with constant coefficients can be written in $a_nx^n + a_{n-1}x^{n-1} + ...a_2x^2 + a_1x + a_0$ form where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers.

1. What are whole numbers?

2. What are real numbers?

3. Decide whether each function below is a polynomial. If it is, write the function in standard form. If it is not, explain why.

a. $f(x) = 2x^3 + 5x^2 + 4x + 8$

b. $f(x) = 2x^2 + x^{-1}$

c. $f(x) = 5 - x + 7x^3 - x^2$

d. $f(x) = \frac{2}{3}x^2 - x^4 + 5 + 8x$

e. $f(x) = 2\sqrt{x}$

g. $f(x) = \frac{1}{3x^2} + \frac{6}{x} - 2$

4. Polynomials can be classified by the number terms as well as by the degree of the polynomial. The degree of the polynomial is the same as the term with the highest degree. Complete the following chart. Make up your own expressions for the last three rows.

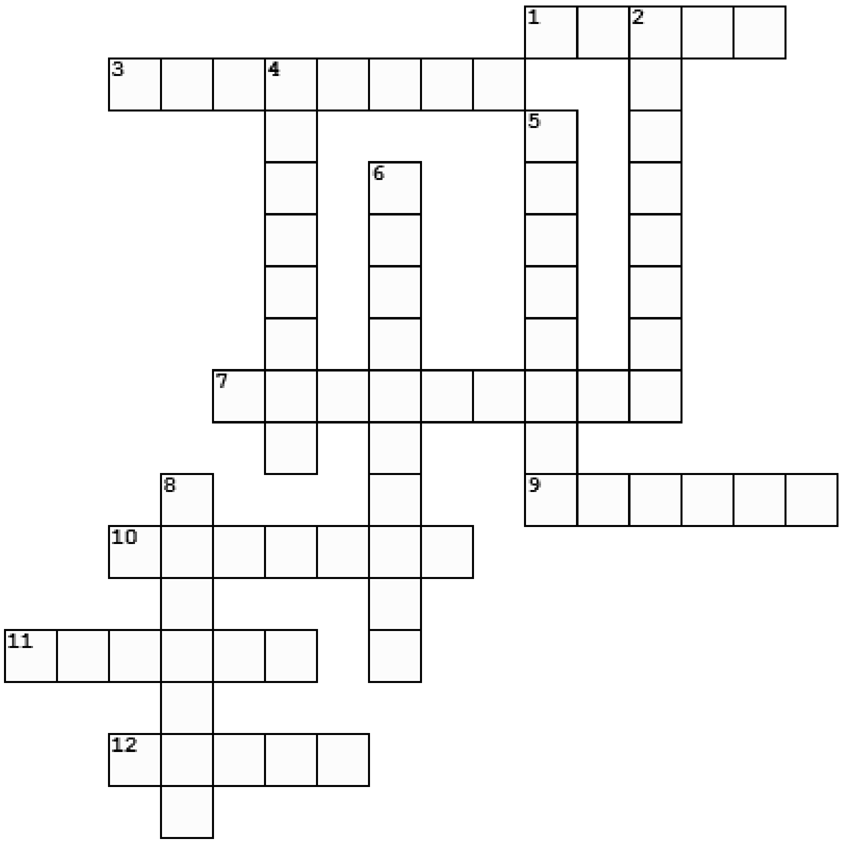
Polynomial	Number of Terms	Classification	Degree	Classification
$f(x)=2$		monomial		constant
$f(x)=3x-1$		binomial		linear
$f(x)=x^2-2x+1$		trinomial		quadratic
$f(x)=8x^3+125$		binomial		cubic
$f(x)=x^4+10x^2+16$		trinomial		quartic
$f(x)=-x^5$		monomial		quintic

Algebra 2 Unit #2 Concept #1

Lesson Name: A2.U2.C1.**A.05.hwk.equations**

Name:____________________

Complete the crossword puzzle below



Created using the Crossword Maker on TheTeachersCorner.net

Across

1. A polynomial with a degree of three.
3. What is the name of just a number?
7. A polynomial with three terms?
9. What is the degree of 3x?
10. What is the degree of $7x^5$?
11. The highest exponent of the polynomial.
12. What is the degree of $3x^6 + 4x^5 + 2x$

Down

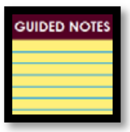
2. A polynomial with two terms.
4. A polynomial written in descending order of exponents is written in _____________ form.
5. A number or product of numbers and variables with whole number exponents, or a polynomial with one term.
6. A monomial or a sum or difference of monomials.
8. A polynomial with a degree of four.

Vocabulary

Term	Definition	Notation	Diagram/Visual
Monomial			
Polynomial			
Degree of monomial			
Leading Coefficient			
Distributive Property			
Like Terms			

Adding and Subtracting Polynomial Functions

Previously, we have discussed how to classify polynomial functions based on their degree and number of terms. In this lesson, we will learn how to add and subtract polynomial functions.



To add polynomials, we need to first identify the ____________. Like terms are terms in which the _______________ are the _________. For example, 7**x**, -4**x**, and **x** are like terms because they all share the same variable **x**. However, 3**x** and 3**y** are ________ like terms because they each consist of a different variable. In addition, 3**x** and $3x^2$ are ______ like terms because their _____________ are different.

After identifying the like terms, we can add them together.

There are two ways we can add polynomial together: _________________ and _________________.

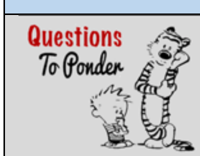
$$(2x^3 + 9 - x) + (5x^2 + 4 + 7x + x^3)$$

Horizontally

$(2x^3 - x + 9) + (x^3 + 5x^2 + 7x + 4)$	Write in Standard Form
$(2x^3 + x^3) + (5x^2) + (-x + 7x) + (9+4)$	Group Like Terms Together
$3x^2 + 5x^2 + 6x + 13$	Add Like Terms

Vertically

$2x^3 \quad - x + 9$	Write in Standard Form
$+ x^3 + 5x^2 + 7x + 4$	Place one equation on top of the other
$3x^2 + 5x^2 + 6x + 13$	Add Like Terms



What would happen if you did not write the equations in standard form first? Could you still arrive at the correct answer? If so, what steps would you take to solve?

Example! **Add $(-36x^2 + 6x - 11)$ and $(6x^2 + 16x^3 - 5)$.**

Horizontally

( ) + ( ) Write in Standard Form

( ) + ( ) + ( ) + ( ) Group Like Terms Together

Add Like Terms

Vertically

Write in Standard Form

+ ________________ Place one equation on top of the other

Add Like Terms

GUIDED NOTES

To subtract polynomials, we begin by ______________the ______________ sign to all the terms within the parenthesis.

Afterwards, we identify the like terms and add them together either horizontally or vertically.

*Note: Two ____________ make a ____________ (or Minus a negative = Plus a Positive)!

Example! **$(3 - 2x^2) - (x^2 + 6 - x)$**

Horizontally

$(3 - 2x^2) + (-x^2 - 6 + x)$ Distribute the Negative

$(-2x^2 + 3) + (-x^2 + x - 6)$ Write in Standard Form

$(-2x^2 - x^2) + (x) + (3 - 6)$ Group Like Terms Together

$-3x^2 + x - 3$ Add Like Terms

Vertically

$(3 - 2x^2) + (-x^2 - 6 + x)$ — Distribute the Negative

$(-2x^2 + 3) + (-x^2 + x - 6)$ — Write in Standard Form

$$\begin{array}{rrrr} & -2x^2 & & +3 \\ + & -x^2 & +x & -6 \\ \hline \end{array}$$

Place one equation on top of the other

$-3x^2 + x - 3$ — Add Like Terms

Example!

$(5x^3 + 12 + 6x^2) - (15x^2 + 3x - 2)$

Horizontally

( ) + ( ) — Distribute the Negative

( ) + ( ) — Write in Standard Form

( ) + ( ) + ( ) + ( ) — Group Like Terms Together

Add Like Terms

Vertically

( ) + ( ) — Distribute the Negative

( ) + ( ) — Write in Standard Form

Place one equation on top of the other

\+

Add Like Terms

Questions To Ponder

Which method of adding and subtracting polynomials is the most efficient? Why?

Add or subtract the polynomials below.

1. $(7x^3 + 3x^2) + (x^2 - 3x^3)$

2. $(1 - 4x^2) - (8 + 5x^2)$

3. $(3x^2 + 8 + 4x^4) + (2x^2 + 5x^4)$

4. $(4x^3 + 8x^2 - 4x) - (4x^3 - 5x^2 - 2)$

5. If P = $5x^4 - 2x^2 + 4x - 3$ and Q = $5x^4 + 3x^3 - 4x + 3$, what is P - Q?

Adding, Subtracting, and Multiplying Polynomials - Application

Adding and Subtracting Polynomials (Task)

We've Got to Operate

Mathematical Goals

- Add, subtract, and multiply polynomials and understand how closure applies

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see x4 – y4 as (x2)2 - (y2)2, thus recognizing it as a difference of squares that can be factored as (x2 – y2) (x2 + y2).

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Standards for Mathematical Practice

7. Make sense of problems and persevere in solving them.

8. Reason abstractly and quantitatively.

9. Construct viable arguments and critique the reasoning of others.

10. Model with mathematics.

11. Look for and make use of structure.

12. Look for and express regularity in repeated reasoning.

Introduction

In this task, we will perform operations on polynomials (addition, subtraction, multiplication) and simplify these expressions by combining like terms and using the distributive property. Finally, we will learn how closure applies to these operations on polynomials.

Materials

- Pencil
- Handout

We've Got to Operate

Previously, you learned how to use manipulatives to add and subtract like terms of polynomial expressions. Now, in this task, you will continue to use strategies that you previously developed to simplify polynomial expressions. To simplify expressions and solve problems, you learned that we sometimes need to perform operations with polynomials. We will further explore addition and subtraction in this task.

Answer the following questions and justify your reasoning for each solution.

1. Bob owns a small music store. He keeps inventory on his xylophones by using x^2 to represent his professional grade xylophones, x to represent xylophones he sells for recreational use, and constants to represent the number of xylophone instruction manuals he keeps in stock. If the polynomial $5x^2+2x+4$ represents what he has on display in his shop and the polynomial $3x^2+6x+1$ represents what he has stocked in the back of his shop, what is the polynomial expression that represents the entire inventory he currently has in stock?

2. Suppose a band director makes an order for 6 professional grade xylophones, 13 recreational xylophones and 5 instruction manuals. What polynomial expression would represent Bob's inventory **after** he processes this order? Explain the meaning of each term.

3. Find the sum or difference of the following using a strategy you acquired in the previous lesson:

a. $\begin{array}{r} 5x^2+2x-8 \\ +3x^2-7x-1 \\ \hline \end{array}$

b. $\begin{array}{r} 2x^2-2x+7 \\ (-)\,x^2+2x+1 \\ \hline \end{array}$

c. $(7x-5)+(2x+8)$

d. $(2a^2-5a+1)+(a^2+3a)$

e. $(-2x^2-5x+9)-(-3x^2+2x+4)$

f. $(5x^2+2xy-7y^2)-(3x^2-5xy+2y^2)$

4. You have multiplied polynomials previously, but may not have been aware of it. When you utilized the distributive property, you were just multiplying a polynomial by a monomial. In multiplication of polynomials, the central idea is the distributive property.

a. An important connection between arithmetic with integers and arithmetic with polynomials can be seen by considering whole numbers in base ten to be polynomials in the base $b=10$. Compare the product 213×47 with the product $(2b^2+1b+3)(4b+7)$:

$2b^2+1b+3$	$200+10+3$	213
$\times 4b+7$	$40+7$	$\times 47$
$14b^2+7b+21$	$1400+70+21$	1491
$8b^3+4b^2+12b+0$	$8000+400+120+0$	8520
$8b^3+18b^2+19b+21$	$8000+1800+190+21$	10011

b. Now compare the product 135×24 with the product $(1b^2+3b+5)(2b+4)$. Show your work!

5. Find the following products. Be sure to simplify results.

a. $3x(2x^2+8x+9)$

b. $-2x^2(5x^2-x-4)$

c. $(2x+7)(2x-5)$

d. $(4x-7)(3x-2)$

e. $(x-3)(2x^2+3x-1)$

f. $(6x+4)(x^2-3x+2)$

g. $(4x-7y)(4x+7y)$

h. $(3x-4)^2$

i. $(x-1)^3$

j. $(x-1)^4$

6. A set has the **closure property** under a particular **operation** if the result of the operation is always an element in the set. If a set has the **closure property** under a particular **operation**, then we say that the set is "**closed** under the **operation.**"

It is much easier to understand a property by looking at examples than it is by simply talking about it in an abstract way, so let's move on to looking at examples so that you can see exactly what we are talking about when we say that a set has the **closure property**.

a. The set of integers is **closed** under the **operation** of addition because the sum of any two integers is always another integer and is therefore in the set of integers. Write a few examples to illustrate this concept:

b. The set of integers is not **closed** under the **operation** of division because when you divide one integer by another, you don't always get another integer as the answer. Write an example to illustrate this concept:

c. Go back and look at all of your answers to problem number 5, in which you added and subtracted polynomials. Do you think that polynomial addition and subtraction is closed? Why or why not?

d. Now, go back and look at all of your answers to problems 6 and 7, in which you multiplied polynomials. Do you think that polynomial multiplication is closed? Why or why not?

Simplify each expression.

1) $(5+5n^3)-(1-3n^3)$

2) $(6a-3a^2)+(2a^2-3a)$

3) $(x^2-x)+(8x-2x^2)$

4) $(2a^2+4a^3)-(3a^3+8)$

5) $(5x^2+4)-(5+5x^3)$

6) $(8n^2-2n^3)+(6n^3-8n^2)$

7) $(8b^3+8)-(6-7b^3)$

8) $(4x^3-6)+(5x^3+3)$

9) $(10p^4+11)-(11p^4+13+16p^2)$

10) $(20v^2-9v^3)-(7v^3-10v^4-14v^2)$

11) $(10x^4 - 16) + (12 - 6x^3 + 11x^4)$

12) $(14 + 12a^3) + (17a^4 + 15 - 5a^3)$

13) $(17v^2 - 8) + (17v^2 + 10 + v^3)$

14) $(20n + 11n^4) - (15n + 16n^2 - 17n^4)$

15) $(10k^4 + 17k^3) - (14k^3 - 2k + 9k^4)$

16) $(9r + 6r^4) + (12r - 2r^4 - 17)$

17) $(6r + 2 + 8r^3) - (5r^3 - 11 - 8r^5) - (6r + 9r^5)$

18) $(9a^4 + 1 - 11a^2) - (a + 8a^2 + 2) - (6a^2 - 9)$

19) $(9k - 9 - 12k^4) - (4k + k^4 + 4) - (10 + 7k)$

20) $(8x^4 - 12 + 3x) - (9x^4 + 7 - 11x) + (9x + 8)$

Vocabulary

Term	Definition	Notation	Diagram/Visual
Monomial	________________ ________________ ________________		
Binomial	________________ ________________ ________________		
Trinomial	________________ ________________ ________________		
Leading Coefficient/Constants	________________ ________________ ________________		
Distributive Property	________________ ________________ ________________		

Key Ideas

Term	Definition	Notation	Diagram/Visual
F.O.I.L.	________________ ________________ ________________		
Box Method	________________ ________________ ________________		

Multiplying Polynomial Functions

So far, we have covered how to add and subtract polynomials. Today, we will learn how to multiply polynomial functions using the distributive property.

GUIDED NOTES

When multiplying polynomials together, the result will be a ________________.
(just like with adding and subtracting)

To find the product, you will need to use the ______________ property.

There are a few common cases you will see when multiplying.

1. Monomial times ___________ (____term x 1 term)
2. __________ times Binomial (1 term x _____terms)
3. __________ times ____________ (2 terms x 1 term)
4. __________ times ___________ (____ terms x ____ terms)
5. Binomial times _______________ (2 terms x _____ terms)

When using the _____________ property, you can multiply horizontally or use a __________.

Example!

Monomial times Monomial

(2xy) (4y)

(2 · 4) (x · y · y) Multiply the leading coefficients (constants) and variables together.

$8xy^2$ Remember: When multiplying variables, we ADD the exponents.

Monomial times Binomial

2x (x + 3xy)

(2x · x) + (2x · 3xy) Multiply the first term to both terms inside of the parenthesis (Distributive Property).

$2x^2 + 6x^2y$ Multiply the leading coefficients and variables together.

Binomial times Monomial

(x + 3xy) 2x

$(x \cdot 2x) + (3xy \cdot 2x)$ — Use the distributive property → multiply both terms inside the parenthesis to the single term on the outside

$2x^2 + 6x^2y$ — Multiply the leading coefficients and variables together.

Binomial times Binomial

(3x + 2) (4x – 5)

First **Outer** **Inner** **Last**

(3x · 4x) + (3x · -5) + (2 · 4x) + (2 · -5) — Multiply the first two terms to the second two terms using F.O.I.L. (**F**irst **O**uter **I**nner **L**ast)

$12x^2$ - 15x + 8x - 10 — Multiply the constants and variables together in their respective parenthesis

$12x^2$ **- 15x** **+ 8x** - 10 — Combine Like Terms

$12x^2$ – 7x – 10 — Simplify

Binomial times Trinomial

(x + 2y)(3x – 4y + 5)

$(x \cdot 3x) + (x \cdot -4y)$ + (x · 5) — Multiply the first term in the binomial to all the terms in the trinomial.

(2y · $3x$) + (2y · $-4y$) + (2y · 5) — Multiply the second term in the binomial to all the terms in the trinomial.

$3x^2 - 4xy + 5x + 6xy - 8y^2 + 10y$ — Multiply the constants and variables together in their respective parenthesis

$3x^2$ **– 4xy** + 5x **+ 6xy** – $8y^2$ + 10y — Combine Like Terms

$3x^2 + 2xy + 5x - 8y^2 + 10y$ — Simplify (*note: 6xy and 6yx mean the same thing)

Does order matter when multiplying two binomials together? What about a binomial to a trinomial? Why or why not?

Monomial times Monomial

(2xy) (7y)

( ) ( ) Multiply the leading coefficients (constants) and variables together.

Remember: When multiplying variables, we ADD the exponents.

Monomial times Binomial

6x (2x + 3)

( ) + ( ) Multiply the first term to both terms inside of the parenthesis (Distributive Property).

Multiply the leading coefficients and variables together.

Binomial times Monomial

(-5v -8) 7v

(   ) + (   )   Use the distributive property → multiply both terms inside the parenthesis to the single term on the outside

Multiply the leading coefficients and variables together

Binomial times Binomial

(4n + 1)(2n + 6)

First   Outer   Inner   Last

(   ) + (   ) + (   ) + (   )   Multiply the first two terms to the second two terms using F.O.I.L. (**F**irst **O**uter **I**nner **L**ast)

___________________________   Multiply the constants and variables together in their respective parenthesis

___________________________   Combine Like Terms

Simplify

Binomial times Trinomial

(4a + 2)(6a^2− a + 2)

(   ) + (   ) + (   )   Multiply the first term in the binomial to all the terms in the trinomial.

(   ) + (   ) + (   )   Multiply the second term in the binomial to all the terms in the trinomial.

___________________________   Multiply the constants and variables together in their respective parenthesis

______________________________ Combine Like Terms

Simplify

Example!

Another way to multiply polynomials is to use the table (also known as box) method.

This is commonly used when doing binomial times binomial, binomial times trinomial, trinomial times trinomial, etc .

(3x + 2) (4x – 5)

	4x	**-5**
3x		
2		

Multiply 3x to 4x and write the answer in the box below.

	4x	**-5**
3x	$12x^2$	
2		

Multiply 3x to -5 and write the answer in the box below.

	4x	**-5**
3x	$12x^2$	-15x
2		

Repeat the same steps for 2 to arrive at the completed box below.

	4x	**-5**
3x		
2		

$12x^2$	-15x
8x	-10

Now, use the terms in the boxes to write your final answer. Be sure to combine like terms if there are any. The final answer for this problem is below.

$12x^2 - 7x - 10$

Example!

Multiply the binomials below using the box method.

(7x + 5) (2x – 4)

Final Answer

Questions To Ponder

Why does the box method also work when multiplying polynomials?

Multiply the polynomials below.

1. $7n^2(n-4)$

2. 3xy (8y)

3. (4h - 2) (3h - 8)

Use the box method to multiply the polynomials below.

4. $(3x-y)(6x^2+5xy-7y^2)$

Adding, Subtracting, and Multiplying Polynomials - Application

Adding and Subtracting Polynomials (Task)

We've Got to Operate

Mathematical Goals

- Add, subtract, and multiply polynomials and understand how closure applies

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see x4 – y4 as (x2)2 - (y2)2, thus recognizing it as a difference of squares that can be factored as (x2 – y2) (x2 + y2).

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Standards for Mathematical Practice

7. Make sense of problems and persevere in solving them.

8. Reason abstractly and quantitatively.

9. Construct viable arguments and critique the reasoning of others.

10. Model with mathematics.

11. Look for and make use of structure.

12. Look for and express regularity in repeated reasoning.

Introduction

In this task, we will perform operations on polynomials (addition, subtraction, multiplication) and simplify these expressions by combining like terms and using the distributive property. Finally, we will learn how closure applies to these operations on polynomials.

Materials

- Pencil
- Handout

We've Got to Operate

Previously, you learned how to use manipulatives to add and subtract like terms of polynomial expressions. Now, in this task, you will continue to use strategies that you previously developed to simplify polynomial expressions. To simplify expressions and solve problems, you learned that we sometimes need to perform operations with polynomials. We will further explore addition and subtraction in this task.

Answer the following questions and justify your reasoning for each solution.

1. Bob owns a small music store. He keeps inventory on his xylophones by using x^2 to represent his professional grade xylophones, x to represent xylophones he sells for recreational use, and constants to represent the number of xylophone instruction manuals he keeps in stock. If the polynomial $5x^2+2x+4$ represents what he has on display in his shop and the polynomial $3x^2+6x+1$ represents what he has stocked in the back of his shop, what is the polynomial expression that represents the entire inventory he currently has in stock?

2. Suppose a band director makes an order for 6 professional grade xylophones, 13 recreational xylophones and 5 instruction manuals. What polynomial expression would represent Bob's inventory **after** he processes this order? Explain the meaning of each term.

3. Find the sum or difference of the following using a strategy you acquired in the previous lesson:

a. $\begin{array}{r} 5x^2+2x-8 \\ \underline{+3x^2-7x-1} \end{array}$

b. $\begin{array}{r} 2x^2-2x+7 \\ \underline{(-)\,x^2+2x+1} \end{array}$

c. $(7x-5)+(2x+8)$

d. $(2a^2-5a+1)+(a^2+3a)$

e. $(-2x^2-5x+9)-(-3x^2+2x+4)$

f. $(5x^2+2xy-7y^2)-(3x^2-5xy+2y^2)$

6. A set has the **closure property** under a particular **operation** if the result of the operation is always an element in the set. If a set has the **closure property** under a particular **operation**, then we say that the set is "**closed** under the **operation.**"

It is much easier to understand a property by looking at examples than it is by simply talking about it in an abstract way, so let's move on to looking at examples so that you can see exactly what we are talking about when we say that a set has the **closure property**.

a. The set of integers is **closed** under the **operation** of addition because the sum of any two integers is always another integer and is therefore in the set of integers. Write a few examples to illustrate this concept:

b. The set of integers is not **closed** under the **operation** of division because when you divide one integer by another, you don't always get another integer as the answer. Write an example to illustrate this concept:

c. Go back and look at all of your answers to problem number 5, in which you added and subtracted polynomials. Do you think that polynomial addition and subtraction is closed? Why or why not?

d. Now, go back and look at all of your answers to problems 6 and 7, in which you multiplied polynomials. Do you think that polynomial multiplication is closed? Why or why not?

i. $(x-1)^3$ j. $(x-1)^4$

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Find each product.

1) $6v(2v+3)$

2) $7(-5v-8)$

3) $2x(-2x-3)$

4) $-4(v+1)$

5) $(2n+2)(6n+1)$

6) $(4n+1)(2n+6)$

7) $(x-3)(6x-2)$

8) $(8p-2)(6p+2)$

9) $(6p+8)(5p-8)$

10) $(3m-1)(8m+7)$

11) $(2a-1)(8a-5)$

12) $(5n+6)(5n-5)$

13) $(4p-1)^2$

14) $(7x-6)(5x+6)$

15) $(6n+3)(6n-4)$

16) $(8n+1)(6n-3)$

17) $(6k+5)(5k+5)$

18) $(3x-4)(4x+3)$

19) $(4a+2)(6a^2-a+2)$

20) $(7k-3)(k^2-2k+7)$

Vocabulary

Term	Definition	Notation	Diagram/Visual
Dividend	______ ______ ______		
Divisor	______ ______ ______		
Quotient	______ ______ ______		
Remainder	______ ______ ______		

Key Ideas

Term	Definition	Notation	Diagram/Visual
Distributive Property	______ ______ ______		

Dividing Polynomial Functions

Previously, we have covered how to add and subtract polynomials. Today, we will learn how to divide polynomials by long division.

Questions To Ponder

What are the steps to complete a long division problem without using a calculator?

GUIDED NOTES

How to Divide Polynomials by Long Division

Step 1
Make sure the polynomial is written in ______________ order. If any terms are missing, use a _____ to fill in the missing term (this will help with the _____________).

Step 2
Divide the term with the ____________ power _____________ the division symbol by the term with the highest power ______________ the division symbol.

Step 3
Multiply (or _____________) the answer obtained in the previous step by the polynomial in front of the division symbol.

Step 4
_______________ and bring ___________ the __________ term.

Step 5
Repeat Steps 2, 3, and 4 until there are ___ __________ terms to bring down.

Last Step
Write the final answer. The term remaining after the _______ subtract ______ is the _____________ and must be written as a ____________ in the final answer.

Example!

$$\frac{2x^2 - 5x - 1}{x - 3}$$

Rewrite the equation to look like an arithmetic long division problem, like this:

$$x - 3\,\overline{)\,2x^2 - 5x - 1}$$

$$x - 3\,\overline{)\,2x^2 - 5x - 1}$$

Step 1 – Written in standard form? Yes**.** Any missing terms? In this case, no.

$$\begin{array}{r} 2x \\ x - 3\,\overline{)\,2x^2 - 5x - 1} \end{array}$$

Step 2 – What is $2x^2$ divided by x? Write the answer on top of division symbol.

Step 3 – Distribute the answer to the polynomial on the outside of the division symbol

$$\begin{array}{r} 2x \\ x - 3\,\overline{)\,2x^2 - 5x - 1} \\ \underline{2x^2 - 6x\quad\;\;} \end{array}$$

$$\begin{array}{r} 2x \\ x - 3\,\overline{)\,2x^2 - 5x - 1} \\ \underline{-(2x^2 - 6x)\quad} \\ 1x - 1 \end{array}$$

Step 4 – Subtract and bring down the next term.

Step 5 – Repeat steps 2, 3, and 4 until there are no more terms to bring down.

$$\begin{array}{r} 2x + 1 \\ x - 3\,\overline{)\,2x^2 - 5x - 1} \\ \underline{-(2x^2 - 6x)\quad} \\ 1x - 1 \end{array}$$

$$\begin{array}{r} 2x + 1 \\ x - 3\,\overline{)\,2x^2 - 5x - 1} \end{array}$$

$-(2x^2 - 6x)$
$1x - 1$

$$\begin{array}{r|l} & 2x + 1 \\ \hline x-3 & 2x^2 - 5x - 1 \\ & -(2x^2 - 6x) \\ \hline & \quad 1x - 1 \\ & \quad -(1x - 3) \\ \hline \end{array}$$

$$\begin{array}{r|l} & 2x + 1 \\ \hline x-3 & 2x^2 - 5x - 1 \\ & -(2x^2 - 6x) \\ \hline & \quad 1x - 1 \\ & \quad -(1x - 3) \\ \hline & \qquad 2 \end{array}$$

Last Step – Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction in the final answer.

Final Answer: $2x + 1 + \dfrac{2}{x-3}$

$$\frac{-y^2 + 2y^3 + 25}{y - 3}$$

Note: You may also see long division problems also written like this:

$$(-y^2 + 2y^3 + 25) \div (y - 3)$$

Step 1 – Written in standard form? **No.** Any missing terms? **Yes**. Therefore, our new problem is:

$(2y^3 - y^2 + 0y + 25) \div (y - 3)$

Now, rewrite the equation to look like an arithmetic long division problem to look like this:

$$y - 3 \overline{\big)\, 2y^3 - y^2 + 0y + 25}$$

Step 2 – What is $2y^3$ divided by y? Write the answer on top of division symbol.

$$y-3\,\overline{)\,2y^3-y^2+0y+25}\quad\text{quotient: } 2y^2$$

Step 3 – Distribute the answer to the polynomial outside of the division symbol.

$$y-3\,\overline{)\,2y^3-y^2+0y+25}\quad\text{quotient: } 2y^2$$
$$2y^3 - 6y^2$$

Step 4 – Subtract and bring down the next term.

$$y-3\,\overline{)\,2y^3-y^2+0y+25}\quad\text{quotient: } 2y^2$$
$$-(2y^3 - 6y^2)$$
$$5y^2+0y$$

Step 5 – Repeat steps 2, 3, and 4 until there are no more terms to bring down.

$$y-3\,\overline{)\,2y^3-y^2+0y+25}\quad\text{quotient: } 2y^2+5y$$
$$-(2y^3 - 6y^2)$$
$$5y^2+0y$$

$$y-3\,\overline{)\,2y^3-y^2+0y+25}\quad\text{quotient: } 2y^2+5y$$
$$-(2y^3 - 6y^2)$$
$$5y^2+0y$$

$$y-3\,\overline{)\,2y^3-y^2+0y+25}\quad\text{quotient: } 2y^2+5y$$

$$\begin{array}{r} -(2y^3 - 6y^2) \\ \hline 5y^2 + 0y \\ -(5y^2 - 15y) \\ \hline 15y + 25 \end{array}$$

$$\begin{array}{r|l} & 2y^2 + 5y + 15 \\ \hline y - 3 & 2y^3 - y^2 + 0y + 25 \\ & -(2y^3 - 6y^2) \\ & \quad 5y^2 + 0y \\ & \quad -(5y^2 - 15y) \\ & \quad\quad 15y + 25 \end{array}$$

$$\begin{array}{r|l} & 2y^2 + 5y + 15 \\ \hline y - 3 & 2y^3 - y^2 + 0y + 25 \\ & -(2y^3 - 6y^2) \\ & \quad 5y^2 + 0y \\ & \quad -(5y^2 - 15y) \\ & \quad\quad 15y + 25 \end{array}$$

$$\begin{array}{r|l} & 2y^2 + 5y + 15 \\ \hline y - 3 & 2y^3 - y^2 + 0y + 25 \\ & -(2y^3 - 6y^2) \\ & \quad 5y^2 + 0y \\ & \quad -(5y^2 - 15y) \\ & \quad\quad 15y + 25 \\ & \quad\quad -(15y - 45) \\ & \quad\quad\quad 70 \end{array}$$

Last Step
Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction in the final answer.

$$2y^2 + 5y + 15 + \frac{70}{y - 3}$$

Example!

Divide the following polynomials.

$$(n^2 + 10n + 18) \div (n + 5)$$

Step 1 – Written in standard form? Any missing terms? If so, rewrite the polynomials.

Now, rewrite the equation to look like an arithmetic long division problem to look like this:

Step 2 – What is n^2 divided by n? Write the answer on top of division symbol.

Step 3 – Distribute the answer to the polynomial outside of the division symbol.

Step 4 – Subtract and bring down the next term.

Step 5 – Repeat steps 2, 3, and 4 until there are no more terms to bring down.

Last Step

Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction in the final answer.

$$(k^3 - 30k - 18 - 4k^2) \div (3 + k)$$

Step 1 – Written in standard form? Any missing terms? If so, rewrite the polynomials.

Now, rewrite the equation to look like an arithmetic long division problem to look like this:

Step 2 – What is k^3 divided by k? Write the answer on top of division symbol.

Step 3 – Distribute the answer to the polynomial outside of the division symbol.

Step 4 – Subtract and bring down the next term.

Step 5 – Repeat steps 2, 3, and 4 until there are no more terms to bring down.

Last Step
Write the final answer. The term remaining after the last subtract step is the remainder and must be written as a fraction in the final answer.

Divide the following polynomials by long division.

1. $(15x^2 + 8x - 12) \div (3x + 1)$	2. $(x^2 + 5x - 28) \div (x - 3)$
Work:	Work:
Answer:	Answer:
3. $(-y^2 + 2y^3 + 25) \div (y - 3)$	4. $(x^6 - 3x + 5) \div (x - 1)$
Work:	Work:
Answer:	Answer:

Dividing Polynomials - Application

Performance Based Learning and Assessment Task

Polynomial Farm

I. **ASSESSSMENT TASK OVERVIEW & PURPOSE:**
This performance task is planned to give students an opportunity to add, subtract, multiply, and divide polynomials in order to solve real-world problems. It is also planned to give students real-world practice factoring completely first- and second-degree binomials and trinomials in one variable. Lastly, this task is designed to encourage students to make connections and to communicate their mathematical thinking clearly and accurately.

II. **UNIT AUTHOR:**
Emily O'Rourke, Northside High School, Roanoke, Virginia

III. **COURSE:**
Algebra I

IV. **CONTENT STRAND:**
Expressions and Operations

V. **OBJECTIVES:**
The student will be able to add, subtract, multiple, and divide polynomials. The student will also be able to factor completely first- and second-degree binomials and trinomials in one variable.

VI. **REFERENCE/RESOURCE MATERIALS:**
Students will need access to class notes, a pencil, a graphing calculator, and a "Polynomial Farm" worksheet.

VII. **PRIMARY ASSESSMENT STRATEGIES:**
The task includes an assessment component that performs two functions: (1) for the student it will be a checklist and provide a self-assessment and (2) for the teacher it will be used as a rubric. Students will be assessed on their understanding of adding, subtracting, multiplying, dividing, and factoring polynomials as it relates to real-life problems. Students will be evaluated on how clearly and accurately they explain their mathematical process and thinking. Students will also be assessed on the connections between their work and their reflection.

VIII. **EVALUATION CRITERIA:**
A self-assessment and a teacher assessment are attached below. A benchmark is also included at the end of the document in order to demonstrate the level of quality that is expected from each group of students.

IX. **INSTRUCTIONAL TIME:**
The performance task should take no longer than one ninety-minute block.

Polynomial Farm

Strand
Expressions and Operations

Mathematical Objective(s)
The mathematical objective for this activity is to give students the opportunity to add, subtract, multiply, divide, and factor polynomials in a real-world context.

Related SOL
The student will perform operations on polynomials, including:

A.2b) adding, subtracting, multiplying, and dividing polynomials
A.2c) factoring completely first- and second-degree binomials and trinomials in one or two variables. Graphing calculators will be used as a tool for factoring and for confirming algebraic factorizations.

NCTM Standards

- Understand meanings of operations and how they relate to one another
- Build new mathematical knowledge through problem solving
- Represent and analyze mathematical situations and structures using algebraic symbols
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others

Materials/Resources
Students will use class notes in order to validate how to add, subtract, multiply, divide, and factor polynomials. The students will use a pencil to record answers on their "Polynomial Farm" worksheet. The student will use a graphing calculator as a tool for factoring and for confirming algebraic factorizations.

Assumption of Prior Knowledge

- Students should understand how to calculate the perimeter and the area of a rectangle. The students should be familiar with adding, subtracting, multiplying, and dividing polynomials. The student should have prior knowledge of how to factor first- and second-degree binomials and trinomials in one variable. Additionally, students must be able to communicate their mathematical processes in an organized and legible way.
- Students have difficulty differentiating when they are able to factor and when they are unable to factor. Students also often forget to look for a greatest common factor. Furthermore, students frequently mix up the following: $x + x = 2x$ and $x \cdot x = x^2$. The teacher may need to prompt students to look for a greatest common factor and provide additional support when the students are factoring. Additionally, teachers may need to provide guidance when recalling $x + x = 2x$ and $x \cdot x = x^2$.

Introduction: Setting Up the Mathematical Task

In this activity, you will investigate the relationship between adding, subtracting, multiplying, dividing, and factoring polynomials in real-world scenarios. The Polynomial Farm Activity should take no longer than one ninety-minute block.

For this activity, you will be adding, subtracting, multiplying, dividing, and factoring in order to help a farmer calculate the perimeter and the area of his produce fields. The goal is to use your knowledge of adding, subtracting, multiplying, dividing, and factoring to answer all of the farmer's questions listed on the "Polynomial Farm" worksheet. Once the worksheet is completed, you will be expected to participate in a classroom reflective discussion about what you learned and also be expected to complete a self-assessment.

A recommendation for the teacher is to begin class with a bell ringer in which you review operations on polynomials and calculating the perimeter and area of a rectangle. After fifteen minutes, the teacher will spend ten minutes calling on students to come up to the board and provide their answers to the bell ringer questions. This will create student awareness of what prior knowledge is expected to successfully complete the task at hand.

Next, the teacher will place students in pairs in order to produce workable, productive groups. The groups will then obtain the "Polynomial Farm" worksheet and have forty-five minutes to complete it with their partner. The teacher will be walking around to answer any and all questions. After the worksheet is completed, ten minutes of class will be spent in a reflective discussion. The students will be asked what challenges they faced, what problem-solving skills they used and developed, and what they enjoyed most about the task. Another question teachers could propose to their students is, "What other real-world scenarios can you think of that may require adding, subtracting, multiplying, dividing, and factoring polynomials?" Within the discussion, students are expected to actively and respectfully participate. Students will also be expected to complete a self-assessment in the last ten minutes of class.

Student Exploration

Student/Teacher Actions:
Students will be collaborating in pairs as determined by the teacher. The teacher will explicitly state the expectations for the day upfront and provide constructive feedback along the way. The teacher will also act as a mentor and coach as students begin answering the more difficult questions about Farmer Bob's fields. Teachers will encourage students to draw on the group's knowledge first, prior to seeking out prompting from the teacher, class notes, and/or the Internet.

The teacher will also be available to facilitate the classroom reflective discussion. Students will communicate their mathematical knowledge by using appropriate vocabulary when presenting and making connections between Farmer Bob's fields and operations on polynomials. The teacher can encourage each group to use technology, specifically GeoGebra, if they would like to change the dimensions of Farmer Bob's produce fields.

Monitoring Student Responses

- Students are to communicate their thinking and their new knowledge by actively participating in the group discussion. Each classmate is expected to provide feedback at least once.
- Students are to communicate with each other actively, respectfully, and supportively.
- Students are encouraged to ask effective questions that do not require a yes or no answer.
- Teachers are to highlight and clarify frequently asked questions to the class as they emerge and provide problem-solving strategies to groups in order to resolve difficult situations.
- Teachers should encourage all students to be engaged within their group and therefore, discourage students from moving forward without their group members. If an entire group is ready to move on, encourage the group to begin brainstorming other ways to use operations on polynomials in the real world.

Again, in order to summarize the Polynomial Farm Activity, the teacher should plan to recap on the strengths and feedback from the classroom discussion. The teacher should also focus on how groups overcame difficult tasks and which problem-solving techniques to carry forward in the classroom. Lastly, the teacher should reflect on the content knowledge that was reviewed and applied.

Assessment List and Benchmarks

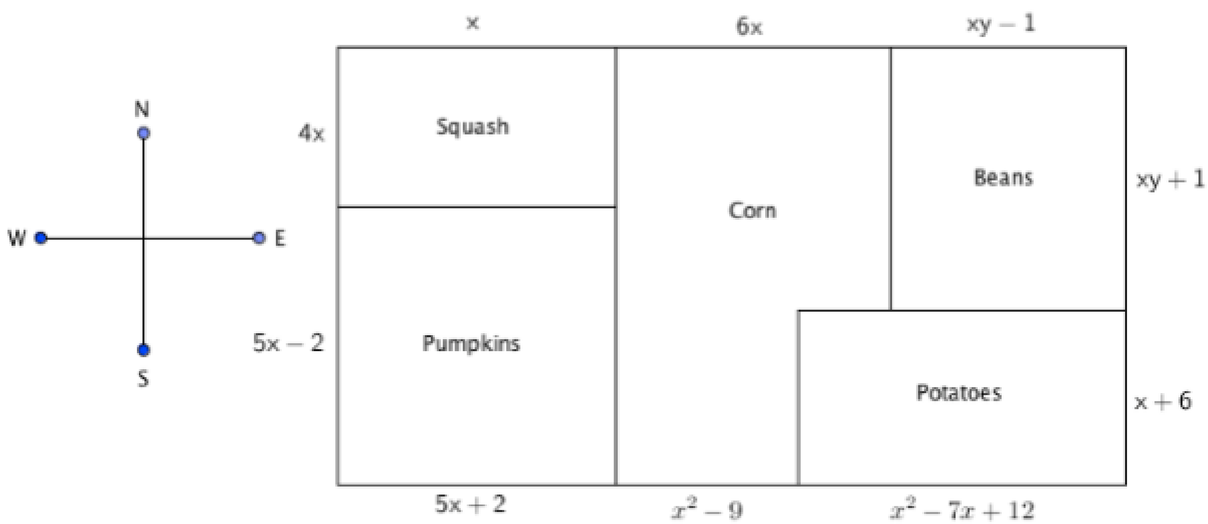
Students will complete the "Polynomial Farm" worksheet with their partner within one ninety-minute block. The teacher will then facilitate a discussion of the challenges, problem-solving strategies, and mathematical content knowledge gained from the Polynomial Farm Activity. Unlike that of the partner worksheet, students will individually complete a self-assessment at the end of the activity. The teacher will use the same rubric to assess each student and will give extra credit for exemplary participation in the classroom discussion and for successfully completing question thirteen on the "Polynomial Farm" worksheet.

Real Mathematics: Application

Performance Task – Expressions & Operations
A.2bc Polynomial Farm

Names: ______________________________
Date: __________________Block: ______

Directions: Farmer Bob is planting a garden this spring. He wants to plant squash, pumpkins, corn, beans, and potatoes. His plan for the field layout in feet is shown in the figure below. Use the figure and your knowledge of polynomials, perimeter, and area to solve the following:



1. Write an expression that represents the length of the south side of the field.

2. Simplify the polynomial expression that represents the south side of the field.

3. Write a polynomial expression that represents the perimeter of the pumpkin field.

4. Simplify the polynomial expression that represents the perimeter of the pumpkin field. State one reason why the perimeter would be useful to Farmer Bob.

5. Write a polynomial expression that represents the area of the potato field.

6. Simplify the polynomial expression that represents the area of the potato field. State one reason why the calculated area would be useful to Farmer Bob.

7. Write and simplify the polynomial expression that represents the area of the bean field if $x = 3$ and $y = 7$. What unit would the area of Bob's bean field have?

8. The farmer would like his bean plants to grow to a height of $(x + 3)$. Write a polynomial expression to find the volume of the bean plants if they reach a height of $(x + 3)$.

9. Simplify the polynomial expression that represents the volume of the bean plants if they reach a height of $(x + 3)$ feet.

10. Farmer Bob would like to plant three additional fields of produce. Using your graphing calculator, find the dimensions of each field given the area.

    a. The area of the strawberry field is $16x^2 + 4x$.

    b. The area of the cucumber field is $x^2 - 4x - 21$.

    c. The area of the tomato field is $x^2 - 36$.

11. Farmer Bob realized he forgot to include a zucchini field into his field layout. He plans to use half the length and half the width of the squash field in order to plant zucchini. Write a polynomial expression that represents the area of the new zucchini field.

12. Simplify the polynomial expression that represents the area of the newly added zucchini field.

13. <u>Extra Credit</u>: Write and simplify polynomial expressions that represent the perimeter and area of the cornfield.

Performance Assessment Task
A.2bc Polynomial Farm
Self-Assessment

Name: ____________________________
Date: ________________Block: ______

			Earned Assessment	
Num	**Element**	**Point Value**	**Self**	**Teacher**
1	The expression that represents the length of the south side of the field is provided.	2		
2	The south side of the field polynomial expression is simplified.	2		
3	The polynomial expression that represents the perimeter of the pumpkin field is provided.	2		
4	The polynomial expression that represents the perimeter of the pumpkin field is simplified.	2		
5	The usefulness of the perimeter in terms of Farmer Bob's fields is provided.	2		
6	The polynomial expression that represents the area of the potato field is provided.	2		
7	The polynomial expression that represents the area of the potato field is simplified.	2		
8	The usefulness of the area in terms of Farmer Bob's fields is provided.	2		
9	The polynomial expression that represents the area of the bean field is written.	2		
10	The polynomial expression that represents the area of the bean field is simplified.	2		
11	The unit for the area of the bean field is provided.	2		
12	The polynomial expression to find the volume of the bean plants if they reach a height of (x+3) is provided.	2		
13	The polynomial expression to find the volume of the bean plants if they reach a height of (x+3) is calculated.	2		
14	The dimensions of the strawberry field area provided.	2		
15	The dimensions of the cucumber field are provided.	2		
16	The dimensions of the tomato field are provided.	2		
17	A polynomial expression that represents the area of the new zucchini field is written.	2		
18	A polynomial expression that represents the area of the new zucchini field is simplified.	2		
19	The "Polynomial Farm" worksheet is completed on time.	2		
20	The student actively and respectfully participated in the reflective discussion.	2		
21	All written work is legible.	2		
22	The mathematical responses are well organized.	2		
23	Self-assessment is completed on time.	2		
	Total	46		

Performance Assessment Task

A.2bc Polynomial Farm

Teacher Assessment

Name: ___________________________

Date: ________________Block: _____

			Earned Assessment	
Num	**Element**	**Point Value**	**Self**	**Teacher**
1	The expression that represents the length of the south side of the field is provided.	2		
2	The south side of the field polynomial expression is simplified.	2		
3	The polynomial expression that represents the perimeter of the pumpkin field is provided.	2		
4	The polynomial expression that represents the perimeter of the pumpkin field is simplified.	2		
5	The usefulness of the perimeter in terms of Farmer Bob's fields is provided.	2		
6	The polynomial expression that represents the area of the potato field is provided.	2		
7	The polynomial expression that represents the area of the potato field is simplified.	2		
8	The usefulness of the area in terms of Farmer Bob's fields is provided.	2		
9	The polynomial expression that represents the area of the bean field is written.	2		
10	The polynomial expression that represents the area of the bean field is simplified.	2		
11	The unit for the area of the bean field is provided.	2		
12	The polynomial expression to find the volume of the bean plants if they reach a height of (x+3) is provided.	2		
13	The polynomial expression to find the volume of the bean plants if they reach a height of (x+3) is calculated.	2		
14	The dimensions of the strawberry field area provided.	2		
15	The dimensions of the cucumber field are provided.	2		
16	The dimensions of the tomato field are provided.	2		
17	A polynomial expression that represents the area of the new zucchini field is written.	2		
18	A polynomial expression that represents the area of the new zucchini field is simplified.	2		
19	The "Polynomial Farm" worksheet is completed on time.	2		
20	The student actively and respectfully participated in the reflective discussion.	2		
21	All written work is legible.	2		
22	The mathematical responses are well organized.	2		
23	Self-assessment is completed on time.	2		
	Total	46		

Performance Assessment Task
A.2bc Polynomial Farm
Category Descriptions

Name: ______________________________
Date: __________________Block: ______

#	Element	0	1	2
1	The expression that represents the length of the south side of the field is provided.	No length provided	Length is incomplete	Length is provided
2	The south side of the field polynomial expression is simplified.	Not simplified	Simplification incomplete	Simplification provided
3	The polynomial expression that represents the perimeter of the pumpkin field is provided.	No perimeter provided	Perimeter is incomplete	Perimeter is provided
4	The polynomial expression that represents the perimeter of the pumpkin field is simplified.	Not simplified	Simplification incomplete	Simplification provided
5	The usefulness of the perimeter in terms of Farmer Bob's fields is provided.	Perimeter usefulness not provided	Perimeter usefulness incomplete	Perimeter usefulness provided
6	The polynomial expression that represents the area of the potato field is provided.	No area provided	Area is incomplete	Area is provided
7	The polynomial expression that represents the area of the potato field is simplified.	Not simplified	Simplification incomplete	Simplification provided
8	The usefulness of the area in terms of Farmer Bob's fields is provided.	Area usefulness not provided	Area usefulness incomplete	Area usefulness provided
9	The polynomial expression that represents the area of the bean field is written.	No area provided	Area is incomplete	Area is provided
10	The polynomial expression that represents the area of the bean field is simplified.	Not simplified	Simplification incomplete	Simplification provided
11	The unit for the area of the bean field is provided.	Area unit not provided	Area unit incomplete	Area unit is provided
12	The polynomial expression to find the volume of the bean plants if they reach a height of (x+3) is provided.	No volume provided	Volume is incomplete	Volume is provided

13	The polynomial expression to find the volume of the bean plants if they reach a height of (x+3) is calculated.	Volume not calculated	Calculation incomplete	Volume calculation is provided
14	The dimensions of the strawberry field area provided.	Dimensions are not included	Dimensions are incomplete	Dimensions are provided
15	The dimensions of the cucumber field are provided.	Dimensions are not included	Dimensions are incomplete	Dimensions are provided
16	The dimensions of the tomato field are provided.	Dimensions are not included	Dimensions are incomplete	Dimensions are provided
17	A polynomial expression that represents the area of the new zucchini field is written.	Area not provided	Area incomplete	Area provided
18	A polynomial expression that represents the area of the new zucchini field is simplified.	Not simplified	Simplification incomplete	Simplification provided
19	The "Polynomial Farm" worksheet is completed on time.	No worksheet	Worksheet is incomplete or not provided on time	Worksheet completed on time
20	The student actively and respectfully participated in the reflective discussion.	Does not actively or respectfully participate	Does not fully participate	Actively and respectfully participates
21	All written work is legible.	Written work illegible	Written work partially legible	Written work legible
22	The mathematical responses are well organized.	No evidence of organization	Not fully organized	Well organized
23	Self-assessment is completed on time.	No self-assessment	Self-assessment is incomplete or not provided on time	Self-assessment provided on time

Divide.

1) $(m^2 - 7m - 11) \div (m - 8)$

2) $(n^2 - n - 29) \div (n - 6)$

3) $(n^2 + 10n + 18) \div (n + 5)$

4) $(k^2 - 7k + 10) \div (k - 1)$

5) $(n^2 - 3n - 21) \div (n - 7)$

6) $(a^2 - 28) \div (a - 5)$

7) $(r^2 + 14r + 38) \div (r + 8)$

8) $(x^2 + 5x + 3) \div (x + 6)$

9) $(2x^2 - 17x - 38) \div (2x + 3)$

10) $(42x^2 - 33) \div (7x + 7)$

11) $(x^2 - 74) \div (x - 8)$

12) $(2p^2 + 7p - 39) \div (2p - 7)$

13) $(n^3 + 7n^2 + 14n + 3) \div (n + 2)$

14) $(p^3 - 10p^2 + 20p + 26) \div (p - 5)$

15) $(v^3 - 2v^2 - 14v - 5) \div (v + 3)$

16) $(x^3 - 13x^2 + 40x + 18) \div (x - 7)$

17) $(k^3 - 30k - 18 - 4k^2) \div (3 + k)$

18) $(-5k^2 + k^3 + 8k + 4) \div (-1 + k)$

19) $(x^3 + 5x^2 - 32x - 7) \div (x - 4)$

20) $(50k^3 + 10k^2 - 35k - 7) \div (5k - 4)$

Answer Key

1) $(m^2 - 7m - 11) \div (m - 8)$

$$m + 1 - \frac{3}{m - 8}$$

2) $(n^2 - n - 29) \div (n - 6)$

$$n + 5 + \frac{1}{n - 6}$$

3) $(n^2 + 10n + 18) \div (n + 5)$

$$n + 5 - \frac{7}{n + 5}$$

4) $(k^2 - 7k + 10) \div (k - 1)$

$$k - 6 + \frac{4}{k - 1}$$

5) $(n^2 - 3n - 21) \div (n - 7)$

$$n + 4 + \frac{7}{n - 7}$$

6) $(a^2 - 28) \div (a - 5)$

$$a + 5 - \frac{3}{a - 5}$$

7) $(r^2 + 14r + 38) \div (r + 8)$

$$r + 6 - \frac{10}{r + 8}$$

8) $(x^2 + 5x + 3) \div (x + 6)$

$$x - 1 + \frac{9}{x + 6}$$

9) $(2x^2 - 17x - 38) \div (2x + 3)$

$$x - 10 - \frac{8}{2x + 3}$$

10) $(42x^2 - 33) \div (7x + 7)$

$$6x - 6 + \frac{9}{7x + 7}$$

11) $(x^2 - 74) \div (x - 8)$

$x + 8 - \frac{10}{x - 8}$

12) $(2p^2 + 7p - 39) \div (2p - 7)$

$p + 7 + \frac{10}{2p - 7}$

13) $(n^3 + 7n^2 + 14n + 3) \div (n + 2)$

$n^2 + 5n + 4 - \frac{5}{n + 2}$

14) $(p^3 - 10p^2 + 20p + 26) \div (p - 5)$

$p^2 - 5p - 5 + \frac{1}{p - 5}$

15) $(v^3 - 2v^2 - 14v - 5) \div (v + 3)$

$v^2 - 5v + 1 - \frac{8}{v + 3}$

16) $(x^3 - 13x^2 + 40x + 18) \div (x - 7)$

$x^2 - 6x - 2 + \frac{4}{x - 7}$

17) $(k^3 - 30k - 18 - 4k^2) \div (3 + k)$

$k^2 - 7k - 9 + \frac{9}{3 + k}$

18) $(-5k^2 + k^3 + 8k + 4) \div (-1 + k)$

$k^2 - 4k + 4 + \frac{8}{-1 + k}$

19) $(x^3 + 5x^2 - 32x - 7) \div (x - 4)$

$x^2 + 9x + 4 + \frac{9}{x - 4}$

20) $(50k^3 + 10k^2 - 35k - 7) \div (5k - 4)$

$10k^2 + 10k + 1 - \frac{3}{5k - 4}$

Vocabulary

Term	Definition	Notation	Diagram/Visual
Remainder Theorem	______ ______ ______		
Factor Theorem	______ ______ ______		

Key Ideas

Term	Definition	Notation	Diagram/Visual
P.E.M.D.A.S.	______ ______ ______		

Remainder Theorem

Previously, we learned how to divide polynomials by long division. Today, we will cover the Remainder Theorem and Factor Theorem.

Questions To Ponder

What do you think the Remainder Theorem is and how do you think it relates to solving for f(x)?

GUIDED NOTES

Remainder Theorem

When the polynomial $P(x)$ is divided by some ____________ factor in the form __________, then the _____________ is the value of $P(c)$.

How to Use the Remainder Theorem

Step 1 - _____________ the expression using polynomial long division.

Step 2 – Find the remainder. _______________ is $P(x)$ evaluated at c, according to the theorem.

To obtain c, look at the _____________factor. According to the Remainder Theorem, the polynomial P(x) is divided by a linear factor in the form ___________.

Therefore, if the linear factor is $x - c$, then the value we evaluate $P(x)$ by is c.

If the linear factor is $x + c$, it can be rewritten as _______________. Therefore, the value we evaluate $P(x)$ by is $-c$.

Step 3 – Evaluate $P(x)$ at c. If your answer is the same as the remainder found after using long division, your work is correct and the remainder theorem is true.

Example!

Step 1 – Divide the expression below using polynomial long division.

$$\frac{2x^2 - 5x - 1}{x - 3}$$

2x +1
$x - 3 \,|\, 2x^2 - 5x - 1$
$-(2x^2 - 6x)$
1x - 1
- (1x - 3)
2

Step 2 – Find the remainder.

$$
\begin{array}{r}
2x + 1 \\
x - 3 \,\overline{)\, 2x^2 - 5x - 1} \\
-(2x^2 - 6x) \\
\hline
1x - 1 \\
-(1x - 3) \\
\hline
2
\end{array}
$$

2 ← **Remainder**

Step 3 - Evaluate $P(x)$ at c. Since the linear factor is $x - 3$, $c = 3$.

$P(3) = 2(3)^2 - 5(3) - 1$
$P(3) = 2(9) - 15 - 1$
$P(3) = 18 - 15 - 1$
$P(3) = 2$

*Remember order of operations!

$P(3)$ and the remainder we found using long division are both equal to 2. This proves the remainder theorem to be true.

Step 1 – Divide the expression below using polynomial long division.

$$\frac{x^4 + 7x^3 + 5x^2 - 4x + 15}{x + 2}$$

Note: You may also see long division problems written like this:

$$\left(x^4 + 7x^3 + 5x^2 - 4x + 15\right) \div \left(x + 2\right)$$

$$
\begin{array}{r}
x^3 + 5x^2 - 5x + 6 \\
x + 2 \,\overline{)\, x^4 + 7x^3 + 5x^2 - 4x + 15} \\
-(x^4 + 2x^3) \\
\hline
5x^3 + 5x^2 \\
-(5x^3 + 10x^2) \\
\hline
-5x^2 - 4x \\
-(-5x^2 - 10x) \\
\hline
6x + 15 \\
-(6x + 12) \\
\hline
3
\end{array}
$$

Step 2 – Find the remainder.

$$\begin{array}{r} 6x + 15 \\ -(6x + 12) \\ \hline 3 \end{array}$$

← **Remainder**

Step 3 - Evaluate $P(x)$ at c. Since the linear factor is $x+2$, it can be rewritten as $x-(-2)$. Therefore, $c = -2$.

$P(-2) = (-2)^4 + 7(-2)^3 + 5(-2)^2 - 4(-2) + 15$
$P(-2)$ = 16 + 7 (-8) + 5 (4) + 8 + 15 *Remember order of operations!
$P(-2)$ = 16 -56 + 20 + 8 + 15
$P(-2)$ = 3

$P(-2)$ and the remainder we found using long division are both equal to 3. This proves the remainder theorem to be true.

Example!

Divide the following polynomial to find the remainder and verify your answer using the Remainder Theorem.

$$(k^2 - 7k + 10) \div (k - 1)$$

Step 1 – Divide the expression using polynomial long division.

Step 2 – Find the remainder.

Step 3 - Evaluate $P(x)$ at c.

Divide the following polynomial to find the remainder and verify your answer using the Remainder Theorem.

$$(v^3 - 2v^2 - 14v - 5) \div (v + 3)$$

Step 1 – Divide the expression using polynomial long division.

Step 2 – Find the remainder.

Step 3 - Evaluate $P(x)$ at c.

What do you think it means when the remainder is 0?

GUIDED NOTES

Factor Theorem

For any polynomial $P(x)$, $(x - c)$ is a factor of $P(x)$ if and only if ______________.

Another way to say this:

If $P(c)$ = 0, then __________ is a factor of $P(x)$.
The _____________ is also true.
If $x - c$ is a factor of $P(x)$, then _____________.

Example!

Determine whether the given binomial is a factor of the polynomial $P(x)$.

$$P(x) = (x^3 + 2x^2 - x - 2) \quad \div \quad (x + 2)$$

Step 1 – Divide the expression using polynomial long division.

$$\begin{array}{r|l} & x^2 + 0x - 1 \\ \hline x+2 & x^3 + 2x^2 - x - 2 \\ & -(x^3 + 2x^2) \\ \hline & \quad 0x^2 - x \\ & \quad -(0x^2 - 0) \\ \hline & \qquad -x - 2 \\ & \qquad -(-x - 2) \\ \hline & \qquad\quad 0 \end{array}$$

Step 2 – Find the remainder.

$$\begin{array}{r|l} & x^2 + 0x - 1 \\ \hline x+2 & x^3 + 2x^2 - x - 2 \\ & -(x^3 + 2x^2) \\ \hline & \quad 0x^2 - x \\ & \quad -(0x^2 - 0) \\ \hline & \qquad -x - 2 \\ & \qquad -(-x - 2) \\ \hline & \qquad\quad 0 \end{array}$$

← **Remainder**

Step 3 – If the remainder is 0, then $x - c$ is a factor of $P(x)$.

Since the remainder is 0, $\boldsymbol{x+2}$ **is a factor of** $\boldsymbol{P(x)}$**.**

Example!

Determine whether the given binomial is a factor of the polynomial $P(x)$.

$$P(x) = (2x^4 + 6x^3 - 5x - 10) \quad \div \quad (x+2)$$

Step 1 – Divide the expression using polynomial long division.

$$\mathbf{2x^3 + 2x^2 - 4x + 3}$$
$$x + 2 \,\big|\, \mathbf{2x^4 + 6x^3 + 0x^2 - 5x - 10}$$
$$-\,(2x^4 + 4x^3)$$
$$2x^3 + 0x^2$$
$$-\,(2x^3 + 4x^2)$$
$$-4x^2 - 5x$$
$$-\,(-4 - 8x)$$
$$3x - 10$$
$$-\,(3x + 6)$$
$$-16$$

Step 2 – Find the remainder.

$$\mathbf{2x^3 + 2x^2 - 4x + 3}$$
$$x + 2 \,\big|\, \mathbf{2x^4 + 6x^3 + 0x^2 - 5x - 10}$$
$$-\,(2x^4 + 4x^3)$$
$$2x^3 + 0x^2$$
$$-\,(2x^3 + 4x^2)$$
$$-4x^2 - 5x$$
$$-\,(-4 - 8x)$$
$$3x - 10$$
$$-\,(3x + 6)$$
$$-16$$

← **Remainder**

Step 3 – If the remainder is 0, then $x - c$ is a factor of $P(x)$.

Since the remainder is **NOT** 0, $\boldsymbol{x+2}$ **is NOT a factor of** $\boldsymbol{P(x)}$**.**

Example!

Determine whether the given binomial is a factor of the polynomial $P(x)$.

$$(k^3 - k^2 - k - 2) \div (k - 2)$$

Step 1 – Divide the expression using polynomial long division.

Step 2 – Find the remainder.

Step 3 – If the remainder is 0, then $x - c$ is a factor of $P(x)$.

Example!

Determine whether the given binomial is a factor of the polynomial $P(x)$.

$$(4x^2 - 26x + 35) \div (x - 5)$$

Step 1 – Divide the expression using polynomial long division.

Step 2 – Find the remainder.

Step 3 – If the remainder is 0, then $x - c$ is a factor of $P(x)$.

State if the given binomial is a factor of the given polynomial.

1) $(7n^3 - 18n^2 - 47n + 28) \div (n - 4)$

2) $(9a^2 + 80a - 100) \div (a + 10)$

3) $(7x^3 + 30x^2 - 22x + 15) \div (x + 5)$

4) $(9v^2 - 44v - 53) \div (v - 6)$

Evaluate each function at the given value.

1) $f(x) = -x^3 + 6x - 7$ at $x = 2$

2) $f(x) = x^3 + x^2 - 5x - 6$ at $x = 2$

3) $f(a) = a^3 + 3a^2 + 2a + 8$ at $a = -3$

4) $f(a) = a^3 + 5a^2 + 10a + 12$ at $a = -2$

5) $f(a) = a^4 + 3a^3 - 17a^2 + 2a - 7$ at $a = 3$

6) $f(x) = x^5 - 47x^3 - 16x^2 + 8x + 52$ at $x = 7$

State if the given binomial is a factor of the given polynomial.

7) $(k^3 - k^2 - k - 2) \div (k - 2)$

8) $(b^4 - 8b^3 - b^2 + 62b - 34) \div (b - 7)$

9) $(n^4 + 9n^3 + 14n^2 + 50n + 9) \div (n + 8)$

10) $(p^4 + 6p^3 + 11p^2 + 29p - 13) \div (p + 5)$

11) $(p^4 - 8p^3 + 10p^2 + 2p + 4) \div (p - 2)$

12) $(n^5 - 25n^3 - 7n^2 - 37n - 18) \div (n + 5)$

13) $(x^5 + 6x^4 - 3x^2 - 22x - 29) \div (x + 6)$

14) $(n^4 + 10n^3 + 21n^2 + 6n - 8) \div (n + 2)$

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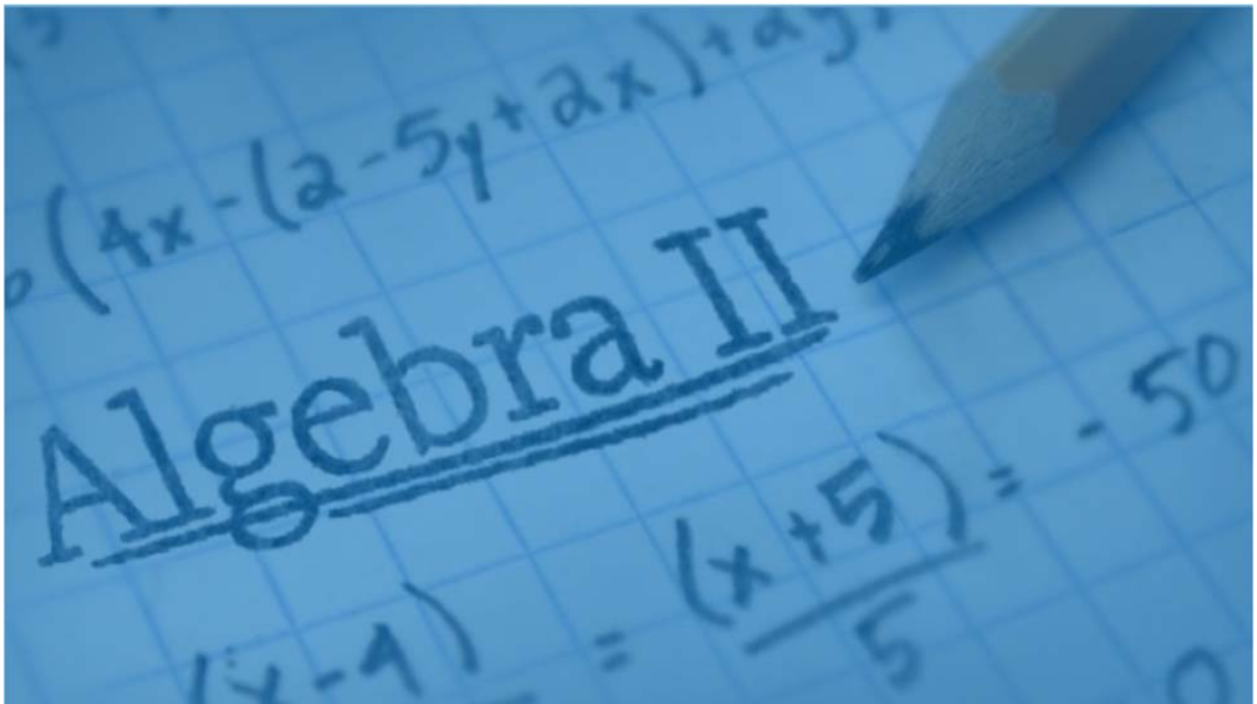


Unit 3
Polynomial Functions

Algebra 2

Unit 3: Polynomial Functions

Concept 1: Solving Polynomials

Lesson A: Fundamental Theorem of Algebra	(A2.U3.C1.A.____.FundThmOfAlg)
Lesson B: Zero Product Theorem With Pre-Factored Polynomials	(A2.U3.C1.B.____.ZeroProductAlreadyFactored)
Lesson C: Find Zeros: Solve Polynomials by Factoring (Difference of Perfect Squares, GCF)	(A2.U3.C1.C.____. ZeroProductFactorable)
Lesson D: Find Zeros: Solve by Factoring (Grouping)	(A2.U3.C1.D.____. ZeroProductGrouping)
Lesson E: Find Zeros: Sum and Difference Perfect Cubes	(A2.U3.C1.E.____.SumDiffPerfectCubes)
Lesson F: Find Zeros: Solving by Quadratic Formula	(A2.U3.C1.F.____.SolveByQuadraticFormula)
Lesson G: Rational Root Theorem: Synthetic Division	(A2.U3.C1.G.____.RationalRootThmSytheticDiv)
Lesson H: Rational Root Theorem: Apply RRT	(A2.U3.C1.H.____.RationalRootTheoremm)

Concept 2: Graphing Polynomial Functions

Lesson I: Graph Polynomials ID Characteristics: By Hand From Factored Form	(A2.U3.C2.I.____.GraphPolyFromFactored)
Lesson J: Graph Polynomials ID Characteristics: By Hand From Standard Form	(A2.U3.C2.J.____. GraphPolyFromStandard)
Lesson K: Graph Polynomials ID Characteristics: Using Technology Part 1	(A2.U3.C2.K.____.GraphPolyWithTechnology1)
Lesson L: Graph Polynomials ID Characteristics: Using Technology Part 2	(A2.U3.C2.L.____. GraphPolyWithTechnology2)
Lesson M: Fundamental Theorem of Algebra Observed Graphically	(A2.U3.C2.M.____.FundThmAlgGraphs)
Lesson N: Polynomial Transformations	(A2.U3.C2.N.____.PolyTransformations)
Lesson O: Even, Odd, Neither	(A2.U3.C2.O.____.EvenOddNeitherFunctions)

Concept 1 Solving Polynomials

Fundamental Theorem of Algebra

A. Determine the number of solutions a polynomial has without graphing it

ID zeros

B. Zero Product Theorem and derived from pre-factored polynomials (cross over from graphs to factorizations) to link factors to solutions.

Find the zeros of polynomials functions (algebraically)

C. Solve by factoring (difference of perfect squares, GCF)
D. Solve by factoring (grouping)
E. Sum and difference of perfect cubes
F. Solving by Quadratic Formula

Rational Root Theorem

G. Synthetic Division
H. Apply Rational Root Theorem

Concept 2 Graphing Polynomial Functions

Graphing Polynomial Functions

I. BY HAND, sketch the graph (in factored form) and tell characteristics (roots, y-int, domain, range, intervals of increase/decrease, number of turning of points, end behaviors, average rate of change on a specific interval)

J. BY HAND, sketch the graph (in standard form) and tell characteristics (roots, y-int, domain, range, intervals of increase/decrease, number of turning of points, end behaviors, average rate of change on a specific interval)

K. USING TECHNOLOGY, sketch the graph (in factored form) and tell characteristics (roots, y-int, domain, range, intervals of increase/decrease, number of turning of points, end behaviors, average rate of change on a specific interval)

L. USING TECHNOLOGY, sketch the graph (in standard form) and tell characteristics (roots, y-int, domain, range, intervals of increase/decrease, number of turning of points, end behaviors, average rate of change on a specific interval)

Relate Fundamental Theorem of Algebra to graphs

M. See Concept 2 Lesson A, B (tie the graphs from previous lesson to formal discuss the relationship with both real and imaginary solutions)

Transform a given function to create a new function

N. FBF3. Transformations of Polynomials

Describe the function

O. Even, Odd, Neither (graphically and algebraically)

Algebra 2 Unit #3 Concept#1A

Lesson Name: A2.U3.C1.**A.01.vocab.FundThmOfAlg**

Term	Definition	Notation	Diagram/Visual
Solution			
Exponent			
X-Intercept			
Decartes Rule of Signs			
Conjugates			

Fundamental Theorem of Algebra

GUIDED NOTES

The number solutions in a graph can be determined by looking at the highest exponent in polynomial. In this lesson, we will learn how to identify the number and type of solutions a polynomial has without graphing it.

Descartes' Rule of Signs

- The number of *positive* real zeros of a polynomial function *P(x),* with real coefficients, is equal to the number of variations in sign of the terms of *f(x)* or is less than this number by a multiple of 2.
- The number of *negative* real zeros is equal to the number of variations in sign of the terms of *f(-x)* or is less than this number by a multiple of 2.

Example!

Find the number of zeros of $f(x) = 16x^2 + x^4 - 8x - 7x^3 - 32$

First, put the terms of the polynomial in descending order by their exponents.

$f(x) = x^4 - 7x^3 + 16x^2 - 8x - 3$

By looking at the first term's exponent, one can find the total number of zeros.
In this case, the total number of zeros is four.
Now we must look at the number of positive solutions by looking at the number of sign changes through each term.
1st term positive
2nd term negative
3rd term positive
4th term negative
5th term negative
From this we have a total number of 3 sign changes. According to Decartes Rule of Signs, there are 3 or 1 positive real zero(s).

Now we should find the number of sign changes for f(-x), to find the number of negative real zeros.
$F(-x) = (-x)^4 - 7(-x)^3 + 16(-x)^2 - 8(-x) - 3$
$f(-x) = x^4 + 7x^3 + 16x^2 + 8x - 3$
1st term positive
2nd term positive
3rd term positive
4th term positive
5th term negative
From this we have 1 sign change which tells us there is only 1 negative real zero.

So we have a total of 4 zeros. Below is the list of possible zeros.

Postitive	Negative	Imaginary	Total
3	1	0	4
1	1	2	4

Example! Find the number and type of possible zeros of $f(x) = 2x^3 - 7x - 8$

First, put the terms of the polynomial in descending order by their exponents.

$$f(x) = 2x^3 - 7x - 8$$

By looking at the first term's exponent, one can find the total number of zeros.
In this case, the total number of zeros is three.
Now we must look at the number of positive solutions by looking at the number of sign changes through each term.
1st term positive
2nd term negative
3rd term negative

From this we have a total number of 1 sign change. According to Decartes Rule of Signs, there is 1 positive real zero.

Now we should find the number of sign changes for f(-x), to find the number of negative real zeros.
$f(-x) = 2(-x)^3 – 7(-x) - 8$
$f(-x) = -2x^3 + 7x - 8$
1st term negative
2nd term positive
3rd term negative
From this we have 2 sign changes which tells us there are 2 or no negative real zero(s).

So we have a total of 3 zeros. Below is the list of possible zeros.

Postitive	Negative	Imaginary	Total
1	2	0	3
1	0	2	3

SELF CHECK Find the number of zeros of the following polynomials.
1) $f(x) = 7x^3 + 3x^2 - 58x – 3$

When finding the number of zeros in a polynomial, why is it important to arrange the polynomial in standard form?

Find the number of zeros in each polynomial.

1) $f(x) = 4x^5 -7x^3 + 6x^2 – 27$
17x + 8

2) $f(x) = 9x^4 + 4x^3 + 2x^2 –$

3) $f(x) = 8x^3 + 3x^5 – 12x^8 – 1$

4) f(x) = 5x3 – 4x2 + 7x –2

5) $f(x) = 16x^2 +3x^5 – 7x^4 – 3x^3 + x - 7$

FUNDAMENTAL THEOREM OF ALGEBRA – GOING BACK TO YOUR ROOTS (TEXAS INSTRUMENTS)

Going Back To Your Roots

Time Required

40 minutes

Activity Overview

In this activity, students apply the Fundamental Theorem of Algebra in determining the complex roots of polynomial functions. The theorem is applied both algebraically and graphically.

Topic: Polynomial Functions

- *Fundamental Theorem of Algebra*
- *Complex roots*
- *Multiplicity*

Teacher Preparation and Notes

- *Problems 1 through 3 are to be done in class. The extension can be used for further exploration of the topic. Additional practice is provided on the associated worksheet for guided practice or homework.*
- ***To download the student worksheet, go to education.ti.com/exchange and enter "11758" in the quick search box.***

Associated Materials

- *PrecalcWeek16_BackToRoots_worksheet_TI84.doc*

Suggested Related Activity

To download any activity listed, go to education.ti.com/exchange and enter the number in the quick search box.

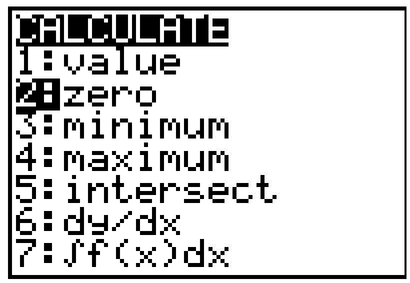
- *Discriminating Against the Zero (TI-84 Plus family) — 11520*

Before beginning the activity, it is important that students understand how to determine the degree of a polynomial and the definition of a complex number.

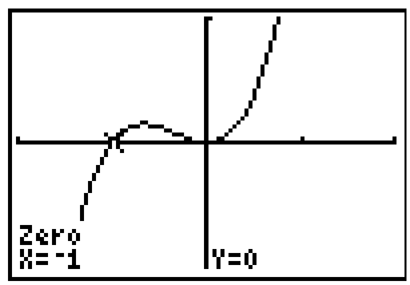
Problem 1 – The Fundamental Theorem of Algebra

In this activity, students are introduced to the Fundamental Theorem of Algebra and apply it in obtaining all complex roots of polynomial functions. After introduction of the theorem, explain to students that all real numbers can be written as complex numbers. For example, the number 3 can be written as 3 + 0*i*. Therefore, every real zero is also a complex zero.

Students will consider the polynomial $f(x) = x^3 + x^2$. To factor this polynomial, they will need to first factor out the greatest common factor of $x^3$and x^2, which is x^2. Further simplification to linear factors yields $f(x) = x^2(x + 1) = x \cdot x \cdot (x + 1)$.

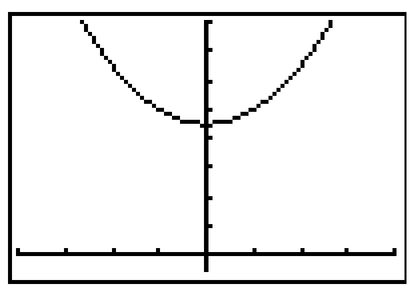


Then students are to graph the functions and use the **zero** command from the CALCULATE menu to determine the roots. At this time, the multiplicity of each root is introduced. Make sure that students understand that multiplicity is the number of times a factor shows up in a polynomial.

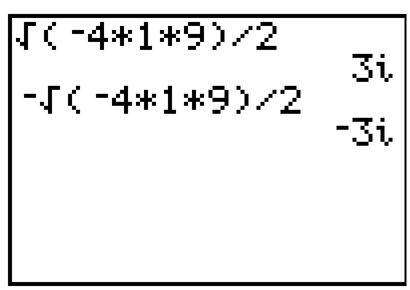


Students should notice that the multiplicities of all roots or zeros add up to the degree of the polynomial. The degree is the highest power on the variable in the expanded form of the polynomial.

Problem 2 – Beyond Real



In this section, students explore the graph of a polynomial with no real roots. They are to graph the function $f(x) = x^2 + 9$ and then use the graph to determine how many complex roots and what type of roots it has. They should see that since the graph does not cross the x-axis it has two imaginary roots.

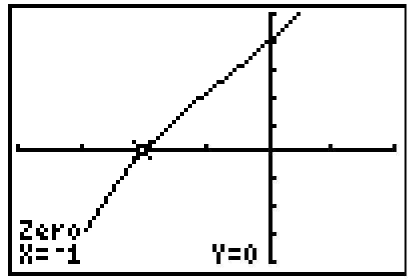


You may need to explain the difference between a complex number and an imaginary number. An imaginary number is a complex number, $a + bi$, where $a = 0$, $b \neq 0$, and $i = \sqrt{-1}$.

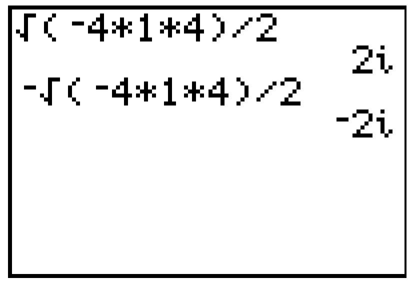
Students are to use the quadratic formula on the Home screen to identify all complex roots for the polynomial.

Problem 3 – The Mixed Case

Students are provided with a polynomial that has both real and imaginary roots. The graph illustrates the real root and other methods must be employed to find the remaining roots.



Students are to determine the roots applying a combination of such methods as synthetic division, polynomial long division, completing the square, and the quadratic formula.

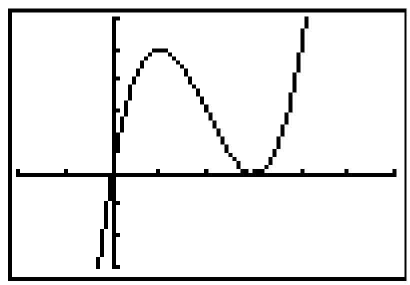


$$\begin{array}{r} x^2+4 \\ x+1\overline{)x^3+x^2+4x+4} \end{array}$$

Confirm with students that the sum of the multiplicities of the roots equals the degree of the polynomial.

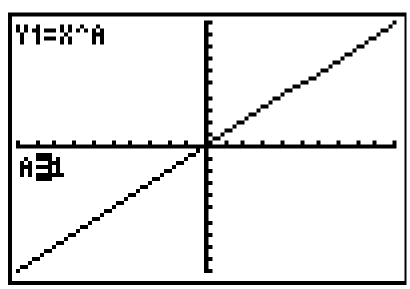
Extension – Even and Odd Multiplicity

In this problem, students explore the effect of even and odd multiplicities of roots by first looking at two specific graphs already in factored form. They need to compare the graph at the *x*-value of the root and the multiplicity of the root.



Students should begin to see that for roots with even multiplicities, the graph touches but does not cross the *x*-axis at that *x*-value. For roots with odd multiplicities, the graph does cross the *x*-axis at that *x*-value.

Students then use the transformational graphing application to explore how the multiplicity value affects the graph of the polynomial. They should understand that the root of $f(x) = x^a$ is $x = 0$.



As part of further exploration, students can change the function to have other roots, such as $f(x) = (x - 2)^a$.

Application & Practice Answers

Polynomial	Factor(s)	Roots	Multiplicities
$f(x) = x^4 - 9x^3 + 27x^2 - 31x + 12$	$x - 4$ $x - 3$ $x - 1$	4 3 1	1 1 2
$f(x) = x^3 - 7x^2 + 11x - 5$	$x - 5$ $x - 1$	5 1	1 2
$f(x) = x^5 + 9x^4 + 31x^3 + 63x^2 + 108x + 108$	$x - 2i$ $x + 2i$ $x + 3$	$2i$ $-2i$ -3	1 1 3

Using Decartes' Rule of Signs, find the number and possible types of solutions.

1) $x^5 - 8x^4 + 16x^3 + 4x^2 - 36x + 24$

2) $x^4 + 2x^3 - x^2 + 2x - 2$

3) $x^5 - 4x^4 + 25x^3 - 92x^2 + 200$

4) $x^4 - 6x^3 + 7x^2 - 6x + 6$

5) $x^3 - 5x^2 + 33x - 29$

6) $x^5 - 4x^4 + 39x^3 - 120x^2 + 350x - 500$

7) $x^2 + 2x + 5$

8) $x^3 + 5x^2 + 5x + 1$

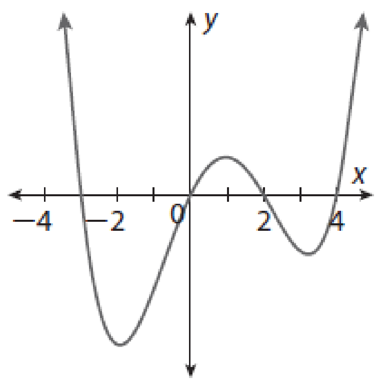
Term	Definition	Notation	Diagram/Visual
Factor			
Zero Product Theorem			
Zeros			

Zeros of a Polynomial Function

GUIDED NOTES

The zeros of the polynomial are the same as the solutions to the factors of a polynomial. Multiplicity is when a factor is repeated. When a factor (x-k) is raised to an odd power, the graph crosses over the x-axis at x = k. When a factor (x-k) is raised to an even power, the graph is tangent (bounces) to the x-axis at x = k.

Graph:



X-intercepts: (-3, 0) (0,0) (2,0) (4,0)

Factorization of Polynomial:

x-intercepts (Zeros/Solutions) : 0, -3, 2, 4

x = 0   x + 3 = 0   x – 2 = 0   x – 4 = 0

0 = x (x+3) (x-2) (x-4)   Zero Product Theorem

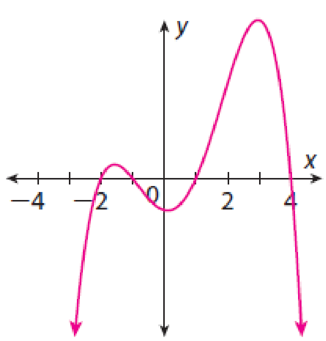
F(x) = x(x+3)(x-2)(x-4)

Given the factorization what are the x intercepts

Function	$f(x) = x^4$	$f(x) = x^3(x-2)$	$f(x) = x^2(x-2)(x+2)$	$f(x) = x(x-2)(x+2)(x+3)$
How many distinct factors?				
What are the *x*-intercepts?				

Example!

Given the graph:



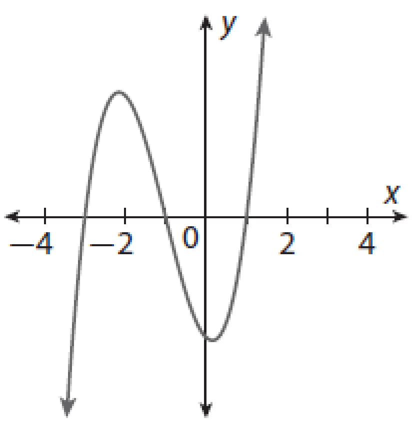
What are the x intercepts:

What is the factorization:

Function	$f(x) = x^3$	$f(x) = x^2(x-2)$	$f(x) = x(x-2)(x+2)$
How many distinct factors does $f(x)$ have?			
What are the graph's x-intercepts?			

SELF CHECK

Given the polynomial:



What are the x-intercepts:

What is the factorization of the polynomial?

Questions To Ponder

If I know all the x-intercepts of a polynomial function, can I come up with the original equation of the polynomial?

Name___________________ **Period**____ **Date**____

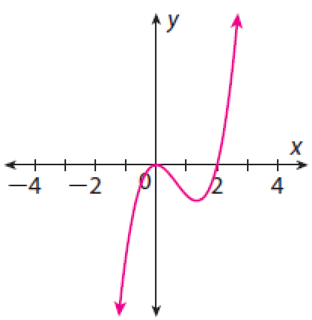
For each factorization identify the zeros

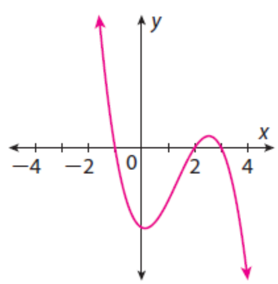
1. $f(x) = -(x+1)(x-2)(x-3)$

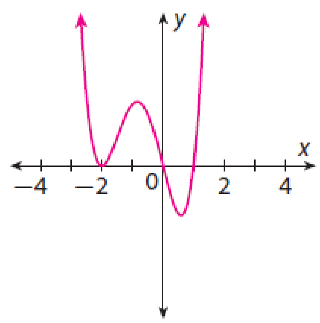
2. $f(x) = x(x+2)^2(x-1)$

3. $f(x) = x^2(x-2)$

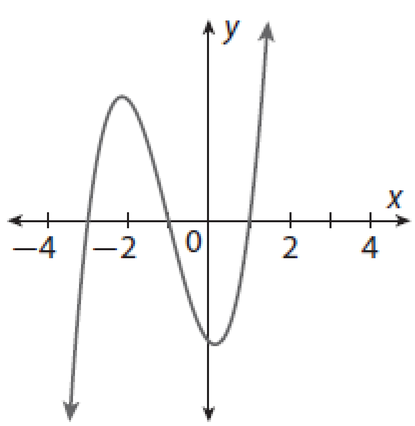
Which could be the graphs for the exercises above



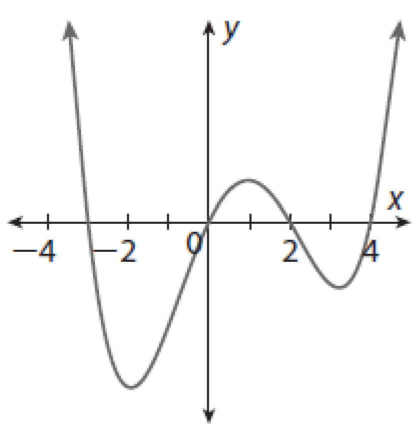




For each graph find the x intercepts and factorization?



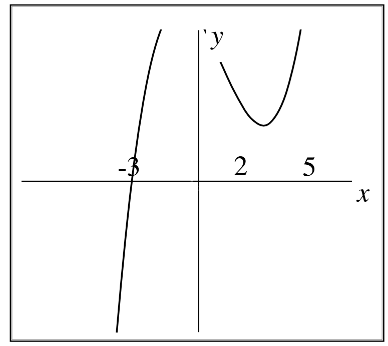
4.



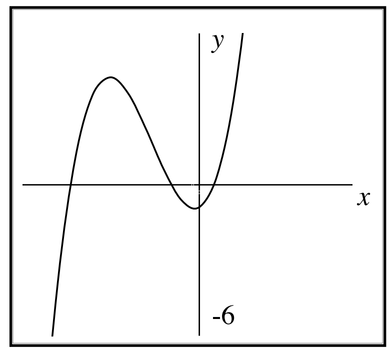
5.

Cubic Graphs and Their Equations

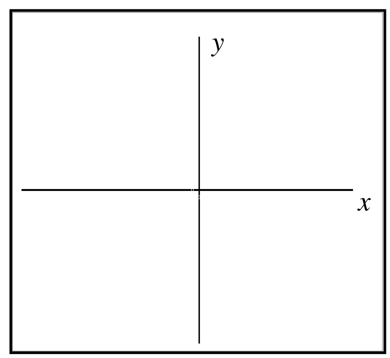
1. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the x-axis at (-3,0), (2,0), and (5,0).



2. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the y-axis at (0,-6).



3. On the axes, sketch a graph of the function $y=(x+1)(x-4)^2$.
   You do not need to plot it accurately!
   Show where the graph crosses the x- and y-axes.



4. Write down the equation of the graph you get after you:

   (i) Reflect $y=(x+1)(x-4)^2$ over the x-axis:

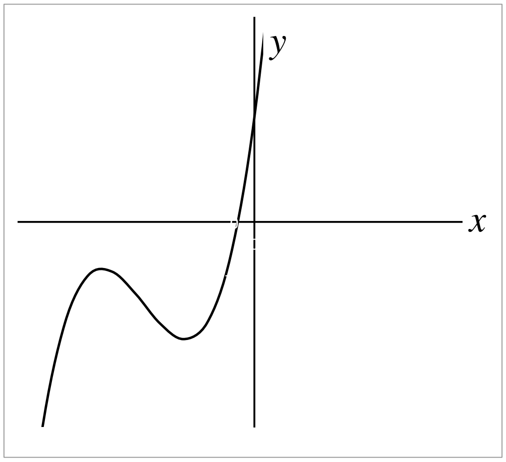
   (ii) Reflect $y=(x+1)(x-4)^2$ over the y-axis:

   (iii) Horizontally translate $y=(x+1)(x-4)^2$ through +2 units:

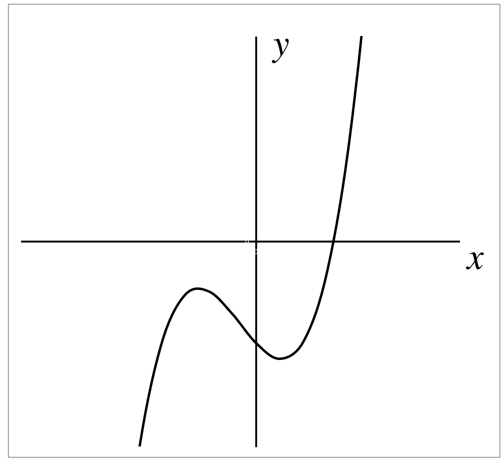
   (iv) Vertically translate $y=(x+1)(x-4)^2$ through +3 units:

Cubic Graphs

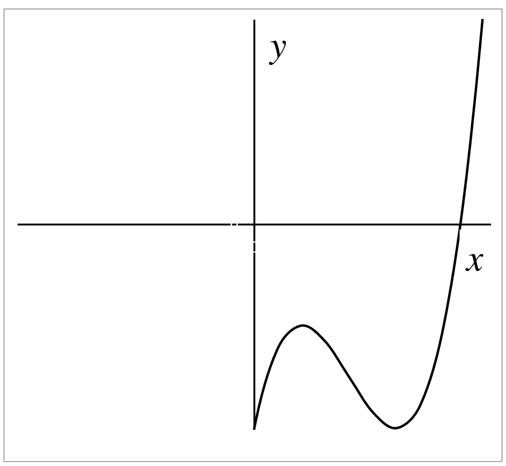
Graph A



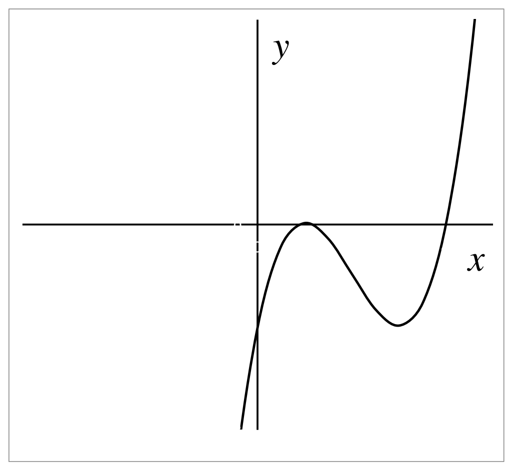
Graph B



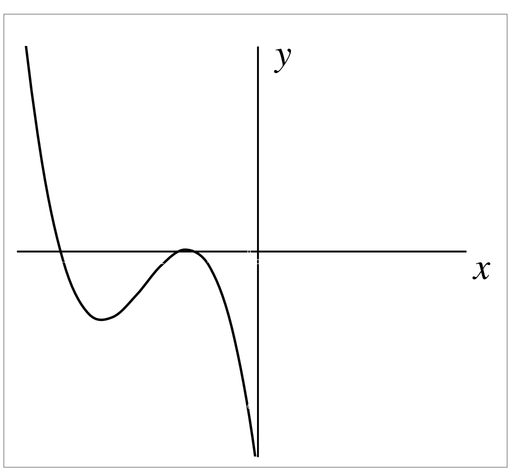
Graph C



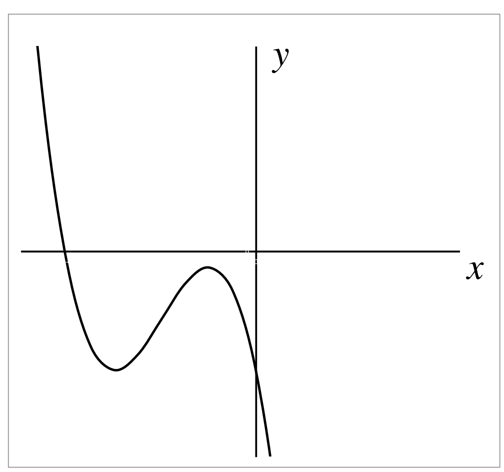
Graph D



Graph E



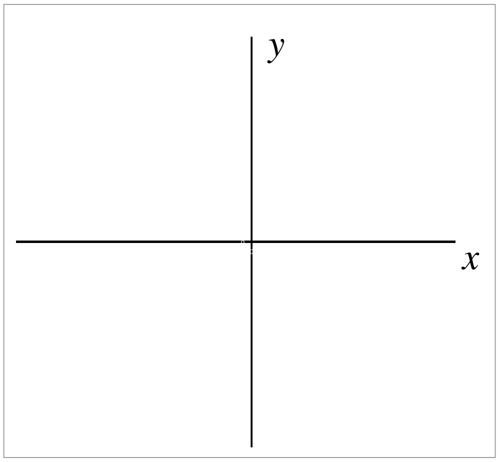
Graph F



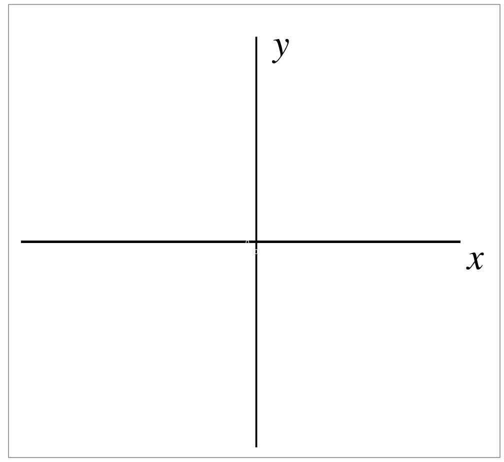
Cubic Graphs

Complete these graphs for the remaining functions.

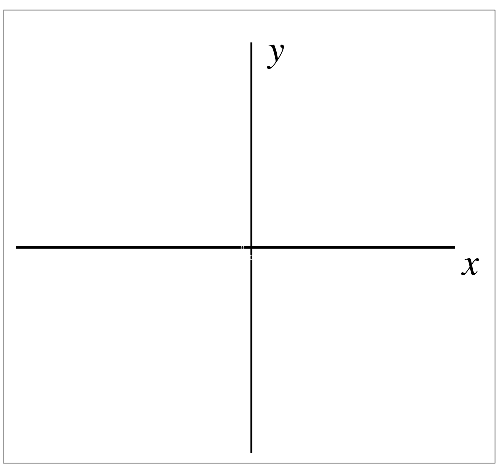
Graph G



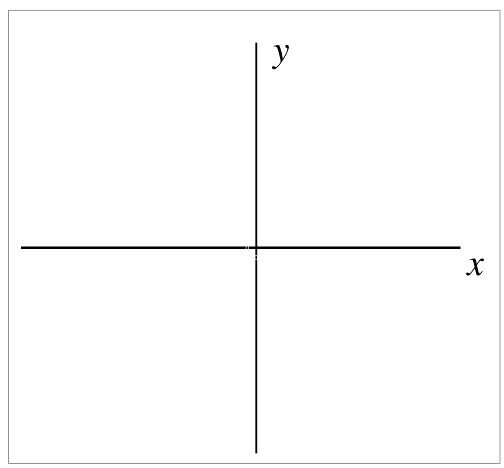
Graph H



Graph I



Graph J



Cubic Functions

1	$y = x(x-1)(x+2)$
2	$y = (x+1)(x+2)(x+4)$
3	$y = -(x+1)(x+2)(x+4)$
4	$y = (x-1)^2(x-4)$
5	$y = -(x-1)^2(x-4)$
6	$y = x(x-3)^2$
7	$y = -(x+1)^2(x+4)+4$
8	$y = -(x+1)^2(x+4)$
9	$y = (x-1)^2(x-4)+4$
10	$y = (x-1)(x-2)(x-4)$
11	$y = -x(x+3)^2$

Statements to Discuss: True or False?

A1

$f(x) = x^3 - 2x^2 - 9x + 18$

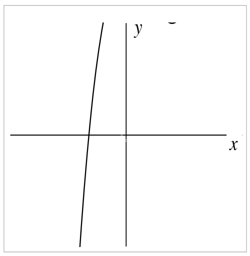
$f(2) = 0 \Rightarrow (x+2)$ is a factor of $f(x)$

A2

$f(x) = 3x^3 - 9x - 6$

$f(2) = 0 \Rightarrow (x-2)$ is a factor of $f(x)$

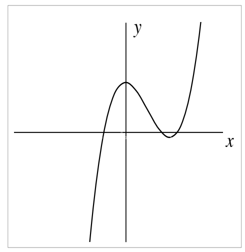
B1



A possible equation for this graph is:

$y = (x+1)(x-2)^2$

B2



A possible equation for this graph is:

$y = (x-1)(x+2)^2$

C1

If $f(x)$ is a cubic function and $f(1) = 0$, $f(3) = 0$ and $f(4) = 0$,

then $f(5) = 8$

C2

If $f(x)$ is a cubic function and $f(1) = 0$, $f(3) = 0$ and $f(4) = 0$,

then $f(5)$ could be anything

D1

$f(x) = (x-2)^2(x-7)$

$g(x) = (x-2)^2(7-x)$

$f(x)$ is a reflection of $g(x)$ over the y axis

D2

$f(x) = x(x-2)^2$

$g(x) = -x(-x-2)^2$

$f(x)$ is a reflection of $g(x)$ over the y axis

E1

$f(x) = g(x+2) \Rightarrow$

$g(x)$ is a horizontal translation of $f(x)$.

E2

$f(x) = g(x) + 2 \Rightarrow$

$g(x)$ is a horizontal translation of $f(x)$

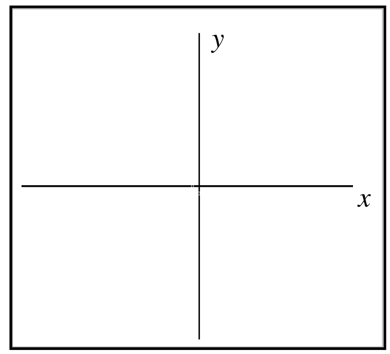
Cubic Graphs and Their Equations (revisited)

1. A cubic function has just two x-intercepts, one at x = 0 and the other at x = 6.

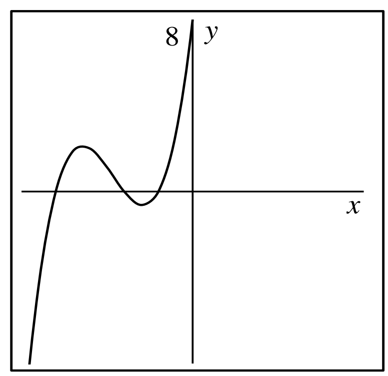
   Write down at least three possible equations with these intercepts:

2. On the axes, sketch a graph of the function $y=(x+2)(x-3)(x-1)$.
   You do not need to plot it accurately!
   Show where the graph crosses the x- and y-axes.

   Write down the equation of the graph after you reflect it over the y-axis.



3. Write down an equation of a cubic function that would give a graph like the one shown here. It crosses the y-axis at (0,8).



4. The function $y=(x-2)(x+3)^2$ is transformed to:

   (i) $y=(x-2)(x+3)^2+4$.
   Describe in words how the function has been transformed.

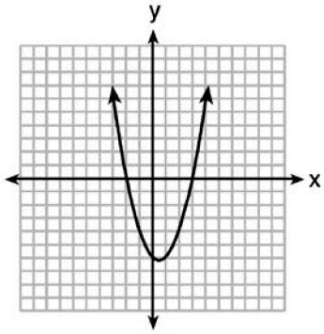
   (ii) $y=-(x-2)(x+3)^2$.
   Describe in words how the function has been transformed.

   (iii) $y=(x+1)(x+6)^2$.
   Describe in words how the function has been transformed.

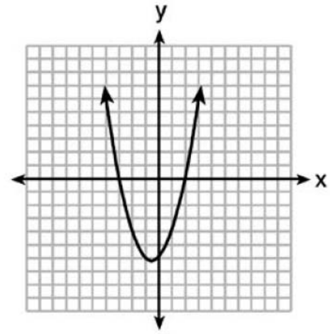
A.

The graphs below represent functions defined by polynomials. For which function are the zeros of the polynomials 2 and -3?

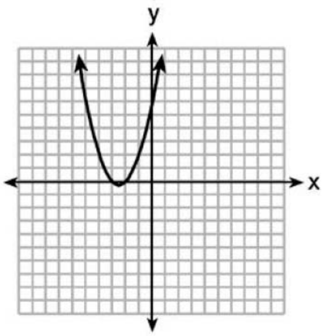
1)



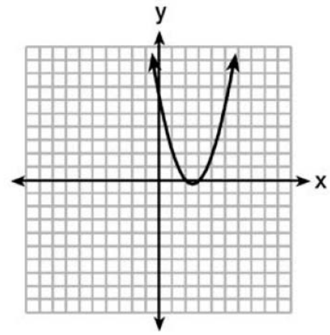
3)



2)



4)



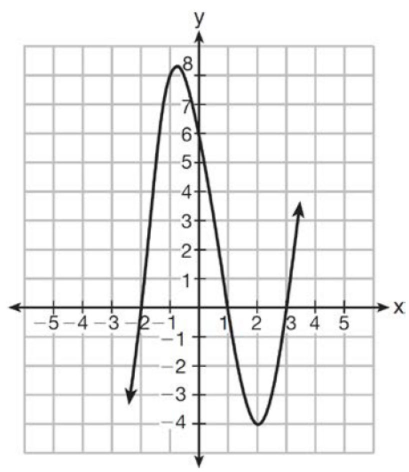
B.

Which equation(s) represent the graph below?

I $y = (x + 2)(x^2 - 4x - 12)$

II $y = (x - 3)(x^2 + x - 2)$

III $y = (x - 1)(x^2 - 5x - 6)$



1) I, only
2) II, only
3) I and II
4) II and III

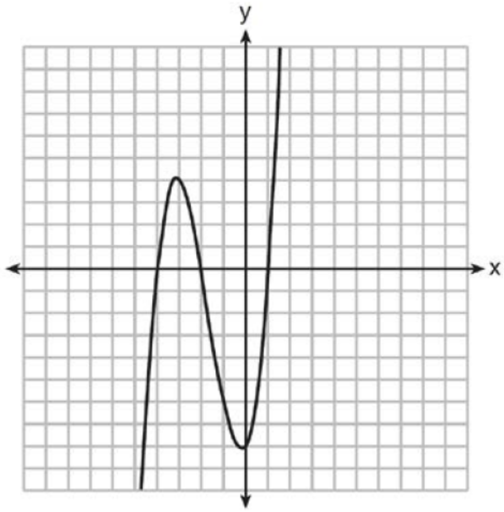
C.

For which function defined by a polynomial are the zeros of the polynomial -4 and -6?

1) $y = x^2 - 10x - 24$
2) $y = x^2 + 10x + 24$
3) $y = x^2 + 10x - 24$
4) $y = x^2 - 10x + 24$

D.

The graph of $f(x)$ is shown below.

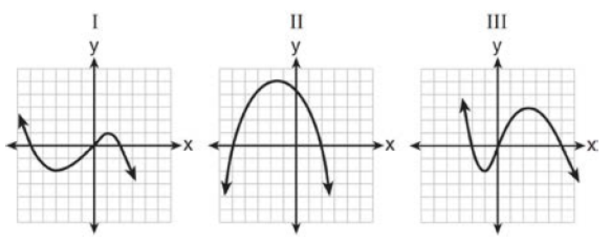


Which function could represent the graph of $f(x)$?

1) $f(x) = (x+2)(x^2+3x-4)$
2) $f(x) = (x-2)(x^2+3x-4)$
3) $f(x) = (x+2)(x^2+3x+4)$
4) $f(x) = (x-2)(x^2+3x+4)$

E.

A polynomial function contains the factors x, $x-2$, and $x+5$. Which graph(s) below could represent the graph of this function?



1) I, only
2) II, only
3) I and III
4) I, II, and III

F.

What are the zeros of the function $f(x) = x^2 - 13x - 30$?

1) -10 and 3
2) 10 and -3
3) -15 and 2
4) 15 and -2

G.

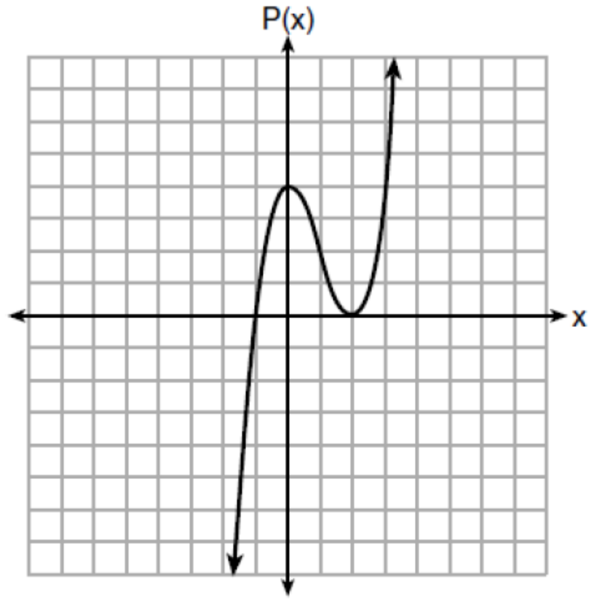
The zeros of the function $f(x) = 2x^3 + 12x - 10x^2$ are

1) $\{2, 3\}$
2) $\{-1, 6\}$
3) $\{0, 2, 3\}$
4) $\{0, -1, 6\}$

H. Which polynomial function has zeros at -3, 0, and 4?

1) $f(x) = (x+3)(x^2+4)$
2) $f(x) = (x^2-3)(x-4)$
3) $f(x) = x(x+3)(x-4)$
4) $f(x) = x(x-3)(x+4)$

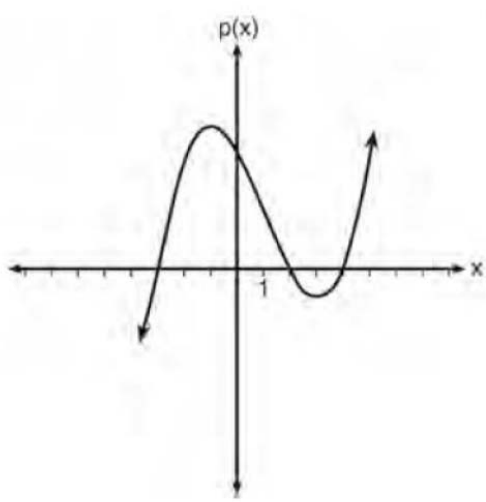
I.



Which equation could represent $P(x)$?

1) $P(x) = (x+1)(x-2)^2$
2) $P(x) = (x-1)(x+2)^2$
3) $P(x) = (x+1)(x-2)$
4) $P(x) = (x-1)(x+2)$

J. Based on the graph below, which expression is a possible factorization of *p(x)*?



1) $(x+3)(x-2)(x-4)$
2) $(x-3)(x+2)(x+4)$
3) $(x+3)(x-5)(x-2)(x-4)$
4) $(x-3)(x+5)(x+2)(x+4)$

ANSWER KEY

A) 3

ANS: 3
Strategy: Look for the coordinates of the x-intercepts (where the graph crosses the x-axis). The zeros are the x-values of those coordinates.

Answer c is the correct choice. The coordinates of the x-intercepts of the graph are (2, 0) and (-3, 0). The zeros of the polynomial are 2 and -3.

B) 2

Strategy: Factor the trinomials in each equation, then convert the factors into zeros and select the equations that have zeros at -2, 1, and 3.

STEP 1.

I	II	III
$y = (x+2)(x^2 - 4x - 12)$	$y = (x-3)(x^2 + x - 2)$	$y = (x-1)(x^2 - 5x - 6)$
$y = (x+2)(x-6)(x+2)$	$y = (x-3)(x+2)(x-1)$	$y = (x-1)(x-6)(x+1)$
Zeros at -2, 6, and -2	Zeros at 3, -2, and 1	Zeros at 1, 6, and -1
(Wrong Choice)	(Correct Choice)	(Wrong Choice)

C) 2

D) 1

Strategy:
STEP 1. Identify the zeros and convert them into factors.
The graph has zeros at -4, -2, and 1. Convert these zeros of the function into the following factors: (x+4)(x+2)(x-1). The functiuon rule is f(x)= (x+4)(x+2)(x-1)
STEP 2. Eliminate wrong answers. Choices b and d can be eliminated because (x-2) is not a factor.

b.	d.
$f(x) = (x-2)(x^2 + 3x - 4)$	$f(x) = (x-2)(x^2 + 3x + 4)$
$(x-2)$ is not a factor.	$(x-2)$ is not a factor.
(Wrong Choice)	(Wrong Choice)

STEP 3. Choose between remaining choices by factoring the trinomials.

a.	c.
$f(x) = (x+2)(x^2 + 3x - 4)$	$f(x) = (x+2)(x^2 + 3x + 4)$
$f(x) = (x+2)(x+4)(x-1)$	$(x^2 + 3x + 4)$ cannot be factored into $(x+4)(x-1)$
Contains all three factors.	(Wrong Choice)
(Correct Choice)	

E) 1

Stategy 1. Convert the factors to zeros, then find the graph(s) with the corresponding zeros.

STEP 1. Convert the factors to zeros.

A factor of $x-0$ equates to a zero of the polynomial at x=0.

A factor of $x-2$ equates to a zero of the polynomial at x=2.

A factor of $x+5$ equates to a zero of the polynomial at x=-5.

STEP 2. Find the zeros of the graphs.

Graph I has zeros at -5, 0, and 2.
Graph II has zeros at -5 and 2.
Graph III has zeros at -2, 0, and 5.

Answer choice *a* is correct.

F) 4

Strategy: Find the factors of $f(x)=x^2-13x-30$, then convert the factors to zeros.

STEP 1. Find the factors of $f(x)=x^2-13x-30$.

$$f(x)=x^2-13x-30$$

$$f(x)=(x-\_\_\_)(x+\_\_\_)$$

The factors of 30 are

1 and 30

2 and 15 (*use these*)

$$f(x)=(x-15)(x+2)$$

STEP 2. Convert the factors to zeros.

If the factors are $(x-15)$ and $(x+2)$,

then the zeros are at $x=15$ and $x=-2$.

G) 3

Strategy #1. Find the factors and use the multiplication property of zero to find the zeros.

$$2x^3+12x-10x^2=0$$

$$2x^3-10x^2+12x=0$$

$$2x(x^2-5x+6)=0$$

$$2x(x-3)(x-2)=0$$

If the factors are 2x, x-3, and x-2, the zeros are 0, 2, and 3.

H) 3

The zeros of a function are the x-values when $y = 0$.
Strategy: Convert the zeros to factors, then combine the factors to write the function.

Zeros	Factors
$x = -3$	$(x + 3)$
$x = 0$	(x)
$x = 4$	$(x - 4)$

$f(x) = (x + 3)(x)(x - 4)$
Check by inputting the function in a graphing calculator and inspecting the zeros

I) 1

Note that the zeros (x-intercepts) occur at -1 and +2. This means that the factors of the equation are (x+1) and (x-2). Eliminate $P(x) = (x - 1)(x + 2)^2$ and $P(x) = (x - 1)(x + 2)$ because they have the wrong factors.

The choice is between $P(x) = (x + 1)(x - 2)^2$ and $P(x) = (x + 1)(x - 2)$. $P(x) = (x + 1)(x - 2)^2$ is a third degree equation and $P(x) = (x + 1)(x - 2)$ is a second degree (quadratic) equation.

The graph is definitely not a parabola, so it cannot be the graph of a quadratic function. Eliminate $P(x) = (x + 1)(x - 2)$. The correct answer is $P(x) = (x + 1)(x - 2)^2$.

J) 1

Strategy: Convert the zeros of the function to factors.

Zeros occur at	Factors are:
(-3, 0)	(x+3)
(2, 0)	(x-2)
(4, 0)	(x-4)

PTS: 2 NAT: A.APR.B.3

Lesson Name: A2.U3.C1.**C.01.vocab.ZeroProductFactorable**

Term	Definition	Notation	Diagram/Visual
Difference of Squares			
Greatest Common Factor			

Solve Polynomials by Factoring

Factoring is the process by which we go about determining what we multiplied to get the given quantity. For example, here are some ways to factor 25.

25 =(1)(25) 25 = (-1)(-25) 25 = (5)(5) 25 = (-5)(-5)

Here are some ways to factor even powers of x.
$X^2 = (x)(x)$
$X^4 = (x)(x)(x)(x) = (x^2)(x^2)$
$X^6 = (x)(x)(x)(x)(x)(x) = (x^3)(x^3)$

Factoring polynomials is done in pretty much the same matter. We determine all the terms that were multiplied together to get the given polynomial. We then try to factor each of the terms we found in the first step. This continues until we simply can't factor anymore.

The first method for factoring polynomials will be factoring out the greatest common factor. When factoring in general this will also be the first thing that we should try as it will often simplify the problem.
To use this method all that we do is look at all the terms and determine if there is a factor that is in common to all the terms. If there is, we will factor it out of the polynomial. Also note that in this case we are really only using the distributive law in reverse.

Here is an example of factoring out the greatest common factor.

$5x + 90 = 5(x + 18)$.

Example! $x^2 – 16 = 0$ In this example a = 1, b = 0, c = -16. The factors of -16 are (1)(-16), (-1)(16), (2)(-8), (-2)(8), (4)(-4). The two factors that add up to 0 are 4 and -4. Therefore, $(x-4)(x+4)= 0$ and the solutions are x = 4 and x = -4

Example! $X^4 – 16 = 0$. Recall $x^4 = (x^2)(x^2) = (x^2)^2$, so $x^4 – 16 =(x^2)^2 - 16\ (x^2 – 4)(x^2+4)$
In this example a = 1, b = 0, c = -16. The factors of -16 are (1)(-16), (-1)(16), (2)(-8), (-2)(8), (4)(-4). The two factors that add up to 0 are 4 and -4. Therefore, $(x^2-4)(x^2+4)= 0$

$X^2-4 = 0$, $(x-2)(x+2) = 0$, $x = +/-2$

$x^2+4 = 0$, $x^2 = -4$, $x = +/-2i$

So the solutions are x = {2, -2, 2i, -2i}

Example!

Factor the polynomial

$8x^4 - 4x^3 + 10x^2$

First, we will notice that we can factor a 2 out of every term.

$$8x^4 - 4x^3 + 10x^2 = 2(4x^4 - 2x^3 + 5x^2)$$

Also note that we can factor an x^2 out of every term. Here is the factoring for this problem.

$$8x^4 - 4x^3 + 10x^2 = 2x^2(4x^2 - 2x + 5)$$

Example!

Factor the polynomial

$x^4 - 2x^2 - 48$

Using u substitution, substitute u for x^2. $u=x^2$ Since $x^4 = (x^2)^2$

$u2 - 2u - 48 = (u - 8)(u + 6)$

$(x^2-8)(x^2 + 6)$

SELF CHECK

Factor each of the following.

1) $3x^4 - 3x^3 - 36x^2$
2) $x^4 - 25$
3) $x^4 + x^2 - 30$

Questions To Ponder

When is it a good time to use u substitution? Should I consider factoring out the greatest common factor first, before I factor a polynomial?

Factor out the greatest common factor from each polynomial.

1) $6x^7 + 3x^4 - 9x^3$

2) $a^3b^9 - 7a^{10}b^4 + 2a^5b^2$

Factor the polynomials

3) $x^2 - 100$

4) $y^2 - 121$

5) $6x^2 - 150$

6) $4x^4 - 144$

7) $z^4 - 3z^2 - 40$

8) $3x^4 + 12x^3 - 96x$

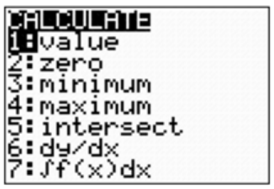
FINDING THE X-INTERCEPTS OF A QUADRATIC EQUATION

Name:

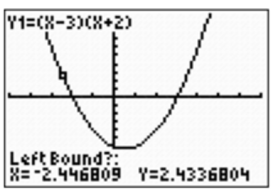
Use the steps below to answer the questions on the next page.

To find the *x*-intercepts:

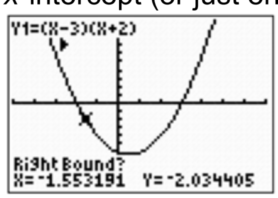
1. Enter the equation in Y_1. $y = x^2 - x - 6$
2. Press **ZOOM** and **6** (Zstandard) to set the scale for your graph. The calculator will then show the parabola.
3. Press **2nd TRACE** 1 to view the Calculate screen.



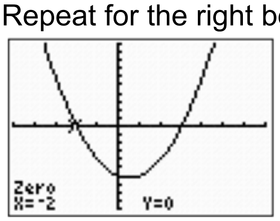
4. Select **2: ZERO**. Your screen should be similar to the following screen.



5. You will be asked to enter a left bound. You can move the cursor to the left of one *x*-intercept (or just enter an *x* value that is to the left of the *x*-intercept). Press **ENTER.**



6. Repeat for the right bound, being sure that you are to the right of the same *x*-intercept.



7. The next screen will say guess. You can guess if you want but it is not necessary. Press **ENTER**. You will get one *x*-intercept.
8. Repeat steps 3 through 7 to get the other *x*-intercept.

Finding the *x*-Intercepts of a Quadratic Equation (continued)

1. Use the graphing calculator to find the *x*-intercepts for each of the following:

Equation	First *x*-intercept	Second *x*-intercept
$y = x^2 - 4x - 12$		
$y = x^2 + 2x - 8$		
$y = x^2 - x + 6$		
$y = (x-1)(x-2)$		
$y = (x+4)(x+3)$		
$y = (x-3)(x+5)$		

2. Can you determine the *x*-intercepts by looking at a quadratic equation? Explain.

3. Which form of the quadratic equation did you find the easiest to use when determining the *x*-intercepts? Explain the connection between the factors and the *x*-intercepts.

Use the difference of squares pattern to factor each polynomial.

1. $X^2 - 1$

2. $X^2 - 25$

3. $x^2 - 100$

4. $X^2 - 4y^2$

5. $9x^2 - y^2$

6. $49y^2 - 64x^2$

7. $1 - 4n^2$

8. $5x^2 - 20$

9. $7x^2 - 7$

10. $4a^4 + 16a^2$

Lesson Name: A2.U3.C1.**D.01.Vocab.ZeroProductGrouping**

Term	Definition	Notation	Diagram/Visual
Common Binomials			
Greatest Common Factor			
Factoring By Grouping			

Factoring by Grouping

GUIDED NOTES

Grouping is a factor method most useful when you have ____ terms.

1) Factor out ______ (if any).
2) Group the first _____ terms, and the ______ two terms. Be sure to group the signs as well and add the groups together.
3) Factor the ______ out of both groups.
4) Remaining binomial should be the ______ for both groups. _______ the common binomial.

Example! $x^3 + 6x^2 – 5x – 30$

There is no GCF to be factored out. Now we can group the first two and last two terms.

$(x^3 + 6x^2) + (-5x – 30)$

Now we factor the GCF out of both groups.

$X^2(x +6) + -5(x +6)$

We factor what is outside of our common binomial (x+6) and now we have

$(x^2 – 5)(x + 6)$

Example! 4a – 8b + 5ax – 10bx

There is no GCF to be factored out. Now we can group the first two and last two terms.

(4a – 8b) + (5ax – 10bx)

Now we factor the GCF out of both groups.

4(a – 2b) + 5x(a – 2b)

We factor what is outside of our common binomial (a – 2b) and now we have

(a – 2b)(4 + 5x)

SELF CHECK

1) $12x^3 + 2x^2 - 30x - 5$

2) $8x^3 - 64x^2 + x - 8$

Questions To Ponder

Find and explain the error.

Problem:	$\mathbf{8xy - 8z + 7z - 7y}$
Step 1:	$(8xy - 8z) + (7z - 7y)$
Step 2:	$8x(y - z) + 7(z - y)$
Step 3:	$(8x + 7)(y - z)$

Factor the following polynomials.

1) $m^3 + 4m^2 - 6m - 24 =$

2) $x^3 - x^2 - 3x + 3$

3) $ab + b + 2a + 2$

4) $6ax - by + 2bx - 3ay$

5) $2rs + 4s - r - 2$

6) $8 + 9y^4 - 6y^3 - 12y$

7) $2x^3y^2 + x^2y^2 - 14xy^2 - 7y^2$

8) $mx + qx + my + qy$

Solve and Graph it!

1. Solve.

a) $x^3 - x^2 - 2x = 0$

b) $0 = x^4 - 2x^3 - 5x^2 + 6x$

c) $0 = x^4 - 1$

d) $0 = 3x^2 + 12x + 12$

e) $0 = -x^3 + 2x^2 + 4x - 8$

f) $-x^4 - 4x^3 - 4x^2 = 0$

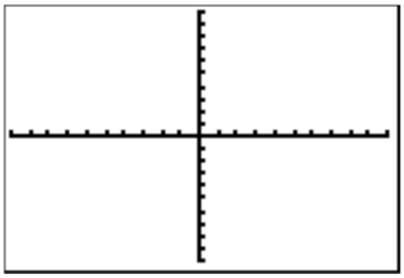
g) $x^3 - 9x^2 + 9x - 27 = 0$

h) $0 = x^4 - 5x^3 + 6x^2 + 4x - 8$

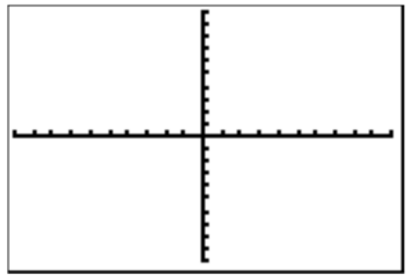
Solve & Graph it! (Continued)

2. Use your graphing calculator to sketch a graph of the functions below. Compare the roots from #1 with the sketches to determine if these roots and x-intercepts match.

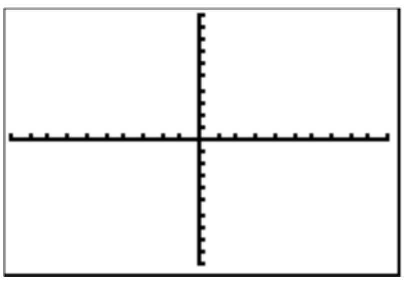
a) $y = x^3 - x^2 - 2x$



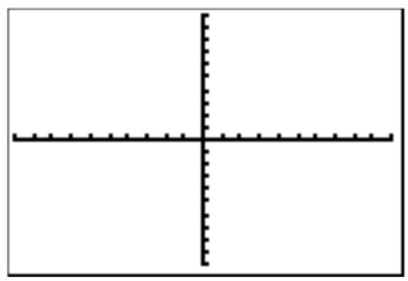
b) $y = x^4 - 2x^3 - 5x^2 + 6x$



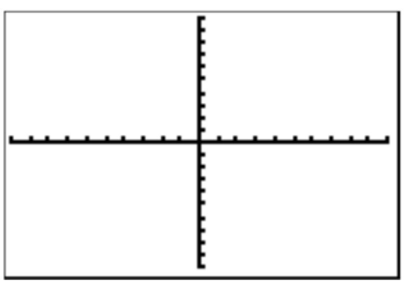
c) $y = x^4 - 1$



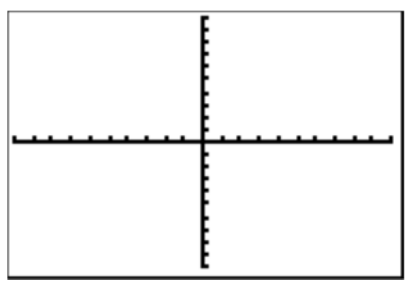
d) $y = 3x^2 + 12x + 12$



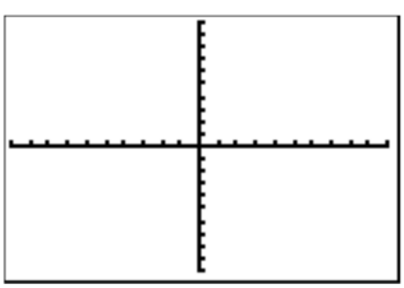
e) $y = -x^3 + 2x^2 + 4x - 8$



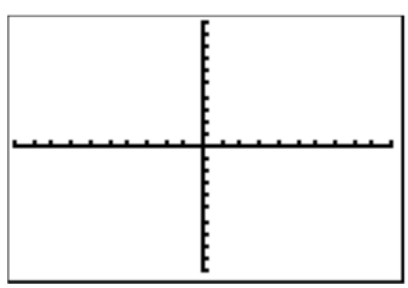
f) $y = -x^4 - 4x^3 - 4x^2$



g) $y = x^3 - 9x^2 + 9x - 27$



h) $y = x^4 - 5x^3 + 6x^2 + 4x - 8$



Solve & Graph it! (Continued)

3. Solve. Factor and use the quadratic formula where needed.

a) $x^4 + 4x^3 + 5x^2 = 0$

b) $0 = x^4 + x^2$

c) $-4x^3 + 10x^2 – 2x = 0$

Factoring by Grouping

1) $8n^3 - 7n^2 + 56n - 49$

2) $5x^3 - 6x^2 - 15x + 18$

3) $9r^3 + 3r^2 - 21r - 7$

4) $25v^3 + 25v^2 - 15v - 15$

5) $120b^3 + 105b^2 + 200b + 175$

6) $120x^3 - 80x^2 - 168x + 112$

GUIDED NOTES

Special Factoring Patterns

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Example!

1) $x^3 + 8 = (x+2)(x^2 - 2x + 4)$

$a = x$ $b = 2$

2) $8x^3 - 1 = (2x-1)(4x^2 + 2x + 1)$

$a = 2x$ $b = 1$

3) $x^3 - 125$

4) $81y^4 + 192y$

5) $128y^6 + 2$

SELF CHECK **Factor**

1. $8x^3 + 64$
2. $125x^3 - 216$
3. $7x^5 - 56x^2$

Questions To Ponder

What does it mean to be "factored completely"?

How does this relate to finding the zeros of a polynomial?

Factor Completely

Name__________________ Period_____ Date___________

1. $a^3 - 1$	2. $b^3 + 1$
3. $c^3 - 27$	4. $d^3 + 125$
5. $x^3 + 64$	6. $8y^3 - 343$
7. $4x^4 - 108x$	8. $2y^3 + 16z^3$

Factor Completely

Name__________________ Period_____ Date___________

1. $64a^3 - 27$	2. $27b^3 + 343$
3. $c^3 - 8$	4. $8d^3 + 729$
5. $24x^3 - 81$	6. $2y^3 + 432$
7. $-3g^3 + 24$	8. $7h^3 + 448$

Bonus: $a^6b^3 + 125$

Solve the following polynomials by factoring then using the Quadratic Formula

Problem 1. $X^3 - 125 = 0$

Problem 2. $27x^3 - 1 = 0$

Problem 3. $64x^3 - 8y^3 = 0$

Problem 4. $125x^3 - 27y^3 = 0$

Problem 5. $32x^3 - 500 = 0$

Problem 6. $-3x^3 + 192y^3 = 0$

Solve for x by factoring and using the quadratic formula.

Problem 1. $108 - 4x^3 = 0$

Problem 2. $125a^3 + 64b^3 = 0$

Problem 3. $27m^3 - 125 = 0$

Problem 4. $2q^3 + 54 = 0$

Problem 5. $54x^3 - 2 = 0$

Problem 6. $375 - 81a^3 = 0$

Problem 7. $432 + 250m^3 = 0$

Problem 8. $81x^3 + 192 = 0$

USING THE QUADRATIC FORMULA

The sum or difference of cubes can be solved by using the quadratic equation. Recall, $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$ and that we can now apply the quadratic formula to the $(a^2 + ab + b^2)$ portion of the equation and solve the equation.

Example!

$3x^3 + 648 = 0$

$3(x^3 + 216) = 0$

$(x + 6)(x^2 + 6x + 36) = 0$

X + 6 = 0, x = -6

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{(-6)^2 - 4(1)(36)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{36 - 144}}{2}$$

$$x = \frac{-6 \pm \sqrt{-108}}{2}$$

$$x = \frac{-6 \pm 6i\sqrt{3}}{2}$$

$X = -3 \pm 3i\sqrt{3}$

Example!

Solve $x^3 - 27 = 0$
$(x-3)(x^2 + 3x + 9) = 0$
X -3 = 0, x =3

$$x^2 + 3x + 9 = 0$$
$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{9 - 36}}{2}$$
$$= \frac{-3 \pm \sqrt{-27}}{2}$$
$$= \frac{-3 \pm 3i\sqrt{3}}{2}$$

SELF CHECK

Solve $4x^3 + 32 = 0$

Once I apply the rules for sum and difference of cubes, is my next step to always solve using the quadratic formula?

Lesson Name: A2.U3.C1.**F.Vocab.SolveByQuadraticFormula**

Term	Definition	Notation	Diagram/Visual
substitution			
Quadratic Formula			

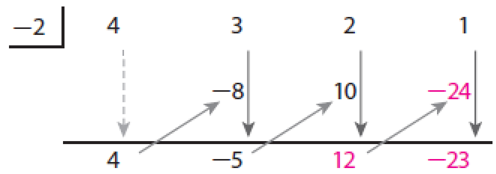
Term	Definition	Notation	Diagram/Visual
Dividend			
Divisor			
Quotient			
Remainder			
Synthetic Division			
Factor Theorem			
Remainder Theorem			

Synthetic Division

GUIDED NOTES

Synthetic Division is another way of dividing polynomials quickly when the divisor is a binomial factor (x – a).

Example of set up of a problem ($4x^4 + 3x^3 + 2x^2 + x$) divided by (x + 2)



Answer: ($4x^3 - 5x^2 + 12x$) – 23
(Note the answer is 1 degree less than original polynomial.)

In Unit 2 you did long division. Compare long division with synthetic substitution

Long Division	Synthetic Substitution
$x - 2 \overline{)\,3x^3 + 4x^2 + 0x + 10}$ with quotient $3x^2 + 10x + 20$ $-(3x^3 - 6x^2)$ $10x^2 + 0x$ $-(10x^2 - 20x)$ $20x + 10$ $-20x - 40$ 50	2 | 3 4 0 10 6 20 40 3 10 20 | 50

$(7x^3 - 6x + 9) \div (x + 5)$

By inspection, $a = -5$. Write the coefficients and a in the synthetic division format.

–5 | 7 0 –6 9

Where did the zero come from?

Bring down the first coefficient. Then multiply and add for each column.

–5	7	0	–6	9
		–35	175	–845
	7	–35	169	–836

Write the result, using the non-remainder entries of the bottom row as the coefficients.

$$(7x^3 - 6x + 9) = (x + 5)(7x^2 - 35x + 169) - 836$$

Divisor

Quotient with Remainder

Check.

$$(7x^3 - 6x + 9) = (x + 5)(7x^2 - 35x + 169) - 836$$
$$= 7x^3 - 35x^2 - 35x^2 - 175x + 169x + 845 - 836$$
$$= 7x^3 - 6x + 9$$

Example!

$$(4x^4 - 3x^2 + 7x + 2) \div (x - \frac{1}{2})$$

a= |
|
|________________________
|___

A). Find a. Then write the coefficients in decreasing order (if a degree is missing place a zero).
B) Bring down the first coefficient. Then multiply and add for each column.
C) Use the coefficients below the line to write the quotient and remainder.
D) Check your work (if necessary)

SELF CHECK

1. $(2x^3 + 5x^2 - x + 7) \div (x - 2)$

2. $(6x^4 - 25x^3 - 3x + 5) \div \left(x + \frac{1}{3}\right)$

Questions To Ponder

Can you use synthetic division to divide a polynomial by $x^2 + 3$?
What conditions must be met in order to use synthetic division?

Synthetic Division (used to find other factors)

GUIDED NOTES

For higher degree polynomials, synthetic division can help you locate all the other factors with repetition when one factor is given (or discovered). This is another way to locate all the zeros of a polynomial when factoring may not be readily accessible.

If $f(x) = x^3 + 3x^2 - 4x - 12$ and it is known that (x + 3) is a factor.

Use synthetic division

-3	1	3	-4	-12
		-3	0	12
	1	0	-4	0

Since the remainder is 0, x+3 is a factor.

With the remaining quotient as a polynomial and factor: $x^2 - 4 = (x + 2)(x - 2)$

So the complete factorization of $f(x) = x^3 + 3x^2 - 4x - 12$ is $(x + 3)(x + 2)(x - 2)$. The zeros/x –intercepts are -3, -2, 2.

Example!

$p(x) = x^4 - 4x^3 - 6x^2 + 4x + 5; (x + 1)$

Use synthetic division.

-1	1	-4	-6	4	5
	1				

Since the remainder is ____, $(x + 1)$ ____ a factor. Write $q(x)$.

$q(x) =$ ______

Now factor $q(x)$ by grouping.

$q(x) =$ ______

$=$ ______

$=$ ______

$=$ ______

So, $p(x) = x^4 - 4x^3 - 6x^2 + 4x + 5 =$ ______.

What are the zeros/x – intercepts: ______________________________

Example 2: find the complete factorization and zeros of the polynomial
$p(x) = 2x^4 + 8x^3 + 2x + 8; (x + 4)$

SELF CHECK Factor and determine the zeros

1. $f(x) = 2x^3 + 5x^2 - 3x ; (x + 3)$

2. $p(x) = x^3 + 2x^2 - x - 2; (x + 2)$

3. $p(x) = x^3 - 22x^2 + 157x - 360; \ (x - 8)$

Questions To Ponder

Long Division	Synthetic Substitution
$\begin{array}{r} 3x^2 + 10x + 20 \\ x - 2 \overline{)\, 3x^3 + 4x^2 + 0x + 10} \\ \underline{-(3x^3 - 6x^2)} \\ 10x^2 + 0x \\ \underline{-(10x^2 - 20x)} \\ 20x + 10 \\ \underline{-20x - 40} \\ 50 \end{array}$	$\begin{array}{r\|rrrr} 2 & 3 & 4 & 0 & 10 \\ & & 6 & 20 & 40 \\ \hline & 3 & 10 & 20 & 50 \end{array}$

How can I tell that x – 2 is not a factor of $3x^3 + 4x^2 + 10$?
or
How do you know when the divisor is a factor of the dividend?

How does knowing one linear factor of a polynomial help find the other factors?

Ticket out the door/Quick Check for basic understanding

Name ___________________________

Complete the following using synthetic division.

Period _____ Date_______________

1. $(x^2 + 4x + 1) \div (x - 5)$

5	1	4	1
		5	45
	A	*B*	*C*

A=_____ B=_____ C=_____

What is the remainder? _________

Write the quotient? ______________________

2. $(x^2 - 8x + 6) \div (x + 2)$

A	*B*	*C*

A=_____ B=_____ C=_____

What is the remainder? _________

Write the quotient? ______________________

3. $(x^2 + 4x - 2) \div (x - 3)$

A	*B*	*C*

A=_____ B=_____ C=_____

What is the remainder? _________

Write the quotient? ______________________

Independent Practice

Name________________________________

Period _____ Date____________________

1. $(9x^2 - 3x + 11) \div (x - 6)$

2. $(3x^4 - 2x^2 + 1) \div (x + 2)$

3. $(6x^5 - 3x^2 + x - 2) \div (x - 1)$

4. $(-x^4 - 7x^3 + 6x^2 - 1) \div (x - 3)$

Name_______________________________ Period_______ Date__________

Determine whether or not the given binomial is a factor of the polynomial.

1. $p(x) = 2x^4 + 6x^3 - 5x - 10; (x + 2)$

2. $p(x) = 4x^3 - 12x^2 + 2x - 5; (x - 3)$

3. $(x-4);\ P(x) = x^2 + 8x - 48$

4. $(x+5);\ P(x) = 2x^2 - 6x - 1$

The given binomial is a factor of the polynomial, find the other factors using synthetic division and the zeros

5. $x^3 + 3x^2 - 6x - 8; x - 2$

6. $3x^3 - 4x^2 - 17x + 6; x + 2$

Divide and Conquer (from DOE) - Learning

9.04 SYNTHETIC DIVISION AND SYNTHETIC SUBSTITUTION

The labor of dividing a polynomial by $x - t$ can be reduced considerably by eliminating the symbols that occur repetitiously in the procedure. Let us consider the following division:

$$\begin{array}{r|l}
 & 4x^3 + 5x^2 + 3x + 2 \\
\hline
x - 2 & 4x^4 - 3x^3 - 7x^2 - 4x - 9 \\
 & \underline{4x^4 - 8x^3} \\
 & \quad 5x^3 - 7x^2 \\
 & \quad \underline{5x^3 - 10x^2} \\
 & \qquad 3x^2 - 4x \\
 & \qquad \underline{3x^2 - 6x} \\
 & \qquad\quad 2x - 9 \\
 & \qquad\quad \underline{2x - 4} \\
 & \qquad\qquad -5
\end{array}$$

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We may streamline this division, as follows, leaving out the various powers of x but maintaining the coefficients in their proper places.

$$\begin{array}{r|rrrrr}
 & 4 & 5 & 3 & 2 & \\
\hline
-2 & 4 & -3 & -7 & -4 & -9 \\
 & & -8 & & & \\
 & & 5 & & & \\
 & & & -10 & & \\
 & & & 3 & & \\
 & & & & -6 & \\
 & & & & 2 & \\
 & & & & & -4 \\
 & & & & & -5
\end{array}$$

The above arrangement may be "collapsed" to give the following:

$$\begin{array}{r|rrrrr}
-2 & 4 & -3 & -7 & -4 & -9 \\
 & & -8 & -10 & -6 & -4 \\
\hline
 & 4 & 5 & 3 & 2 & -5
\end{array}$$

Note that:

$$\begin{aligned} -8 &= 4(-2) \\ -10 &= 5(-2) \\ -6 &= 3(-2) \\ -4 &= 2(-2) \end{aligned}$$

Since it is generally easier to add than to subtract, we shall replace -2 by 2 and add, rather than subtract, in each column beginning with the second from the left. Hence we have the final streamlined division known as *synthetic division*:

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & -7 & -4 & -9 \\ & & 8 & 10 & 6 & 4 \\ \hline & 4 & 5 & 3 & 2 & -5 \end{array}$$

There are several points to be noted in connection with this procedure:

(1) The number in the upper left-hand corner is "t", if we are dividing by $x - t$.

(2) The top row consists of coefficients of terms of the dividend polynomial in order of descending degree. Any missing term in the sequence must be indicated by a zero coefficient. For example, we shall treat $5x^4 + 3x$ as $5x^4 + 0x^3 + 0x^2 + 3x + 0$.

(3) The left-hand coefficient in the top row is merely "brought down" to the third row.

(4) The procedure is then one of "multiply by t and add."

(5) The third row, except for the right-hand number, consists of the coefficients of powers of x in the quotient polynomial, in order of descending degree.

(6) The right-hand number in the third row is the remainder, when the divisor is $x - t$, which, by the Remainder Theorem, also represents the value of the dividend polynomial at $x = t$.

(7) In view of the Remainder Theorem the process is known equally well as *synthetic substitution*.

Real Mathematics:
Application

. Your teacher will now guide you through several of these to practice synthetic division.

a. $x-2\overline{)x^3+2x^2-6x-9}$

b. $x+3\overline{)x^3+2x^2-6x-9}$

c. $(x^4-16x^2+x+4)\div(x+4)$

d. $(4x^4-3x^3-7x^2-4x-9)\div(x-2)$

8. One way to evaluate polynomial functions is to use direct substitution. For instance, $f(x)=2x^4-8x^2+5x-7$ can be evaluated when $x=3$ as follows:
$f(3)=2(3)^4-8(3)^2+5(3)-7=162-72+15-7=98$.

However, there is another way to evaluate a polynomial function called synthetic substitution. Since the Remainder Theorem states that the remainder of a polynomial f(x) divided by a linear divisor (x – c) is equal to f(c), the value of the last number on the right corner should give an equivalent result. Let's see.

3	2	0	-8	5	-7
		6	18	30	105
	2	6	10	35	98

We can see that the remainder is equivalent to the solution to this problem! Use synthetic substitution to evaluate the following. You can confirm your results with direct substitution using a calculator.

a. $f(x)=2x^4+x^3-3x^2+5x-8, x=-1$

b. $f(x)=-3x^3+7x^2-4x+8, x=3$

c. $f(x)=3x^5-2x^2+x, x=2$

d. $f(x)=-x^4+8x^3+13x-4, x=-2$

Zeros – Application (scaffolded)

A. Given that the height of a rectangular prism is $x + 2$ and the volume is $x^3 - x^2 - 6x$, write an expression that represents the area of the base of the prism.

B. The volume of a rectangular prism whose dimensions are binomials with integer coefficients is modeled by the function $V(x) = x^3 - 8x^2 + 19x - 12$.
Given that $x - 1$ and $x - 3$ are two of the dimensions, find the missing dimension of the prism.

C. **Physics** A Van de Graaff generator is a machine that produces very high voltages by using small, safe levels of electric current. One machine has a current that can be modeled by $I(t) = t + 2$, where $t > 0$ represents time in seconds. The power of the system can be modeled by $P(t) = 0.5t^3 + 6t^2 + 10t$. Write an expression that represents the voltage of the system. Recall that $V = \frac{P}{I}$.

A.

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Divide by using synthetic division.

1. $(3x^2 - 8x + 4) \div (x - 2)$

2. $(5x^2 - 4x + 12) \div (x + 3)$

3. $(9x^2 - 7x + 3) \div (x - 1)$

4. $(-6x^2 + 5x - 10) \div (x + 7)$

5.

Explain the Error Two students used synthetic division to divide $3x^3 - 2x - 8$ by $x - 2$. Determine which solution is correct. Find the error in the other solution.

A.	B.
$\begin{array}{r\|rrrr} 2 & 3 & 0 & -2 & -8 \\ & & 6 & 12 & 20 \\ \hline & 3 & 6 & 10 & 12 \end{array}$	$\begin{array}{r\|rrrr} -2 & 3 & 0 & -2 & -8 \\ & & -6 & 12 & -20 \\ \hline & 3 & -6 & 10 & -28 \end{array}$

Name ______________________________________ Period_______ Date_____________

The given binomial is a factor of the polynomial, find the other factors using synthetic division and determine the zeros

1. $x^3 + 7x^2 + 7x - 15; x - 1$

2. $x^3 - x^2 - 14x + 24; x + 4$

3. $4x^3 - 12x^2 - x + 3; x - 3$

4. $x^5 + x^4 - 5x^3 - 5x^2 + 4x + 4; x + 1$

Lesson Name: A2.U3.C1.**H.01.Vocab.RationalRootTheorem**

Term	Definition	Notation	Diagram/Visual
Constant Term			
Leading Coefficient			
Rational Root Theorem			

RATIONAL ROOT THEOREM

GUIDED NOTES

If the polynomial has integer coefficients, then every rational zero has the following form $\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$

For each of the following polynomial functions, list all the possible values of p, all the possible values of q, and all the possible rational zeros $\frac{p}{q}$

1. $f(x) = x^3 - 2x^2 - 11x + 12$

   Find all the possible values of p: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

   Find all the possible values of q: ± 1

   Find all the possible values of $\frac{p}{q}$: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

The possible values of $\frac{p}{q}$ gives you a starting place to find the zeros of a function when a root is not given. Again you can use synthetic division to quickly test the possibilities to find all rational zeros. You may not be successful with the first guess but once you get one, you have a great starting place to find the rest.

Test

```
1 | 1   -2   -11    12
  |      1    -1   -12
  ---------------------
    1   -1   -12     0
```

$(x^2 - x - 12)$

1 is a zero because the remainder is zero, so the factor is $(x - 1)$.

$f(x) = x^3 - 2x^2 - 11x + 12 = (x-1)(x^2 - x - 12)$

Factor completely $= (x-1)(x+3)(x-4)$

Zero Factor Theorem $0 = (x-1)(x+3)(x-4)$

$0 = (x-1)$ $0 = (x+3)$ $0 = (x-4)$

The zeros of the function are 1, -3, and 4.

2. $f(x) = 3x^3 - 17x^2 + 18x + 8$

   Find all the possible values of p: $\pm 1, \pm 2, \pm 4, \pm 8$

   Find all the possible values of q: $\pm 1, \pm 3$

   Find all the possible values of $\frac{p}{q}$: $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$

Test

```
-1/3 | 3   -17   18    8
     |      -1    6   -2
     --------------------
       3   -18   24    0
```

$(3x^2 - 18x + 24)$

$-\frac{1}{3}$ is a zero because the remainder is zero, so the factor is $\left(x + \frac{1}{3}\right)$

$f(x) = 3x^3 - 17x^2 + 18x + 8 = \left(x + \frac{1}{3}\right)(3x^2 - 18x + 24)$

$= \left(x + \frac{1}{3}\right)(3)(x^2 - 6x + 8)$

$= (3x+1)(x-2)(x-4)$

$0 = (3x+1)$ $0 = (x-2)$ $0 = (x-4)$

The zeros of the function are $-\frac{1}{3}, 2, 4$.

Example!

1. Find all real zeros of $f(x) = x^3 - 4x^2 - 7x + 10$

List the possible zeros. The leading coefficient is ____ and the constant term is _____. So, the possible rational zeros are: x= ____, _____, _____, ______.

Test these zeros using synthetic division.

Write the factored form of the polynomial ____________________________________
Zero Factor Theorem ______________ ______________ _________________
The zeros are _________, _________, _______.

2. Find all real zeros of $f(x) = 9x^4 + 12x^3 - 26x^2 - 11x + 6$

*Remember: Quadratic Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

List the possible zeros.

The leading coefficient is ____ and the constant term is _____.
So, the possible rational zeros are: x= ____

Test these zeros using synthetic division.

Find the remaining zeros by using quadratic formula

The real zeros are ____

3. You are building a wooden square sandbox for a local playground. You want the volume of the box to be 16 cubic feet. You want the height of the box to be x feet and the length of each side of the square base to be x + 3. What are the dimensions?

   Write an equation and substitute (V=Bh)

   List the possible rational solutions:

   Test the possible rational solutions. Only positive x-values make sense RIGHT!
   Check for other possible solutions.

   Summarize in context. The value of x is _____. The height of the sandbox should be ___ ft and each side of the base should be _____.

SELF CHECK **Find the rational zeros of the polynomial**

1. $f(x) = x^3 + 3x^2 - 10x - 24$

2. $f(x) = 8x^4 + 2x^3 - 21x^2 - 7x + 3$

Can you find the rational zeros of the polynomial $f(x) = 0.4x^2 - 3x + 2.3$? Explain why or why not?

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List the possible rational zeros of the function using the rational root theorem.

1. $f(x) = x^3 - 5x + 16$

2. $g(x) = x^5 + 2x^4 - 3x - 24$

3. $h(x) = 2x^3 - 5x^2 - 9$

Find all real zeros of the function.

4. $f(x) = x^3 - 8x^2 - 23x + 30$

5. $g(x) = x^3 - 7x^2 + 2x + 40$

6. $h(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$

7. $f(x) = x^4 + 3x^3 - 21x^2 - 43x + 60$

8. $j(x) = x^3 + x^2 - 2x - 2$

Factors, Zeros, and Roots: Oh My! (DOE) - Application

Factors, Zeros, and Roots: Oh My!

Solving polynomials that have a degree greater than those solved in previous courses is going to require the use of skills that were developed when we previously solved quadratics. Let's begin by taking a look at some second degree polynomials and the strategies used to solve them. These equations have the form $ax^2+bx+c=0$, and when they are graphed the result is a parabola.

1. Factoring is used to solve quadratics of the form $ax^2+bx+c=0$ when the roots are rational. Find the roots of the following quadratic functions:

   a. $f(x)=x^2-5x-14$

   b. $f(x)=x^2-64$

   c. $f(x)=6x^2+7x-3$

   d. $f(x)=3x^2+x-2$

2. Another option for solving a quadratic whether it is factorable but particularly when it is not is to use the quadratic formula. Remember, a quadratic equation written in $ax^2+bx+c=0$ has solution(s) $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Also remember that b^2-4ac is the discriminant and gives us the ability to determine the nature of the roots.

$$b^2-4ac\begin{cases} >0 & 2 \quad real\,roots \\ =0 & 1 \quad real\ root \\ <0 & 0 \quad real\,roots(imaginary) \end{cases}$$

Find the roots for each of the following. Also, describe the number and nature of these roots.

   a. $f(x)=4x^2-2x+9$

   b. $f(x)=3x^2+4x-8$

   c. $f(x)=x^2-5x+9$

3. Let's take a look at the situation of a polynomial that is one degree greater. When the polynomial is a third degree, will there be any similarities when we solve?

Suppose we want to find the roots of $f(x) = x^3 + 2x^2 - 5x - 6$. By inspecting the graph of the function, we can see that one of the roots is distinctively 2. Since we know that x = 2 is a solution to $f(x)$, we also know that $x - 2$ is a factor of the expression $x^3 + 2x^2 - 5x - 6$. This means that if we divide $x^3 + 2x^2 - 5x - 6$ by $x - 2$ there will be a remainder of zero. Let's confirm this with synthetic substitution:

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

Let's practice synthetic division before we tackle how to solve cubic polynomials in general. Do the following division problems synthetically.

a. $\dfrac{10x^3 - 17x^2 - 7x + 2}{x - 2}$

b. $\dfrac{x^3 + 3x^2 - 10x - 24}{x + 4}$

c. $\dfrac{x^3 - 7x - 6}{x + 1}$

The main thing to notice about solving cubic polynomials (or higher degree polynomials) is that a polynomial that is divisible by $x-k$ has a root at k. Synthetic division applied to a polynomial and a factor result in a zero for the remainder. This leads us to the Factor Theorem, which states: A polynomial $f(x)$ has a factor $x-k$ if and only if $f(k)=0$.

Solving cubic polynomials can be tricky business sometimes. A graphing utility can be a helpful tool to identify some roots, but in general there is no simple formula for solving cubic polynomials like the quadratic formula aids us in solving quadratics.

There is however a tool that we can use for helping us to identify Rational Roots of the polynomial in question.

4. The Rational Root Theorem states that any rational solutions to a polynomial will be in the form of $\frac{p}{q}$ where p is a factor of the constant term of the polynomial (the term that does not show a variable) and q is a

factor of the leading coefficient. This is actually much simpler than it appears at first glance.

a. Let us consider the polynomial $f(x)=x^3-5x^2-4x+20$

Identify p (all the factors of the constant term 20) = ______________________________

Identify q (all the factors of the leading coefficient 1) = ___________________________

Identify all possible combinations of $\frac{p}{q}$: ______________________________________

If $f(x)=x^3-5x^2-4x+20$ is going to factor, then one of these combinations is going to "work", that is, the polynomial will divide evenly. So the best thing to do is employ a little trial and error. Let's start with the smaller numbers, they will be easier to evaluate.
Substitute for x: 1, -1, 2, -2, 4, -4 …20, -20.

Why would substituting these values in for x be a useful strategy?

Why do we not have to use synthetic division on every one?

Define what the Remainder Theorem states and how it helps us.

Hopefully, you did not get all the way to -20 before you found one that works. Actually, 2 should have worked. Once there is one value that works, we can go from there.
Use the factor $(x-2)$ to divide $f(x)$. This should yield:

$f(x)=x^3-5x^2-4x+20=(x-2)(x^2-3x-10)$

By factoring the result we can find all the factors: $f(x)=x^3-5x^2-4x+20=(x-2)(x+2)(x-5)$

Therefore the roots are 2, -2, and 5.

What could be done if this portion was not factorable?

5. Use the Quadratic Formula

For each of the following find each of the roots, classify them and show the factors.

a. $f(x)=x^3-5x^2-4x+20$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ___________________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

b. $f(x) = x^3 + 2x^2 - 5x - 6$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ______________________________________

Roots and Classification

_________ Rational Irrational Real Imaginary
_________ Rational Irrational Real Imaginary
_________ Rational Irrational Real Imaginary

c. $f(x) = 4x^3 - 7x + 3$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ______________________________________

Roots and Classification

_________ Rational Irrational Real Imaginary
_________ Rational Irrational Real Imaginary
_________ Rational Irrational Real Imaginary

6. What happens when we come to a function that is a 4th degree?

Well, just like the cubic there is no formula to do the job for us, but by extending our strategies that we used on the third degree polynomials, we can tackle any quartic function.

1st Develop your possible roots using the $\frac{p}{q}$ method.

2nd Use synthetic division with your possible roots to find an actual root. If you started with a 4th degree, that makes the dividend a cubic polynomial.

3rd Continue the synthetic division trial process with the resulting cubic. Don't forget that roots can be used more than once.

4th Once you get to a quadratic, use factoring techniques or the quadratic formula to get to the other two roots.

For each of the following find each of the roots, classify them and show the factors.

a. $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ____________________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

b. $f(x) = x^4 - 11x^3 - 13x^2 + 11x + 12$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula(or Factoring):

Complete Factorization: ______________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

c. $f(x) = x^5 - 12x^4 + 49x^3 - 90x^2 + 76x - 24$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ______________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

d. $f(x) = x^5 - 5x^4 + 8x^3 - 8x^2 + 16x - 16$

Possible rational roots:

Show work for Synthetic Division and Quadratic Formula (or Factoring):

Complete Factorization: ________________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

7. Let's consider a scenario where the roots are imaginary.

Suppose that you were asked to find the roots of $f(x) = x^4 - x^3 + 3x^2 - 4x - 4$.

There are only 6 possible roots: $\pm 1, \pm 2, \pm 4$. In the light of this fact, let's take a look at the graph of this function.

It should be apparent that none of these possible solutions are roots of the function. And without a little help at this point we are absolutely stuck. None of the strategies we have discussed so far help us at this point.

a. But consider that we are given that one of the roots of the function is $2i$. Because roots come in pairs (think for a minute about the quadratic formula); an additional root should be $-2i$. So, let's take these values and use them for synthetic division.

b. Though the values may not be very clean, this process should work just as it did earlier. Take a moment and apply what you have been doing to this function.

Complete Factorization: ________________________________

Roots and Classification

_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary
_________	Rational	Irrational	Real	Imaginary

Crafts - Application of Rational Roots

You have 18 cubic inches of wax, and you want to make a candle in the shape of a pyramid with a square base. The height of the pyramid will be $(x + 3)$ and the length of each side of the base is x.

A). Write an equation that shows that the volume of the candle is 18 cubic inches. $V=\frac{1}{3}Bh$

B) Use the rational root theorem to list all the possible rational solutions of the equation from part A.

C) Find all real values of x that are valid as a dimension of the candle.

D) Find the dimensions of the candle.

Name___________________________________ Period_____ Date ________________

List the possible rational zeros of the function using the rational root theorem.

1. $f(x) = x^4 + 8x^2 - 18$

2. $g(x) = x^8 - 2x^5 + x^4 - 3x + 20$

3. $h(x) = 3x^3 + 7x + 12$

Find all real zeros of the function.

4. $f(x) = x^3 + 2x^2 - 11x - 12$

5. $g(x) = x^3 + 9x^2 - 4x - 36$

6. $h(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

7. $f(x) = x^4 + x^3 - 11x^2 - 9x + 18$

8. $j(x) = 2x^4 + 3x^3 - 6x^2 - 6x + 4$

Inequalities Notation and Interval Notation can be used to describe subsets of real numbers when identifying things such as domain, range, intervals of increase and decrease.

Term	Inequality Notation	Interval Notation	Diagram/Visual
Bounded Intervals	$a \leq x \leq b$		
	$a < x < b$		
	$a \leq x < b$		
	$a < x \leq b$		
Unbounded Intervals	$x > a$		
	$x \geq a$		
	$x < b$		
	$x \leq b$		

Represent the inequality using interval notation.

1. $x > 8$
2. $x \leq 0$
3. $4 < x \leq 6$

Represent the interval using inequity notation.

4. $[-2, 3]$
5. $(-1, 1)$
6. $[5,\ \infty)$

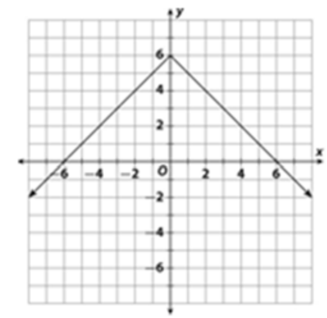
Characteristics Review

- **X intercept**/ zeros/ solutions (where the graph crosses the x-axis) Factor of $(x-5)$ has an x intercept at (5,0). Remember, multiplicity is when a factor is repeated. When a factor (x-k) is raised to an odd power, the graph crosses over the x-axis at x = k. When a factor (x-k) is raised to an even power, the graph is tangent (bounces) to the x-axis at x = k.
- **Domain**: set of all possible "x" values
- **Range**: set of all possible "y" or *f(x)* values
- **Average Rate of Change** = slope on a particular interval, varies from point to point

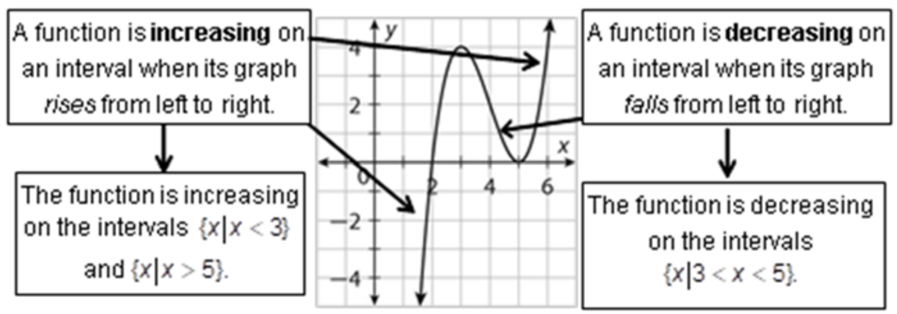
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- **Intervals of Increase and Decrease**: Increasing Uphill (from left–right) Interval: $(-\infty, 0)$

  Decreasing Downhill (from left–right) Interval: $(0, +\infty)$



A function may change from increasing to decreasing or from decreasing to increasing at turning points.



NOTE: Interval of Increase and decrease can be written in interval notation: Increases $(-\infty, 3)U(5, \infty)$
Decreases $(3, 5)$

- Identify the leading coefficient, degree, and end behavior of $f(x) = -x(x+4)(x-1)$.

x^3 is the term of greatest degree.

-1 is the **leading coefficient**. $-1 < 0$ 3 is the **degree**. 3 is **odd**.

End behavior: As $x \to -\infty$, $f(x) \to +\infty$ and as $x \to +\infty$, $f(x) \to -\infty$.

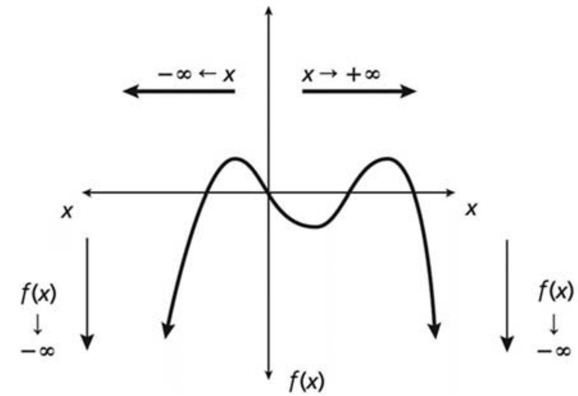
The key is to Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

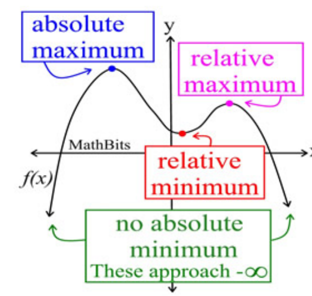
✓ The **end behavior** of a function tells you how the function behaves as its x-value approaches positive or negative infinity. You can find the end behavior of a function by looking at its degree and its leading coefficient.

The degree is the sum of the exponents of the polynomial factors.(rephrase)

To sketch $f(x)=a(x-x_1)(x-x_2)...(x-x_n)$:

n = degree a = constant factor	End Behavior	Graph Description	x-intercepts
n odd $a>0$	as $x\to-\infty$, $f(x)\to-\infty$ as $x\to+\infty$, $f(x)\to+\infty$	Uphill	$(x-x_1)^{odd}$ Crosses x-axis at x_1
n odd $a<0$	as $x\to-\infty$, $f(x)\to+\infty$ as $x\to+\infty$, $f(x)\to-\infty$	Downhill	
n even $a>0$	as $x\to-\infty$, $f(x)\to+\infty$ as $x\to+\infty$, $f(x)\to+\infty$	Opens up	$(x-x_2)^{even}$ Tangent to x-axis at x_2
n even $a<0$	as $x\to-\infty$, $f(x)\to-\infty$ as $x\to+\infty$, $f(x)\to-\infty$	Opens down	

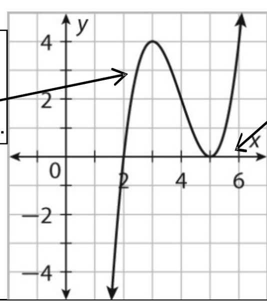




✓ Since the graph of $f(x)$ intersects the x-axis only at its x-intercepts, the graph must move away from and then move back toward the x-axis between each pair of successive x-intercepts, which means that the graph has a ***turning point*** between those x-intercepts. You may have to use a t-chart to find other points in between the x-intercepts to determine if the graph is positive or negative (either side of x axis). Also, instead of crossing the x-axis at an x-intercept, the graph can be tangent to the x-axis, and the point of tangency becomes a turning point because the graph must move toward the x-axis and then away from it near the point of tangency.

A **relative maximum** occurs where the function changes from increasing to decreasing.

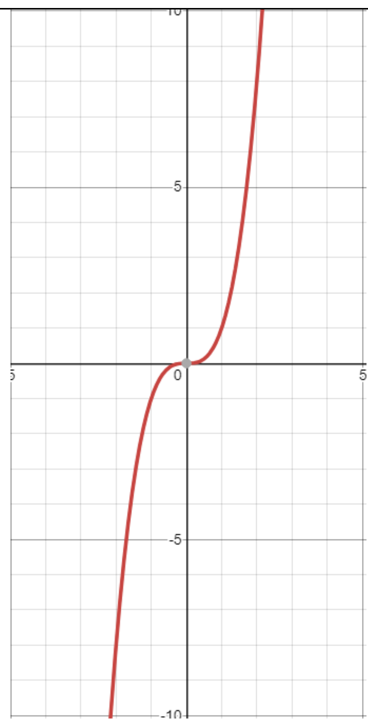
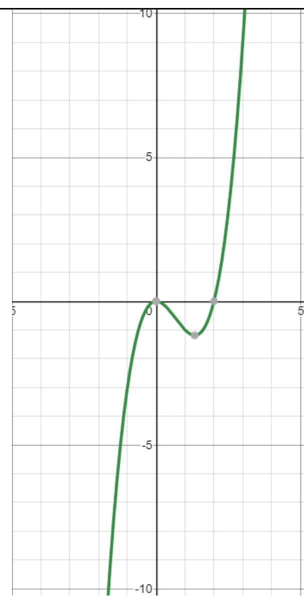
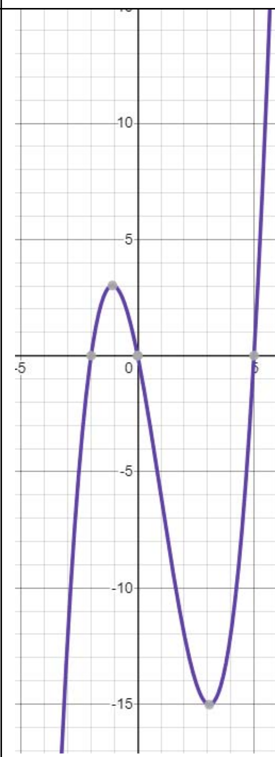
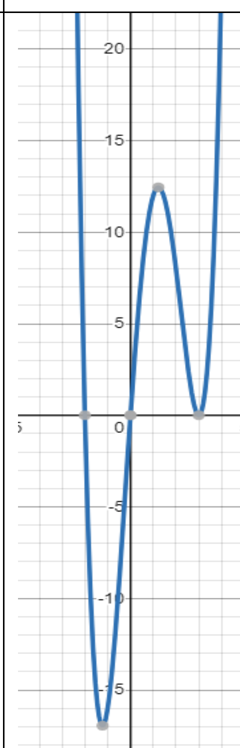
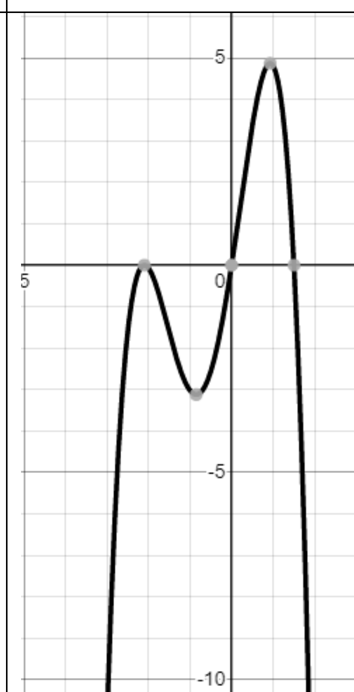
A local maximum occurs at (3, 4).



A **relative minimum** occurs where the function changes from decreasing to increasing.

A local minimum occurs at (5, 0).

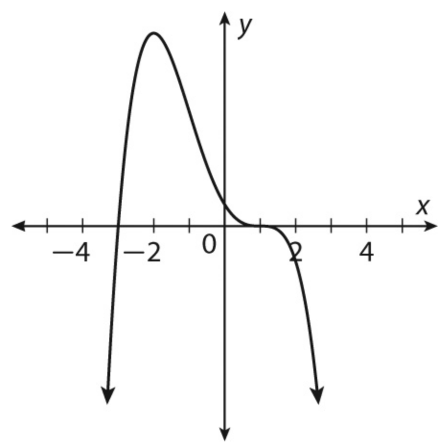
✓ The y-coordinate of each turning point is a maximum or minimum value of the function at least near that turning point. A maximum or minimum value is called global or **absolute** if the function never takes on a value that is greater than the maximum or less than the minimum. A local maximum or local minimum, also called a **relative maximum** or **relative minimum**, is a maximum or minimum within some interval around the turning point that need not be (but may be) a global maximum or global minimum.

FUNCTION	$f(x) = x^3$	$f(x) = x^2(x-2)$	$f(x) = x(x-2)(x+2)$	$f(x) = x(x-3)^2(x+2)$	$f(x) = -x(x+2.1)^2(x-1.5)$
How many distinct factors does *f(x)* have?					
Graph					

What are the graph's x-ints?					
Is the graph tangent to the x-axis or does it cross the x-axis at each x-int?					
How many turning points does the graph have?					
Relative Max? If so where?					
Relative Min? If so where?					
Absolute Max? If so where?					
Absolute Min? If so where?					

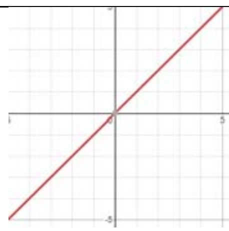
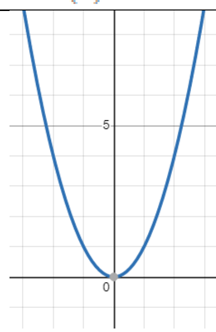
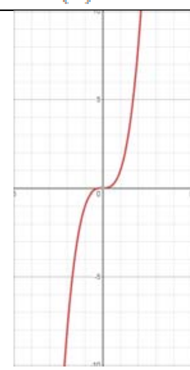
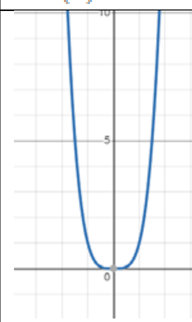
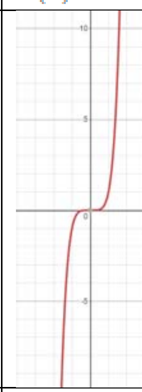
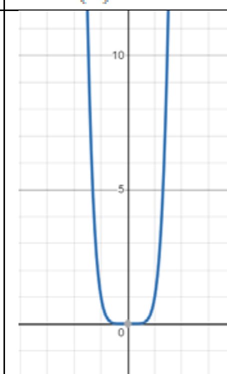
Example!

Example 1 Sketch the graph of the polynomial function $f(x)=\left(-\frac{1}{5}\right)(x+3)(x-1)^3$.

$n=4$ (even), $a=-\frac{1}{5}$ $(a<0) \rightarrow$ Opens down	
$(x+3)$ raised to an odd power $\rightarrow$ crosses at $x=-3$	
$(x-1)$ raised to an odd power $\rightarrow$ crosses at $x=1$	

What are the other characteristics you can tell by the graph? What are some things I can't tell by the graph or equation?

Lesson Name: A2.U3.C2.**I.02.Notes.GraphPolyFromFactored**

FUNCTION	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = x^4$	$f(x) = x^5$	$f(x) = x^6$	Make a Prediction $f(x) = x^{84}$	Make a Prediction $f(x) = x^{99}$
Basic Graph								
X and Y intercept								
Domain								
Range								
End Behavior								
Interval of Increase								

Interval of Decrease								
Average Rate of Change from 1 to 2								

Questions To Ponder

What patterns do you notice?

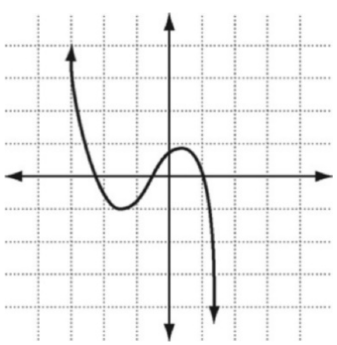
FUNCTION	$f(x) = -x$	$f(x) = -x^2$	$f(x) = -x^3$	$f(x) = -x^4$	$f(x) = -x^5$	$f(x) = -x^6$	Make a Prediction $f(x) = -x^{84}$	$f(x) = -x^{99}$
Basic Graph								
X and Y intercept								
Domain								
Range								
End Behavior								
Interval of Increase								

Interval of Decrease								
Average Rate of Change from 1 to 2								

Questions To Ponder What patterns do you notice?

SELF CHECK

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

1. 

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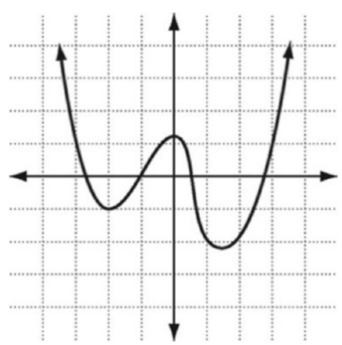
2. 

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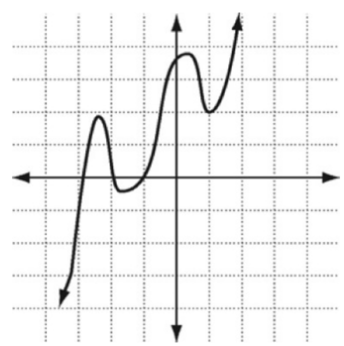
3. 

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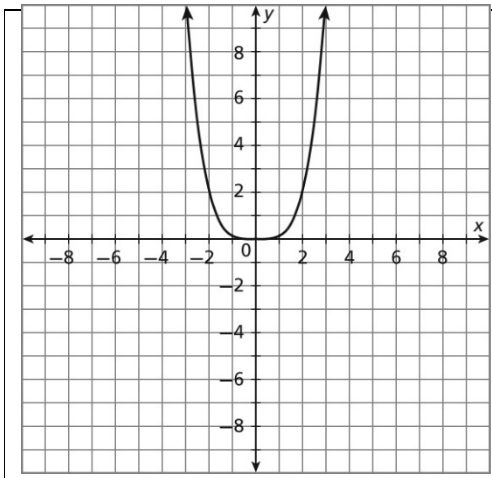
4. Fill in the blanks for the attributes of the functions shown in the graph below.



$f(x)$ has a domain of ____________

$f(x)$ has a range of ____________

$f(x)$ has a zero at $x =$ ____________

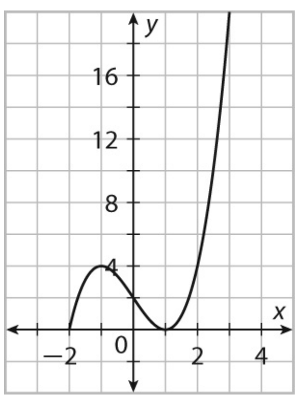
$f(x)$ is increasing on the interval ____________

$f(x)$ is decreasing on the interval ____________

$f(x)$ has a absolute/relative minimum of _____ at $x =$ ______

The average rate of change on the interval 2 to 3 is ____________.

5. Use the graph to answer Problems A-D.



A. On which intervals is the function increasing and decreasing?

____

____

B. What are the local maximum and minimum values?

____

____

C. What are the zeros of the function?

____

D. What is the domain and range?

Lesson Name: A2.U3.C2.**I.02.Notes.GraphPolyFromFactored**

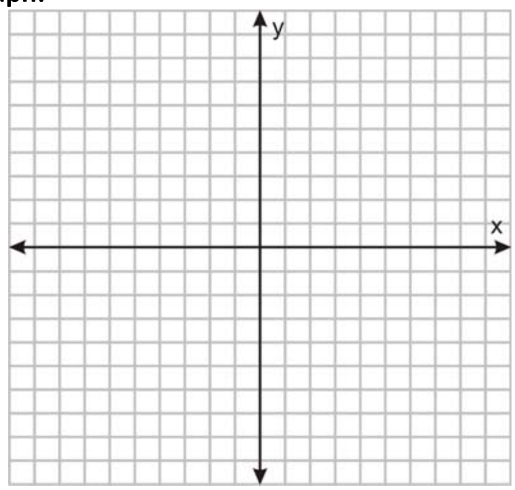
THE BIG IDEA

____

____

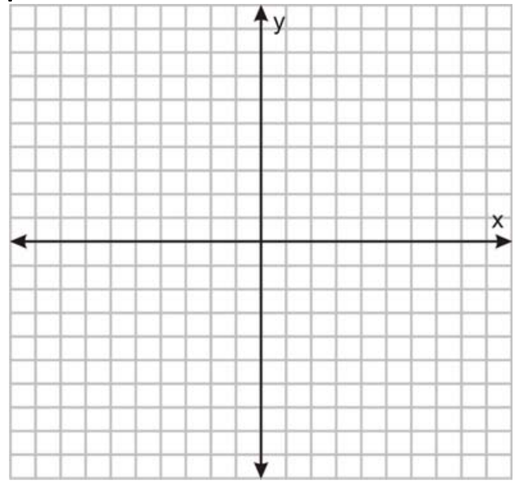
Questions To Ponder

1. Guided Example:

Equation: $f(x) = -2(x+3)^3(x-2)^2$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

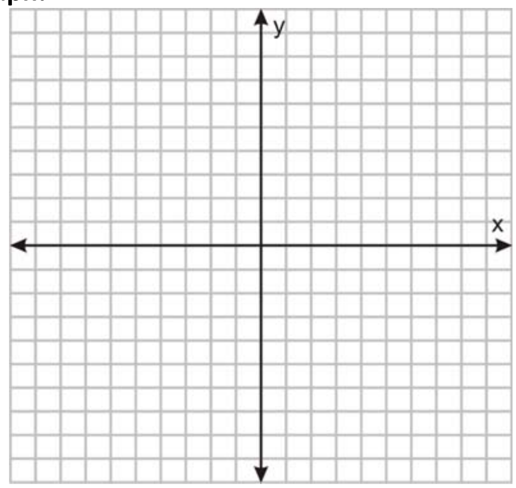
Workspace:

2.

Equation: $\boldsymbol{f(x) =}$ $f(x) = -x^2(x-2)(x+1)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

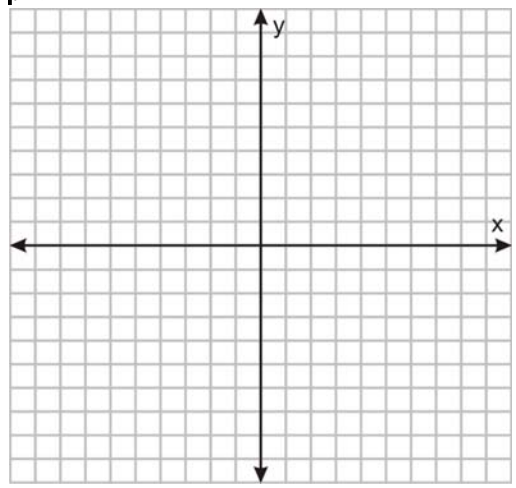
Workspace:

3.

Equation: $\boldsymbol{f(x)=}$ $f(x)=-(x-1)^2(x+3)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

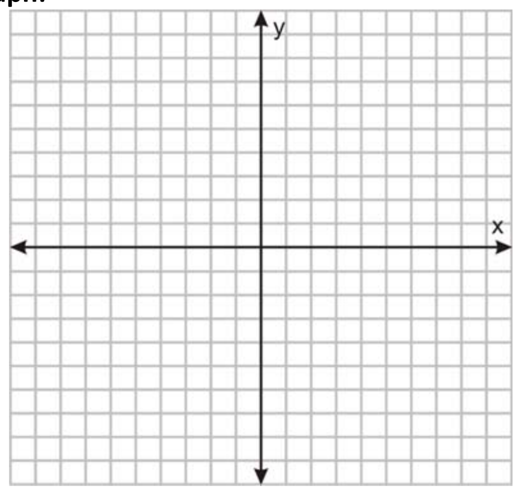
Workspace:

4.

Equation: $f(x) =$ $f(x)=(x+2)(x-3)(x-1)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

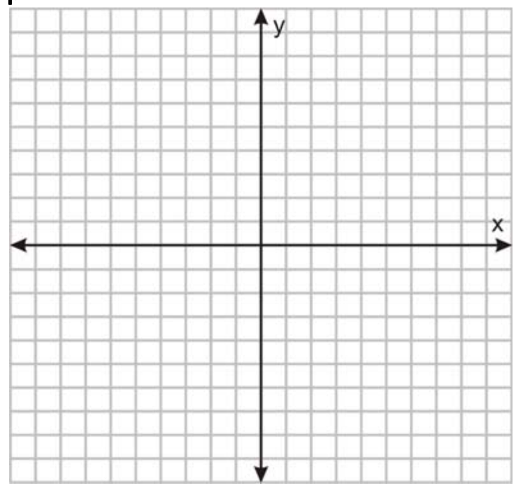
Workspace:

5.

Equation: $f(x) = x(x+2)(x+1)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

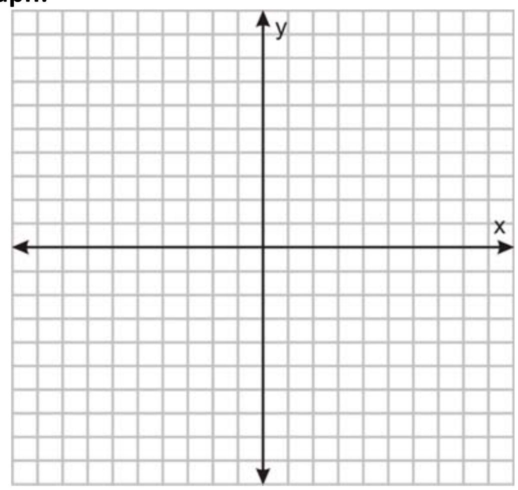
Workspace:

6.

Equation: $f(x) = (x-2)^2(x+2)(x+3)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

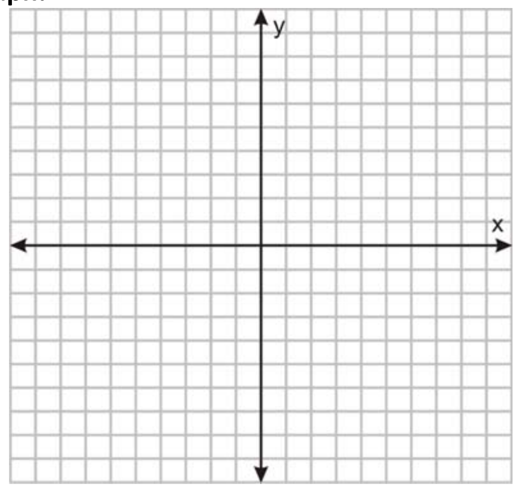
Workspace:

7.

Equation: $f(x) = (x+1)^2(x-2)(x-3)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

8.

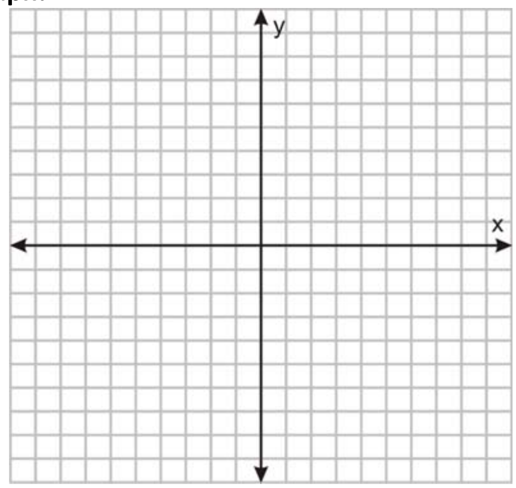
Equation: $f(x) = x(x-4)^2$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

Independent Practice

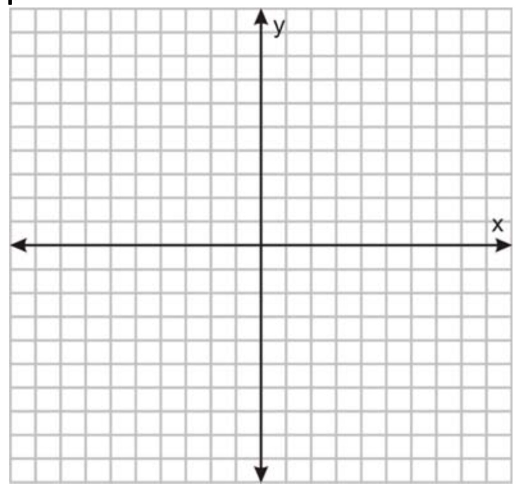
Name____ Period_______ Date____________

1.

Equation: $f(x) = -5x^4 - 10x^3 + 75x^2$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Factored Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

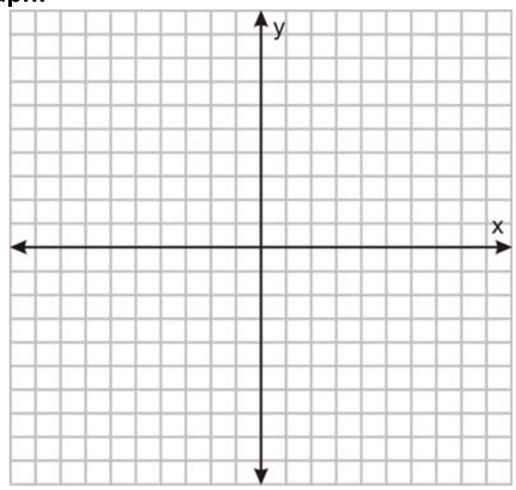
Workspace:

2.

Equation: $f(x) = -x^3 + 3x^2 + 4x$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Factored Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

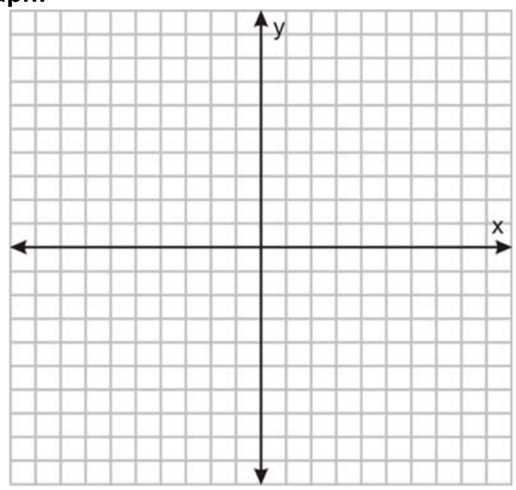
Workspace:

3.

Equation: $f(x) = 2x^3 - 12x^2 - 18x$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Factored Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

4.

Equation: $f(x) = -x^4 - 4x^3 - 4x^2$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Factored Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

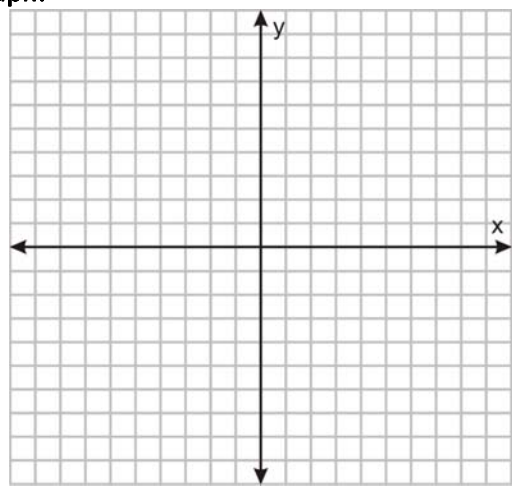
5.

<table>
<tr><td>Equation: $f(x) = -3x^3 + 12x$

Degree:

Number of Turns:

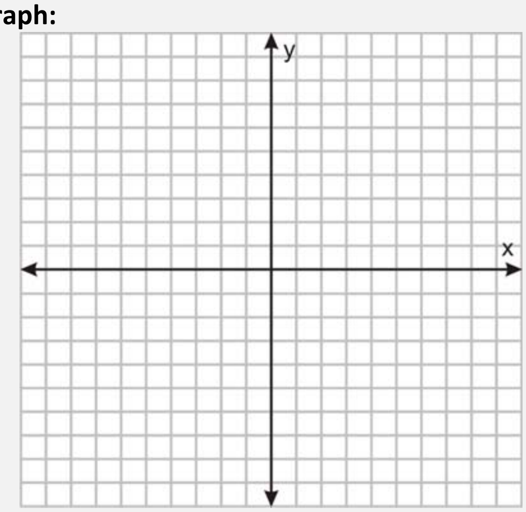
Leading Coefficient:

Factored Form:</td><td>Graph:
</td></tr>
<tr><td>Roots/Zeros/x-intercepts</td><td>y-intercepts</td></tr>
<tr><td>Domain:</td><td>Range:</td></tr>
<tr><td>End Behavior:</td><td>Average Rate of Change on the Interval TBD</td></tr>
<tr><td>Intervals of Increase:</td><td>Intervals of Decrease:</td></tr>
<tr><td>Relative Minimum:</td><td>Relative Maximum:</td></tr>
<tr><td>Absolute Minimum:</td><td>Absolute Maximum:</td></tr>
</table>

Workspace:

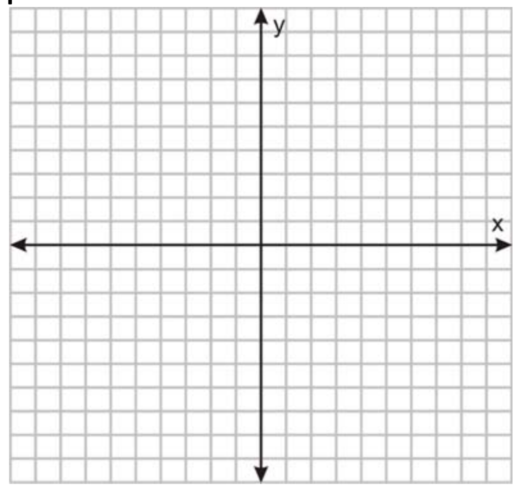
In some cases, it may be easier to graph a polynomial in standard form by looking at the factored form.

Example!

Equation: $f(x) = x^4 - 10x^2 + 9$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Factored Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

SELF CHECK

Equation: $f(x) = x^3 - 16x$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Factored Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

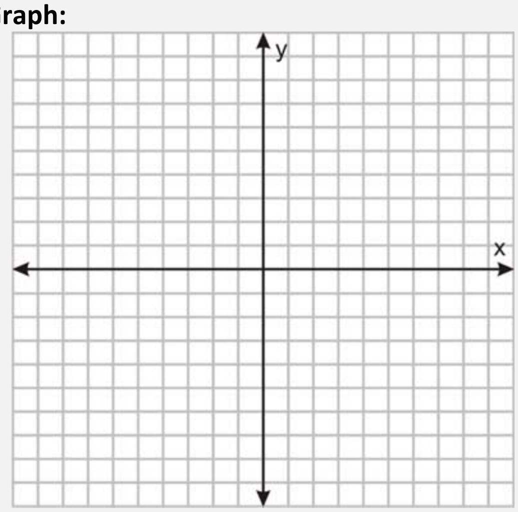
Questions To Ponder

What do I do if I can't factor the polynomial?

GUIDED NOTES

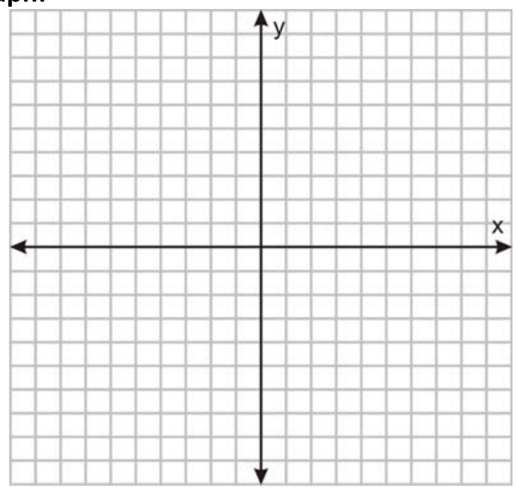
USING GRAPHING CALCULATOR

Example!

Equation: $f(x) = (x-1)(x-2)(x+3)(x+4)^2$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

SELF CHECK

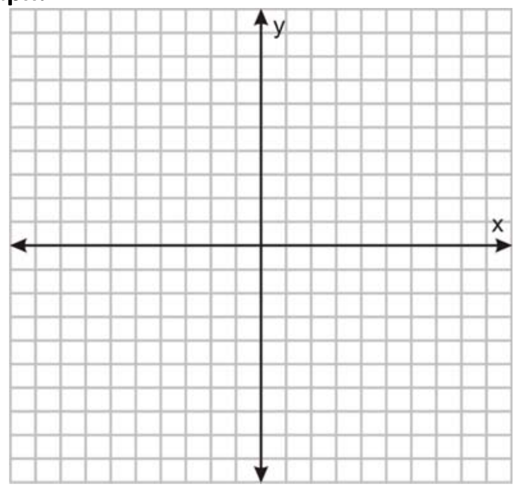
Equation: $f(x) = (x^2 + 1)(x - 2)(x - 3)$ **Degree:** **Number of Turns:** **Leading Coefficient:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

Why is the factor $(x^2 + 1)$ never zero and how does this affect the graph of f?

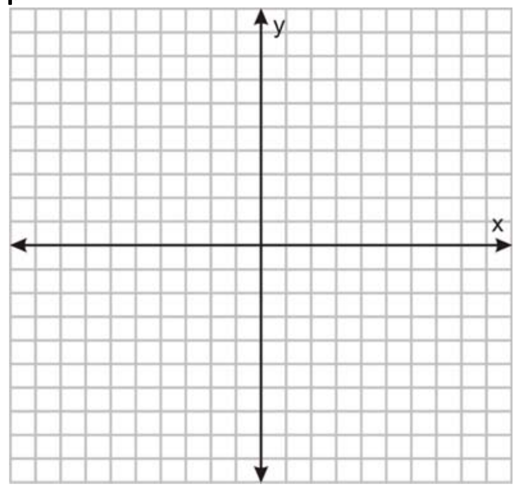
Name ____ Period______ Date_________________

1.

Equation: $f(x) = -(x-1)^3(x+2)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

2.

Equation: $f(x) = x^5(x-3)(x+2)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
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End Behavior:	**Average Rate of Change on the Interval TBD**
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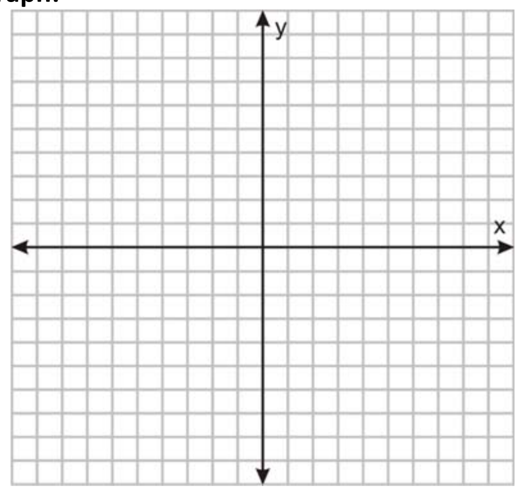
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

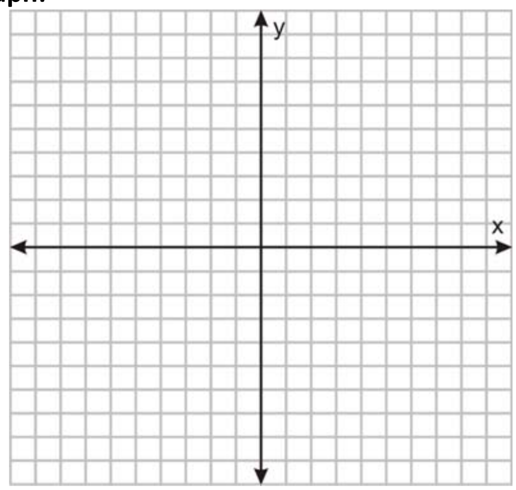
Workspace:

3.

Equation: $f(x) = x(x+3)(x+1)(x-1)(x-3)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

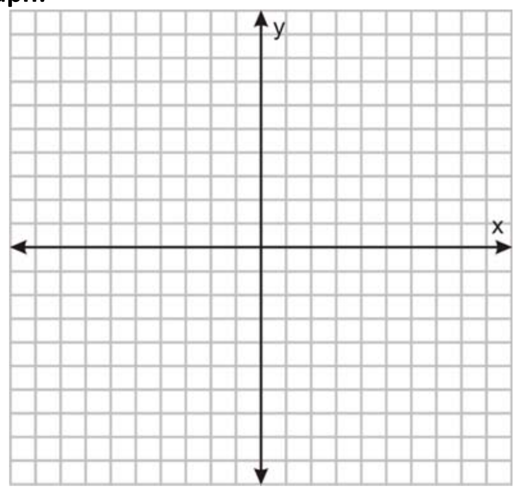
Workspace:

4.

Equation: $f(x) = -(x-1)^3(x+2)^2(x-3)$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

5.

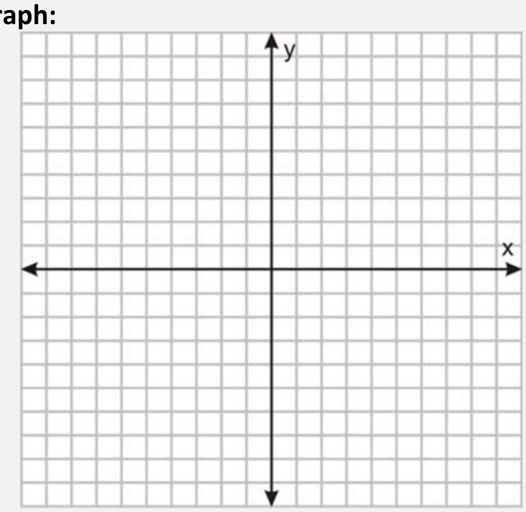
Equation: $f(x) = (x-1)^3(x+4)^2$ **Degree:** **Number of Turns:** **Leading Coefficient:** **Standard Form:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
Relative Minimum:	**Relative Maximum:**
Absolute Minimum:	**Absolute Maximum:**

Workspace:

GUIDED NOTES

USING GRAPHING CALCULATOR

Example!

Equation: $f(x) = x^5 - 4x^3 + 4x - 1$ **Degree:** **Number of Turns:** **Leading Coefficient:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
Domain:	**Range:**
End Behavior:	**Average Rate of Change on the Interval TBD**
Intervals of Increase:	**Intervals of Decrease:**
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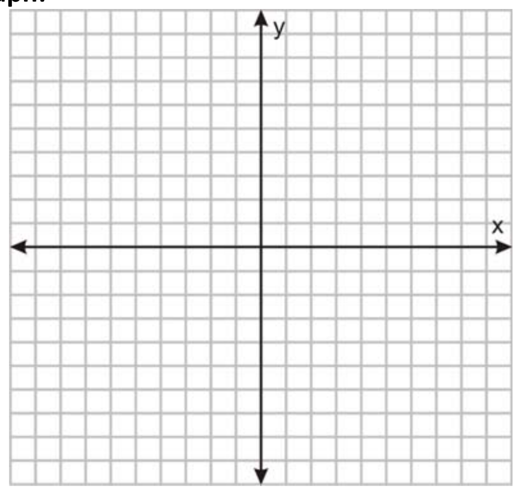
Absolute Minimum:	**Absolute Maximum:**

Workspace:

SELF CHECK

Equation: $f(x) = x^4 - x^2 + x$ **Degree:** **Number of Turns:** **Leading Coefficient:**	**Graph:** 
Roots/Zeros/x-intercepts	**y-intercepts**
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End Behavior:	**Average Rate of Change on the Interval TBD**
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Workspace:

Questions To Ponder

POLYNOMIAL GRAPHS – ROLLER COASTER POLYNOMIALS PART 1

Pre-Calculus TASK

ROLLER COASTER POLYNOMIALS

Name(s): ________________ ________________

________________ ________________

Date: ___________ **Period:** ________

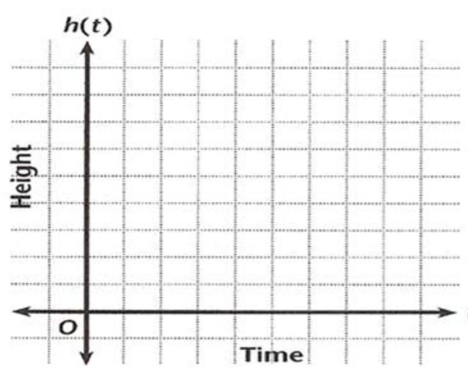
APPLICATION PROBLEMS:

Fred, Elena, Michael, and Diane enjoy roller Coasters. Whenever a new roller Coaster opens near their town, they try to be among the first to ride.

One Saturday, the four friends decide to ride a new coaster. While waiting in line, Fred notices that part of this coaster resembles the graph of a polynomial function that they have been studying in their Pre-Calculus class.

1. The brochure for the coaster says that, for the first 10 seconds of the ride, the height of the coaster can be determined by $h(t) = 0.3t^3 - 5t^2 + 21t$, where t is the time in seconds and h is the height in feet. Classify this polynomial by degree and by number of terms.

2. Graph the polynomial function for the height of the roller coaster on the coordinate plane at the right.



3. Find the height of the coaster at t = 0 seconds.

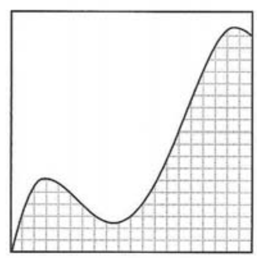
Explain why this answer makes sense.

4. Find the height of the coaster 9 seconds after the ride begins. Explain how you found the answer.

5. Evaluate h(60). Does this answer make sense? Identify practical (valid real life) domain of the ride for this model. CLEARLY EXPLAIN your reasoning. (Hint.: Mt. Everest is 29,028 feet tall.)

6. Next weekend, Fred, Elena, Michael, and Diane visit another roller coaster. Elena snaps a picture of part of the coaster from the park entrance. The diagram at the right represents this part of the coaster.

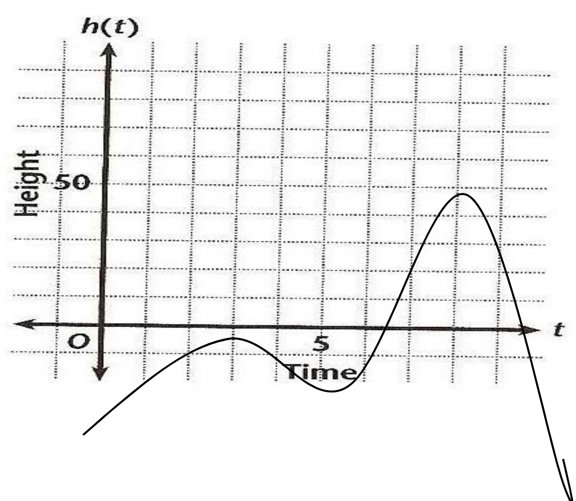
Do you think quadratic, cubic, or quartic function would be the best model for this part of the coaster? Clearly explain your choice.



7. The part of the coaster captured by Elena on film is modeled by the function below.

$$h(t) = -0.2t^4 + 4t^3 - 24t^2 + 48t$$

Graph this polynomial on the grid at the right.



8. Color the graph blue where the polynomial is increasing and red where the polynomial is decreasing.
Identify increasing and decreasing intervals.

9. Use your graphing calculator to approximate relative maxima and minima of this function. Round your answers to three decimal places.

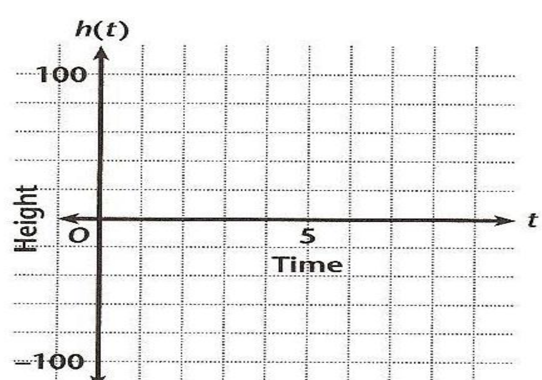
10. Clearly describe the end behavior of this function and the reason for this behavior.

11. Suppose that this coaster is a 2-minute ride. Do you think that $h(t) = -0.2t^4 + 4t^3 - 24t^2 + 48t$ is a good model for the height of the coaster throughout the ride? Clearly explain and justify your response.

12. Elena wants to find the height of the coaster when t = 8 seconds, 9 seconds, 10 seconds, and 11 seconds. Use synthetic division to find the height of the coaster at these times. Show all work.

Diane loves coasters that dip into tunnels during the ride. Her favorite coaster is modeled by $h(t) = -2t^3 + 23t^2 - 59t + 24$. This polynomial models the 8 seconds of the ride after the coaster comes out of a loop.

13. Graph this polynomial on the grid at right.



14. Why do you think this model's practical domain

is only valid from t = 0 to t = 8?

15. At what time(s) is this coaster's height 50 feet? Clearly explain how you found your answer.

Diane wants to find out when the coaster dips below the ground.

16. Use the Rational Zeros Test to identify all possible rational zeros of $h(t) = -2t^3 + 23t^2 - 59t + 24$.

17. Locate all real zeros of this function. Clearly interpret the real-world meaning of these zeros.

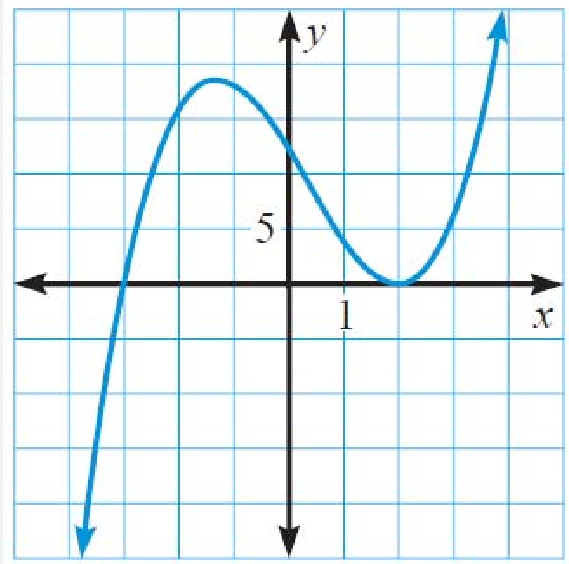
18. Are there any non-real zeros for this polynomial? If so, identify them. Clearly explain your reasoning/ show work.

GUIDED NOTES

FUNDAMENTAL THEOREM OF ALGEBRA

Example!

Given the graph of $f(x) = x^3 - x^2 - 8x + 12$ is shown

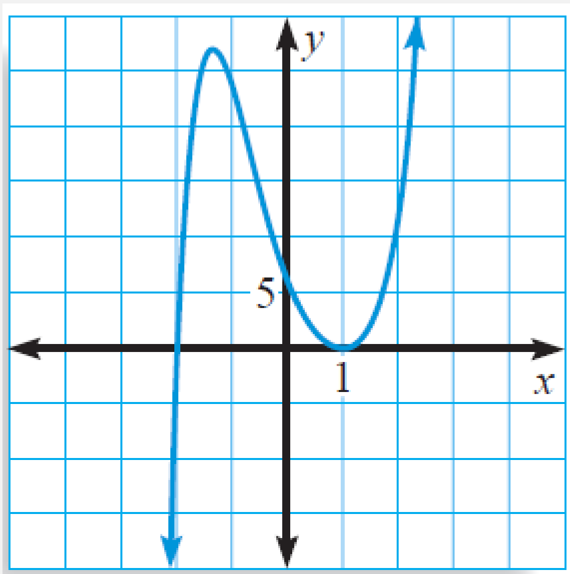


How many real zeros does the function have?

How many imaginary zeros does the function have?

What are the zeros of the function? (any multiplicities?)

Given the graph of $f(x) = x^5 - 2x^4 + 8x^2 - 13x + 6$

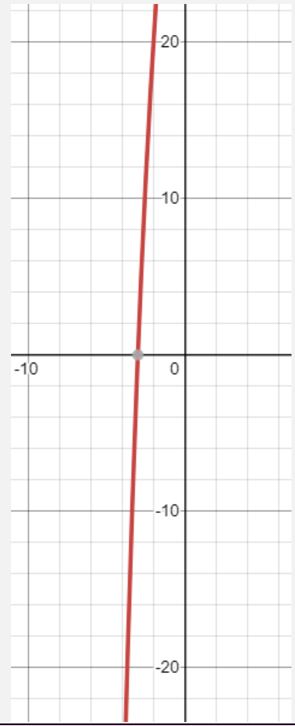


How many real zeros does the function have?

How many imaginary zeros does the function have?

What are the zeros of the function? (any multiplicities?)

Given the graph of $f(x) = x^3 + 3x^2 + 16x + 48$

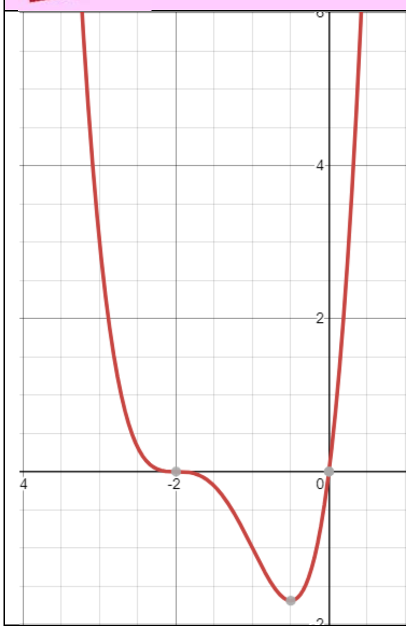


How many real zeros does the function have?

How many imaginary zeros does the function have?

What are the zeros of the function?

SELF CHECK



Given the graph of $f(x) = x^4 + 6x^3 + 12x^2 + 8x$

How many real zeros does the function have?

How many imaginary zeros does the function have?

What are the zeros of the function?

ANSWER KEY

1.

2.

Find all the zeros of $f(x) = x^5 - 2x^4 + 8x^2 - 13x + 6$.

SOLUTION

The possible rational zeros are ± 1, ± 2, ± 3, and ± 6. Using synthetic division, you can determine that 1 is a repeated zero and that -2 is also a zero. You can write the function in factored form as follows:

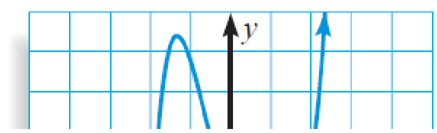
$$f(x) = (x - 1)(x - 1)(x + 2)(x^2 - 2x + 3)$$

Complete the factorization, using the quadratic formula to factor the trinomial.

$$f(x) = (x - 1)(x - 1)(x + 2)[x - (1 + i\sqrt{2})][x - (1 - i\sqrt{2})]$$

▶ This factorization gives the following five zeros:

$$1, 1, -2, 1 + i\sqrt{2}, \text{ and } 1 - i\sqrt{2}$$



1. The equation $x^3 + 3x^2 + 16x + 48 = 0$ has three solutions: -3, $4i$, and $-4i$.

Self Check

The function $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ has four zeros: -2, -2, -2, and 0.

GUIDED NOTES

You can transform the graph of f (x) to obtain the graph of $g(x) = a * f(bx - h) + k$ by combining transformations. Let's examine how each of the given values effect the graph of the function f (x).

Transformation of the graph of ***f(x)***	
Value of ***a***	$a * f(x)$
$a > 1$	Stretch vertically by a factor of *a*
$0 < a < 1$	Compress vertically by a factor of *a*
a is negative	Reflects the graph over the x-axis
Value of ***b***	$f(b * x)$
$b > 1$	Stretch horizontally by a factor of *b*
$0 < b < 1$	Compress horizontally by a factor of *b*
Value of ***h***	$f(x - h)$
h is positive	Translates *h* units right
h is negative	Translates *h* units left
Value of **k**	$f(x) + k$
k is positive	Translates k units up
k is negative	Translates k units down

1. **Describe how to transform the graph of $f(x) = x^2$ to obtain the graph of the related function $g(x)$. Then draw the graph of $g(x)$.**

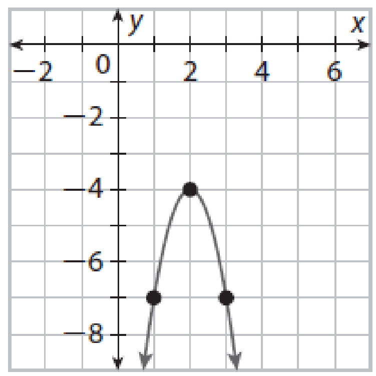
$g(x) = -3f(x - 2) - 4$

Parameter and Its Value	Effect on the Parent Graph
$a = -3$	vertical stretch of the graph of $f(x)$ by a factor of 3 and a reflection across the x-axis
$b = 1$	Since $b = 1$, there is no horizontal stretch or compression.
$h = 2$	horizontal translation of the graph of $f(x)$ to the right 2 units
$k = -4$	vertical translation of the graph of $f(x)$ down 4 units

Applying these transformations to a point (x, y) on the parent graph results in the point $(x + 2, -3y - 4)$. The table shows what happens to the three reference points on the graph of $f(x)$.

Point on the Graph of $f(x)$	Corresponding Point on $g(x)$
$(-1, 1)$	$(-1 + 2, -3(1) - 4) = (1, -7)$
$(0, 0)$	$(0 + 2, -3(0) - 4) = (2, -4)$
$(1, 1)$	$(1 + 2, -3(1) - 4) = (3, -7)$

Use the transformed reference points to graph $g(x)$.



Example 2: Identify the transformations that produce the graph of $g(x) = 2(x+1)^3 - 2$. Then, graph $g(x)$ by applying the transformations to the reference points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.

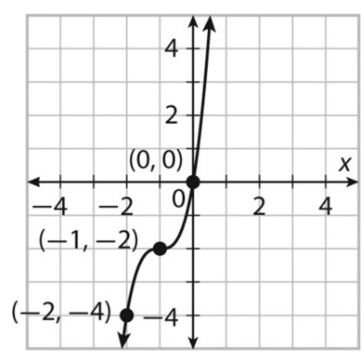
<u>Transformations</u>

$a = 2$ Vertical Stretch by 2
$b = 1$ No Horizontal Stretch or Compression
$h = -1$ Translate Left 1
$k = -2$ Translate Down 2

<u>Reference Points</u>

Original Points	x	y
$(-1, -1)$	$-1 + (-1) = -2$	$-2 + (-2) = -4$
$(0, 0)$	-1	-2
$(1, 1)$	$1 + (-1) = 0$	$2 + (-2) = 0$

<u>Graph</u>



Problem 3

Using the graph of $f(x) = x^2$ as a guide, describe and then graph the transformation.

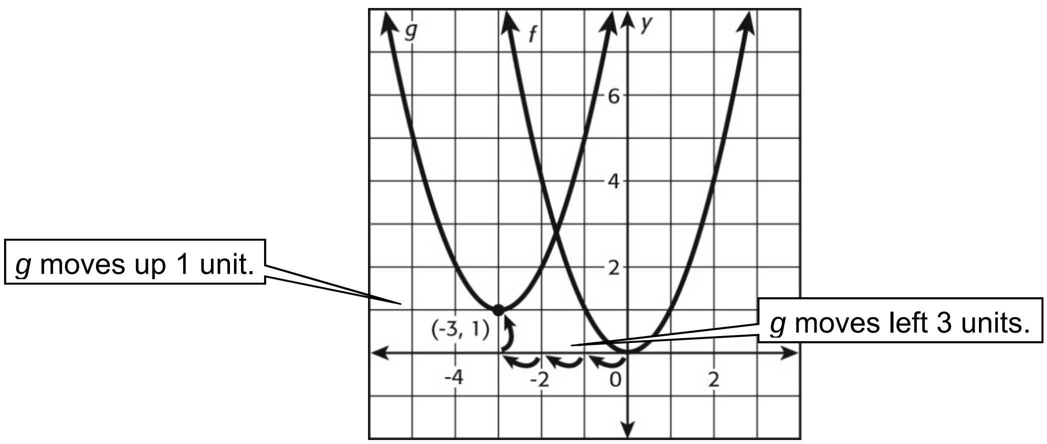
$$g(x) = (x+3)^2 + 1$$

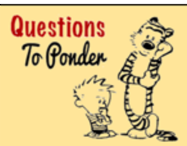
Change the function to $f(x) = a(x-h)^2 + k$ form.

$$g(x) = (x-(-3))^2 + 1$$

This is h. This is k.

-3 means "Move left 3 units." 1 means "Move up 1 unit."





In Problem 3, what would happen to the coordinates for the graph of $g(x)$ if it were reflected across the x-axis?

Problem 4

Using the description as a guide, write an equation for the transformation of $f(x) = x^2$.

"The graph of $f(x)$ is stretched vertically by a factor of 3 and shifted down 2 units."

The transformation equation will have the form

$$g(x) = a(x-h)^2 + k$$

"Stretched vertically by a factor of 3" means $a = 3$.

"Shifted down 2 units" means $k = -2$.

$$g(x) = 3(x-0)^2 + (-2)$$
$$= 3x^2 - 2$$

Example!

Describe how to transform the graph of $f(x) = x^2$ to obtain the graph of the related function $g(x)$. Then draw the graph of $g(x)$.

A.

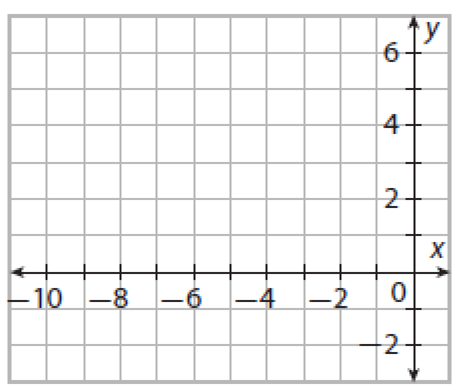
$g(x) = f\left(\frac{1}{2}(x+5)\right) + 2$

Parameter and Its Value	Effect on the Parent Graph
$a =$ ☐	Since $a = 1$, there is no vertical stretch, no vertical compression, and no reflection across the x-axis.
$b =$ ☐	The parent graph is stretched/compressed horizontally by a factor of ________. There is no reflection across the y-axis.
$h =$ ☐	The parent graph is translated ________ units horizontally/vertically.
$k =$ ☐	The parent graph is translated ________ units horizontally/vertically.

Applying these transformations to a point on the parent graph results in the point $(2x - 5, y + 2)$. The table shows what happens to the three reference points on the graph of $f(x)$.

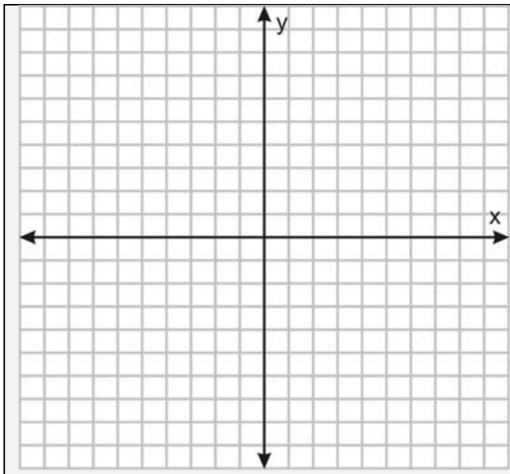
Point on the Graph of $f(x)$	Corresponding Point on the Graph of $g(x)$
$(-1, 1)$	$(2(-1) - 5, 1 + 2) = (\square, \square)$
$(0, 0)$	$(2(0) - 5, 0 + 2) = (\square, \square)$
$(1, 1)$	$(2(1) - 5, 1 + 2) = (\square, \square)$

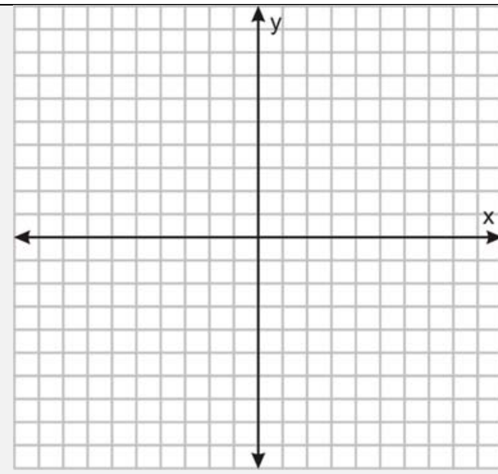
Use the transformed reference points to graph $g(x)$.



B) $g(x) = -\frac{f(x+4)}{3}$

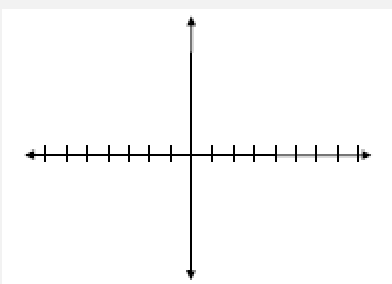
C) $g(x) = -\frac{f(x+4)}{3}$



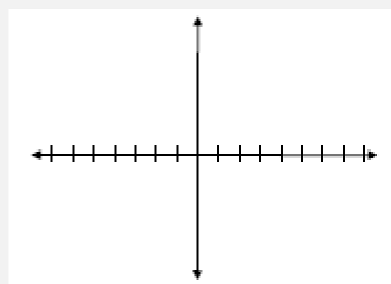


Another example:
Given $f(x) = (x+1)(x-2)^2$. Sketch the following functions and label the indicated points.

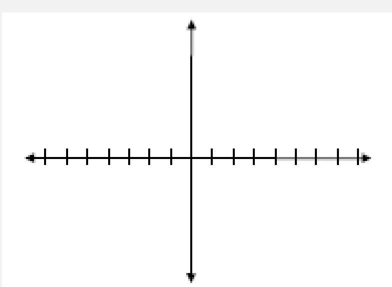
A) *f(x); x & y intercepts*



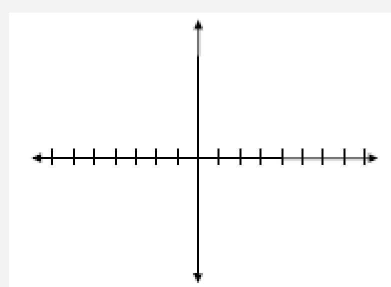
B) f(x)+3; x int &relative minimums



C) f(x+3); x int &relative minimums



D) -f(x); x & y intercepts

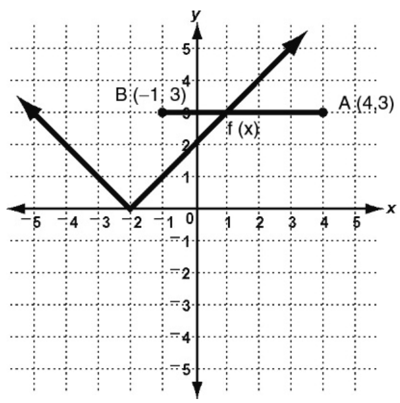


SELF CHECK

Describe the change, $g(x)$, in terms of $f(x)$ for the transformation described. The first one is done for you.

1. vertical translation 8 units down $g(x) = f(x) - 8$
2. horizontal stretch by a factor of 4 ______________________
3. vertical compression by a factor of $\frac{1}{4}$ ______________________
4. horizontal translation 5 units left ______________________
5. reflection across the *y*-axis ______________________

Use the graph to perform each transformation. The first one is done for you.



6. Plot point *A* at (4, 3). Translate point *A* left 5 units. Label this point *B*. Give the coordinates (*x*, *y*) of point *B*. $(-1,\ 3)$
7. Plot point *C* at (1, 1). Translate point *C* right 2 units and down 3 units. Label this point *D*. Give the coordinates (*x*, *y*) of point *D*.

   ____

8. Transform $y = f(x)$ by translating it right 2 units. Label the new function $g(x)$. Compare the coordinates of the corresponding points that make up the 2 functions. Which coordinate changes, *x* or *y*?

   ____

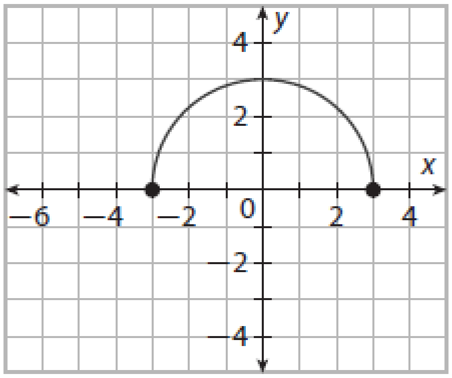
9. Transform $y = f(x)$ by reflecting it across the *x*-axis. Label the new function $h(x)$. Compare the coordinates of the corresponding points that make up the two functions. Which coordinate changes, *x* or *y*?

   ____

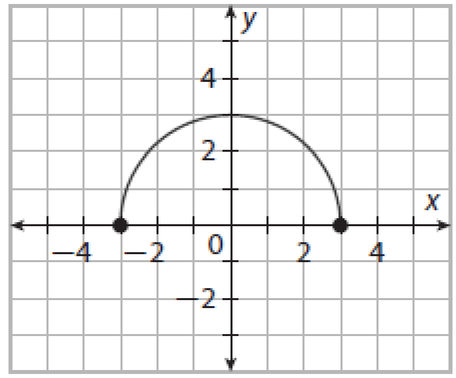
Name ________________________________ **Period**_____ **Date**_____________

Write $g(x)$ in terms of $f(x)$ after performing the given transformation of the graph of $f(x)$.

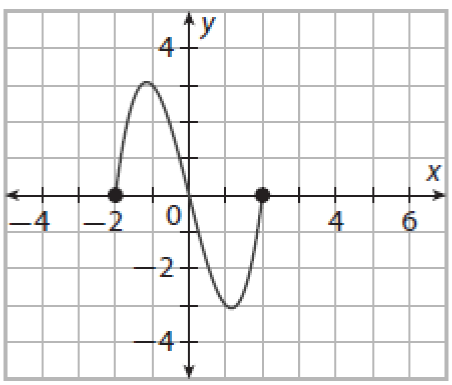
1. Translate the graph of $f(x)$ to the left 3 units.



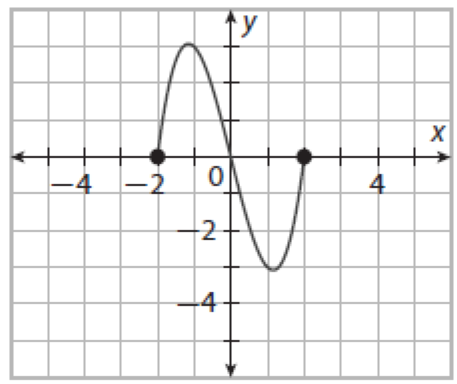
2. Translate the graph of $f(x)$ up 2 units.



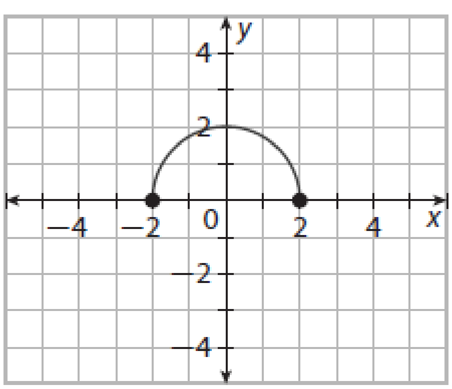
3. Translate the graph of $f(x)$ to the right 4 units.



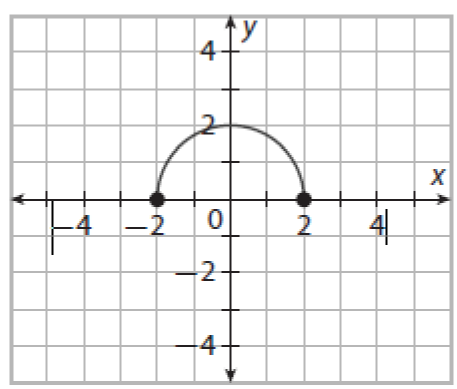
4. Translate the graph of $f(x)$ down 3 units.



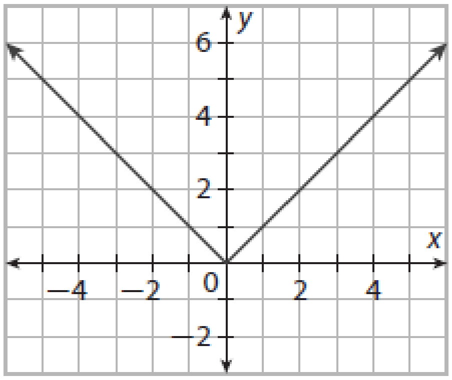
5. Stretch the graph of $f(x)$ horizontally by a factor of 3.



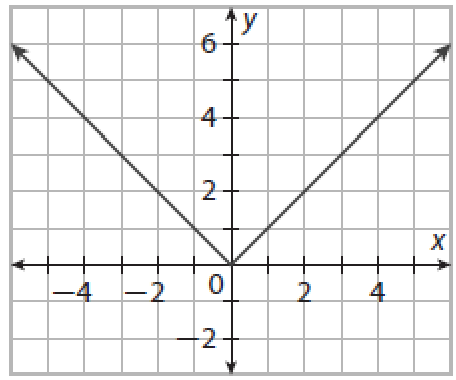
6. Stretch the graph of $f(x)$ vertically by a factor of 2.



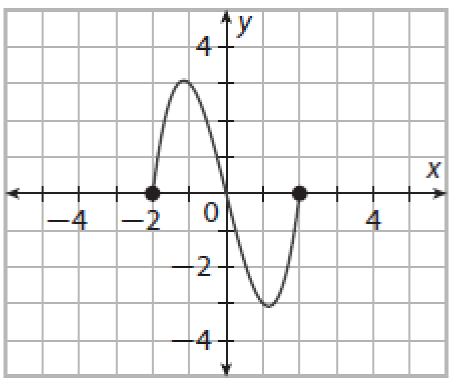
7. Compress the graph of $f(x)$ horizontally by a factor of $\frac{1}{3}$.



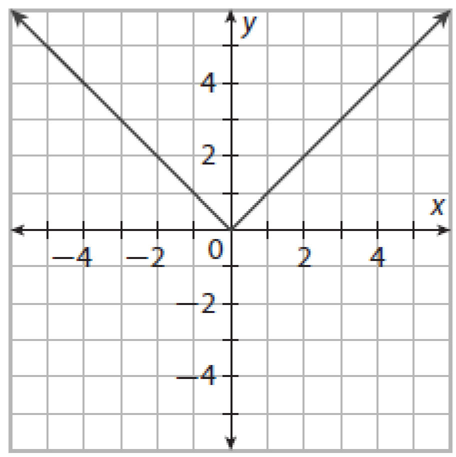
8. Compress the graph of $f(x)$ vertically by a factor of $\frac{1}{2}$.



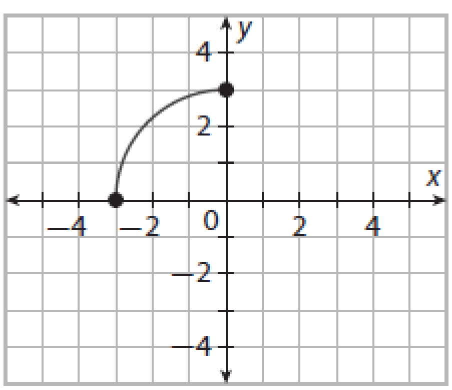
9. Reflect the graph of $f(x)$ across the y-axis.



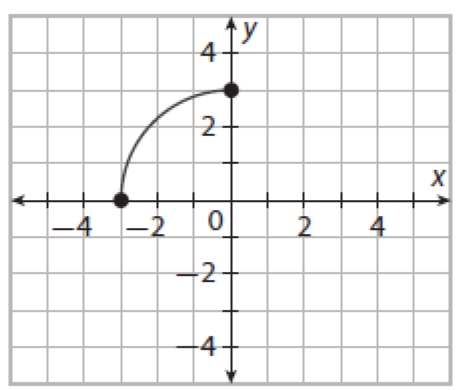
10. Reflect the graph of $f(x)$ across the x-axis.



11. Reflect the graph of $f(x)$ across the y-axis.

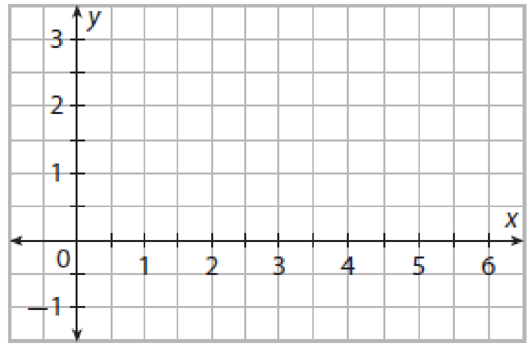


12. Reflect the graph of $f(x)$ across the x-axis.

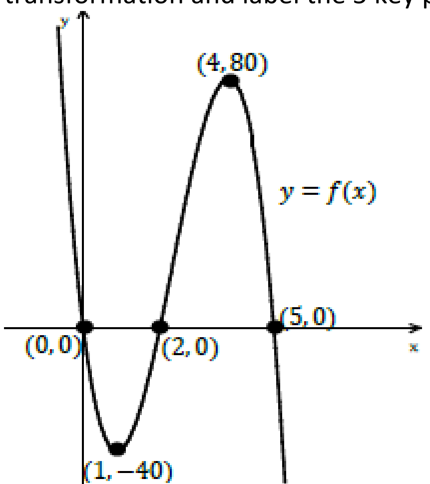


13.

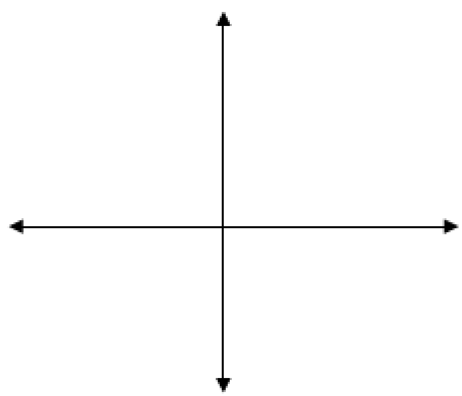
Describe how to transform the graph of $f(x) = x^2$ to obtain the graph of the related function $g(x) = f\left(-4(x - 3)\right) + 1$. Then draw the graph of $g(x)$.



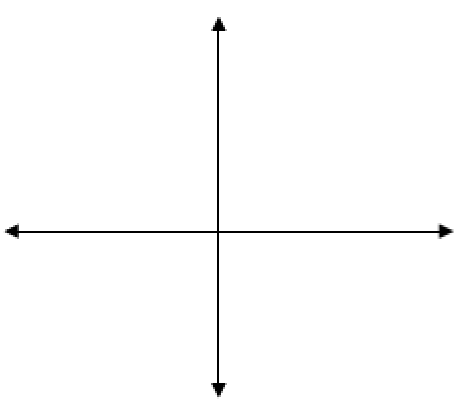
Given the graph of (x) with the five key labeled points. Describe the transformation in words, then sketch each transformation and label the 5 key points $(x,)$.



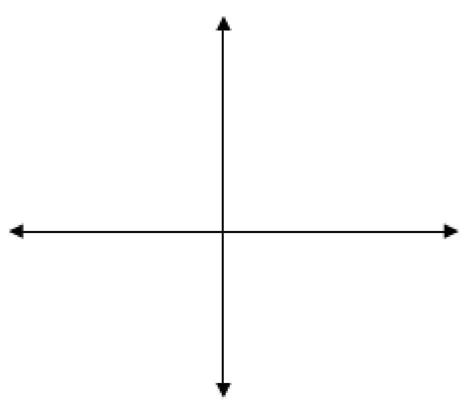
14. $f(x + 2)$
Describe the transformation:



15. $f(-x)$
Describe the transformation:



16. $f(x - 3) + 5$
Describe the transformation:

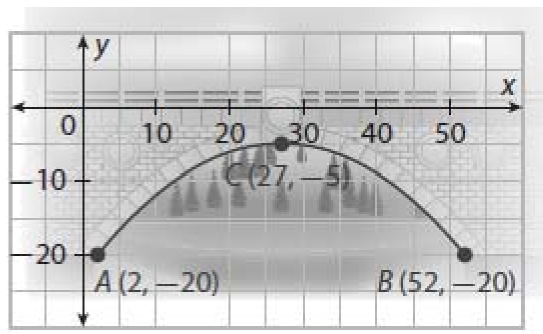


Using Transformations to Write the Equation of a Polynomial- Application

You can model real-world objects that have a parabolic shape using a quadratic function. In order to fit the function's graph to the shape of the object, you will need to determine the values of the parameters in the function $g(x) = a \cdot f\left(\frac{1}{b}(x - h)\right) + k$ where $f(x) = x^2$. Note that because $f(x)$ is simply a squaring function, it's possible to pull the parameter b outside the function and combine it with the parameter a. Doing so allows you to model real-objects using $g(x) = a \cdot f(x - h) + k$, which has only three parameters.

When modeling real-world objects, remember to restrict the domain of $g(x) = a \cdot f(x - h) + k$ to values of x that are based on the object's dimensions.

An old stone bridge over a river uses a parabolic arch for support. In the illustration shown, the unit of measurement for both axes is feet, and the vertex of the arch is point C. Find a quadratic function that models the arch, and state the function's domain.



Analyze Information

Identify the important information.

- The shape of the arch is a ________________.
- The vertex of the parabola is ________________.
- Two other points on the parabola are ________________ and ________________.

Formulate a Plan

You want to find the values of the parameters a, h, and k in $g(x) = a \cdot f(x - h) + k$ where $f(x) = x^2$. You can use the coordinates of point ______ to find the values of h and k. Then you can use the coordinates of one of the other points to find the value of a.

Solve

The vertex of the graph of $g(x)$ is point C, and the vertex of the graph of $f(x)$ is the origin. Point C is the result of translating the origin 27 units to the right and 5 units down. This means that $h = 27$ and $k = -5$. Substituting these values into $g(x)$ gives $g(x) = a \cdot f(x - 27) - 5$. Now substitute the coordinates of point B into $g(x)$ and solve for a.

$g(x) = a \cdot f(x - 27) - 5$	Write the general function.
$g(\square) = a \cdot f(52 - 27) - 5$	Substitute 52 for x.
$-20 = a \cdot f(52 - 27) - 5$	Replace $g(52)$ with -20, the y-value of B.
$-20 = a \cdot f(\square) - 5$	Simplify.
$-20 = a(625) - 5$	Evaluate $f(25)$ for $f(x) = x^2$.
$a = \square$	Solve for a.

Substitute the value of a into $g(x)$.

$$g(x) = -\frac{3}{125} f(x - 27) - 5$$

The arch exists only between points A and B, so the domain of $g(x)$ is $\{x \mid 2 \leq x \leq 52\}$.

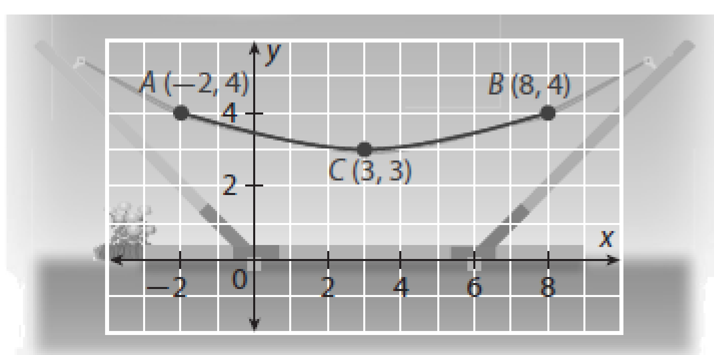
Justify and Evaluate

To justify the answer, verify that $g(2) = -20$.

$g(x) = -\frac{3}{125} f(x - 27) - 5$	Write the function.
$g(\square) = -\frac{3}{125} f(\square - 27) - 5$	Substitute 2 for x.
$= -\frac{3}{125} f(\square) - 5$	Subtract.
$= -\frac{3}{125} \cdot (\square) - 5$	Evaluate $f(-25)$.
$= -20$ ✓	Simplify.

YOUR TASK

The netting of an empty hammock hangs between its supports along a curve that can be modeled by a parabola. In the illustration shown, the unit of measurement for both axes is feet, and the vertex of the curve is point C. Find a quadratic function that models the hammock's netting, and state the function's domain.



Algebra 2 Unit # 3 Concept# N

Lesson Name: A2.U3.C2.**N.05.HW.PolyTransformations**

Name ________________________________ Period________ Date__________

Let *g*(*x*) be the transformation of *f*(*x*). Write the rule for *g*(*x*) using the change described.

1. reflection across the *y*-axis followed by a vertical shift 3 units up ________________

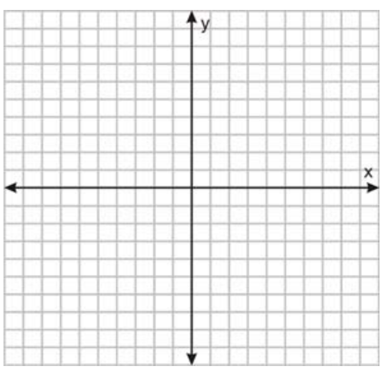
2. horizontal stretch by a factor of 5 followed by a horizontal shift right 2 units ________________

3. vertical compression by a factor of $\frac{1}{8}$ followed by a vertical shift down 6 units ________________

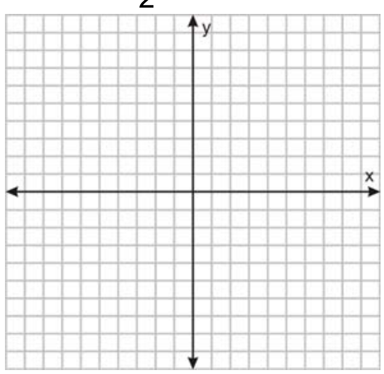
4. reflection across the *x*-axis followed by a vertical stretch by a factor of 2, a horizontal shift 7 units left, and a vertical shift 5 units down ________________

Identify the transformations that produce the graph of the given function. Then, graph the function by applying the transformations to the reference points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.

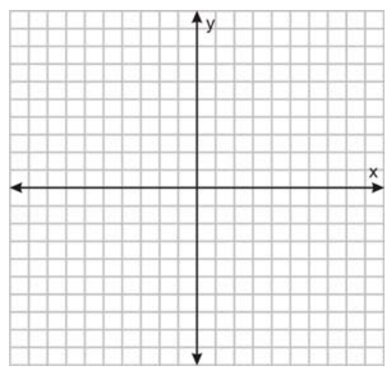
5. $g(x) = -3(x-4)^3 + 1$



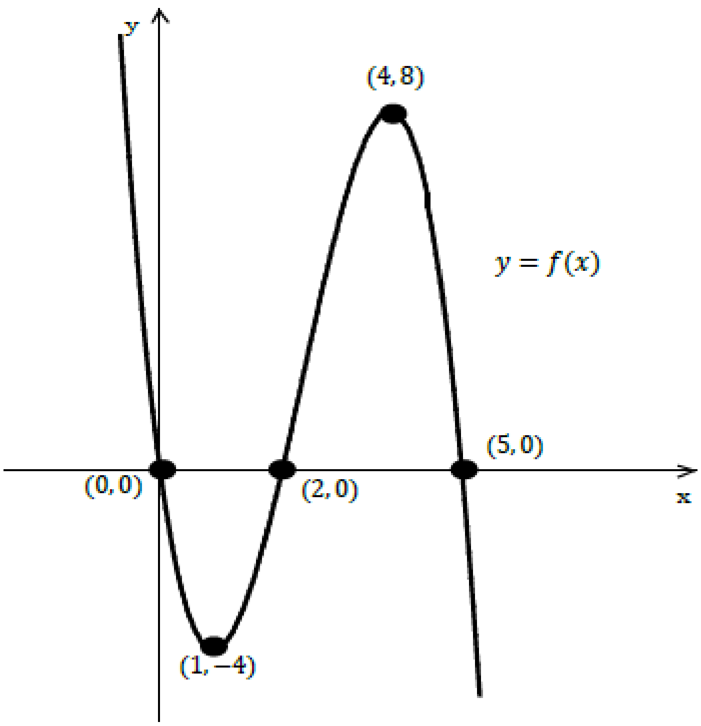
6. $g(x) = \frac{1}{2}(x-2)^3 - 4$

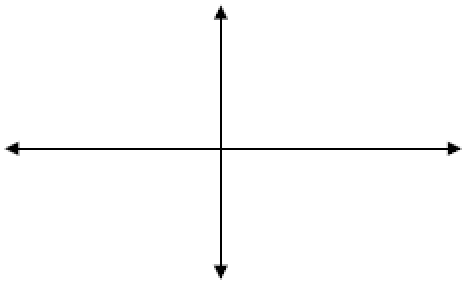
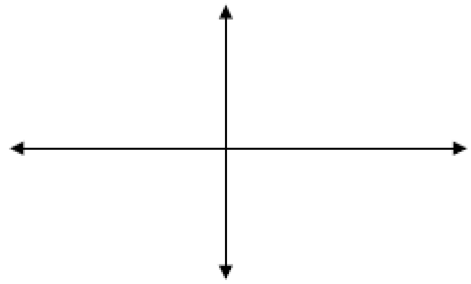
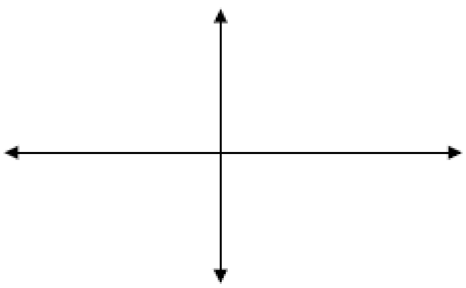
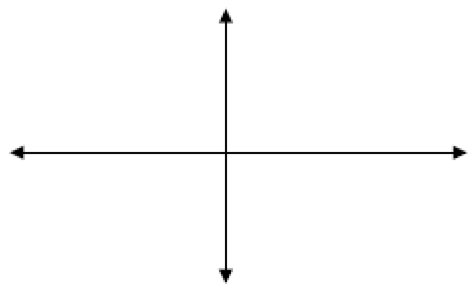
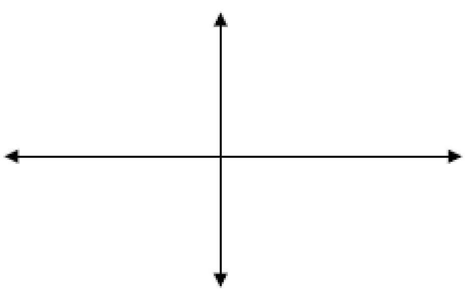
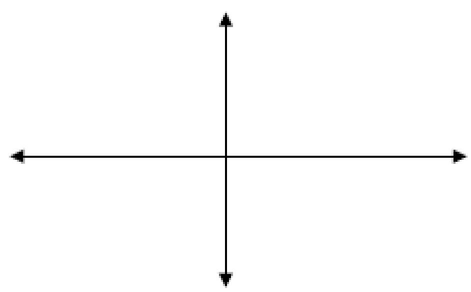
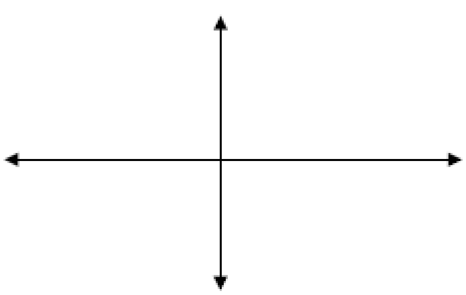
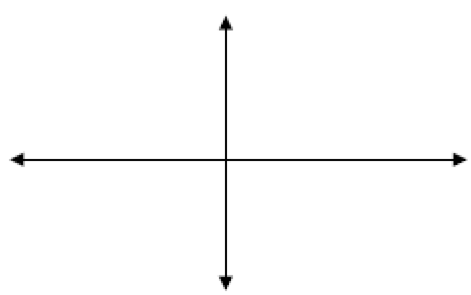


7. $g(x) = -(x+3)^3 + 2$



Given the graph of (x) with the five key labeled points. Describe the transformation in words, then sketch each transformation and label the 5 key points $(x,)$.



8. $f(x+5)$ Transformation: 	9. $f(x-2)$ Transformation: 
10. $f(x)+4$ Transformation: 	11. $f(x)-8$ Transformation: 
12. $-f(x)-4$ Transformation: 	13. $-f(x)+8$ Transformation: 
14. $f(x+1)-4$ Transformation: 	15. $f(x-3)+5$ Transformation: 

Symmetry in Polynomials

GUIDED NOTES

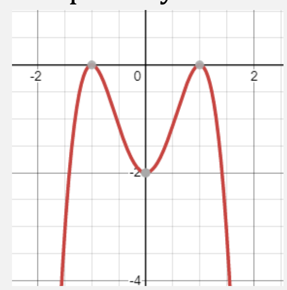
- A function f is **even** if $f(-x) = f(x)$. The graph of an even function is symmetric with respect to the y-axis. This means we could fold the graph on the axis, and it would line up perfectly on both sides!
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function is symmetric with respect to the origin. This means we can flip the image upside down and it will appear exactly the same!
- If none of these conditions apply, the function is neither even nor odd.

Example!

1. Analyze the function: $f(x) = -2x^4 + 4x^2 - 2$
   Algebriacally: replace x with −x
   $f(-x) = -2(-x)^4 + 4(-x)^2 - 2$
   $= -2x^4 + 4x^2 - 2$

   So, $f(x) = f(-x)$
   The function is even and symmetric with respect to the y-axis.

Graphically:

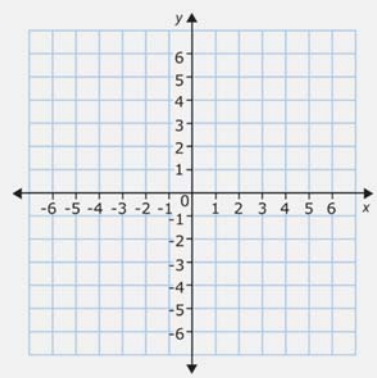


2. Analyze the function: $f(x) = -2x^5 + 8x$
   Algebriacally: replace x with −x
   $f(-x) = -2(-x)^5 + 8(-x)$

   So, ________________________________

   The function is ________________________________

Graphically: sketch



3. Analyze the function: $f(x) = -x^3 + 2x^2 + 2x - 1$
   Algebriacally: replace x with −x

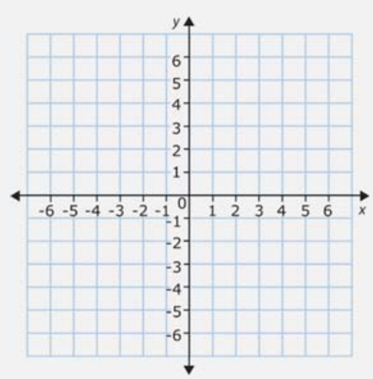
   $f(-x) = -(-x)^3 + 2(-x)^2 + 2(-x) - 1$

   =____________________________

   So, ________________________________
   The function is ________________________________

Graphically: make table

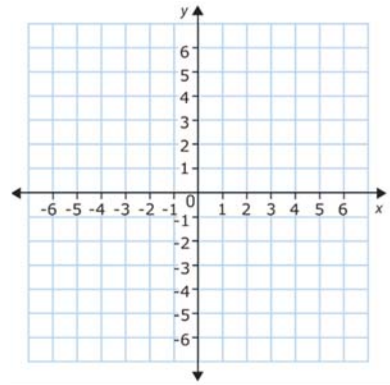
x	-1	0	1	2	3
$f(x)$					



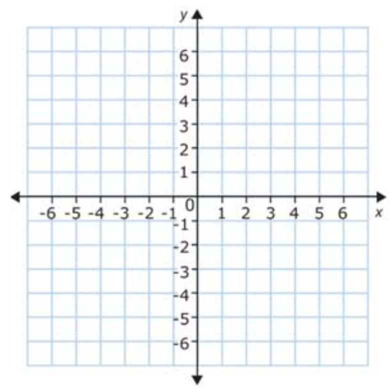
SELF CHECK

Decide whether or not the function is even, odd, or neither?
Describe any symmetries in the graph.

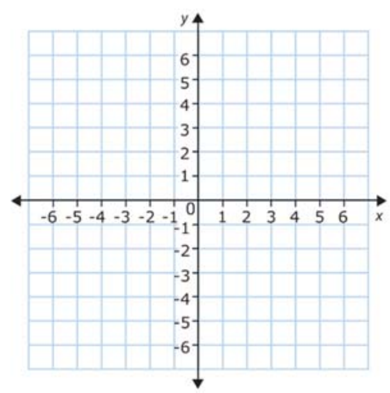
1. $f(x) = x^3$



2. $f(x) = 2x^4 - 2x^2$



3. $f(x) = x^3 + 1$



Questions To Ponder

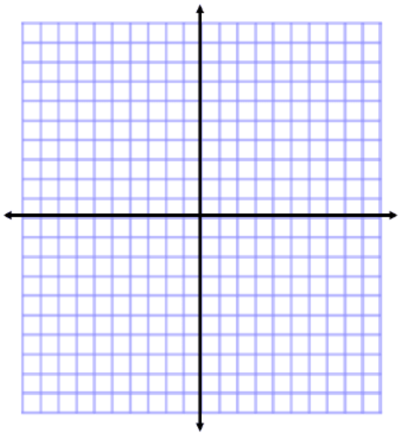
A2.U3.C2.**O.03.ClassWk.EvenOddNeitherFunctions**

Name___________________________________ Period_________ Date___________

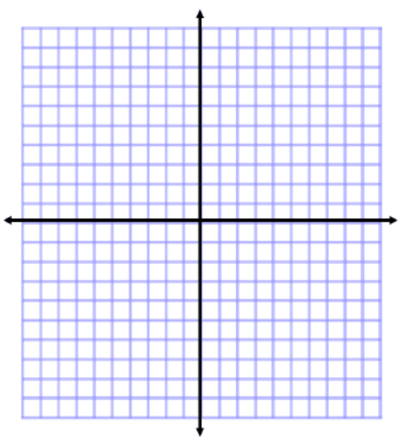
Decide whether or not the function is even, odd, or neither?

Describe any symmetries in the graph.

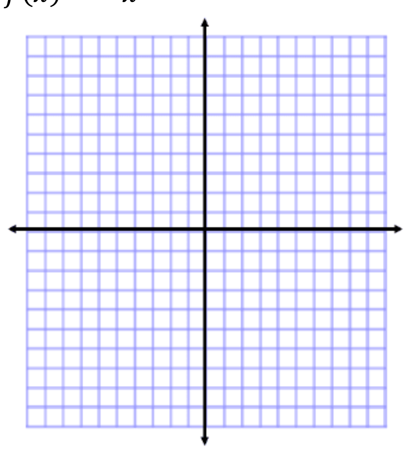
1. $f(x) = x^2 - 3$



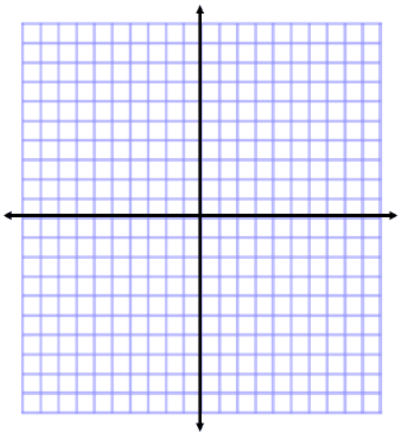
2. $f(x) = x^3 - 5x$



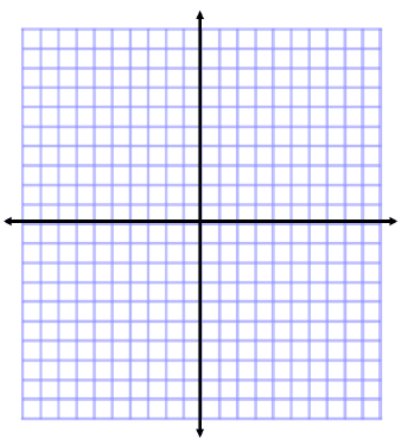
3. $f(x) = -x^3$



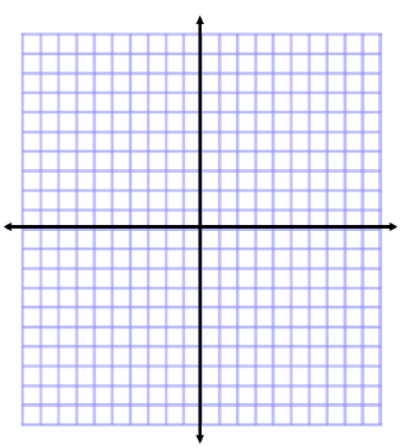
4. $f(x) = x^4 + 1$



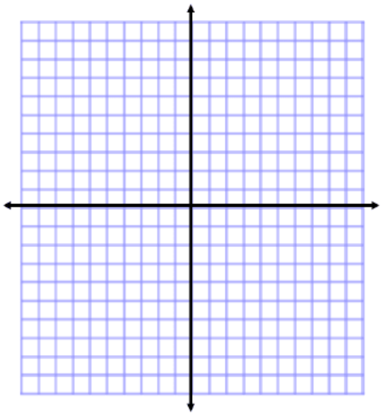
5. $f(x) = 3x^3 - 39x + 36$



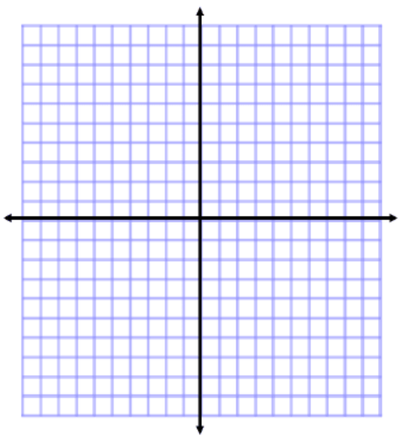
6. $f(x) = -x^4 + 6x^2$



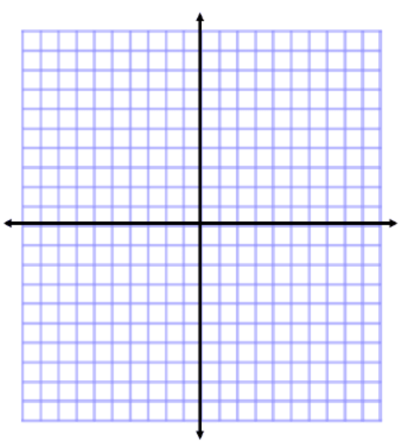
7. $f(x) = 2x^4 - 4x^2 + 8$



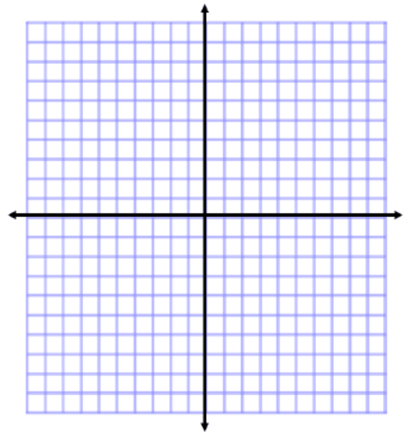
8. $f(x) = -x^3 + 3x^2 - 2x + 5$



9. $f(x) = \frac{1}{4}x^4 - 2x^2 + 16$



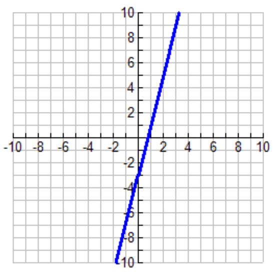
10. $f(x) = x^3 - x^2 - 2x$



Name ________________________________ **Period** ________ **Date** ______________

Directions: Determine graphically using possible symmetry, whether the following functions are even, odd, or neither.

1.



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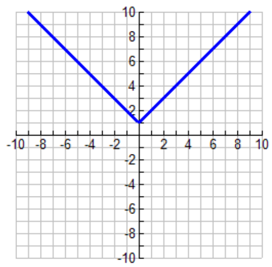

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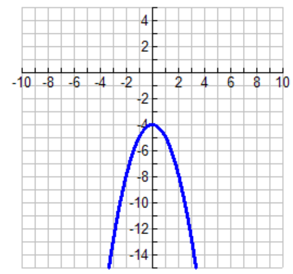

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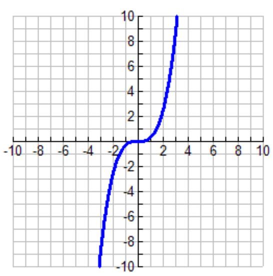

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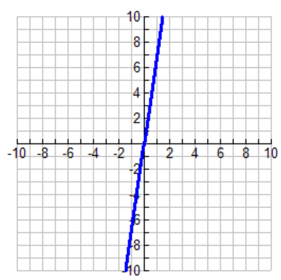

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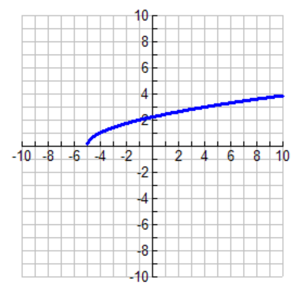


Directions: Verify algebraically whether each function is even, odd, or neither!

1. $f(x) = x^3 - 6x$
2. $g(x) = x^4 - 2x^2$
3. $h(x) = x^2 + 2x + 1$
4. $f(x) = x^2 + 6$
5. $g(x) = 7x^3 - x$
6. $h(x) = x^5 + 1$

Culminating Task (DOE) - Application

Georgia Department of Education
Georgia Standards of Excellence Frameworks
GSE Algebra II/ Advanced Algebra • Unit 3

Polynomial Project Culminating Task: Part 1

I. Finding and Exploring Zeros of Polynomial Functions

A. Find all of the zeros of the polynomial functions

1. $f(x) = x^4 + 5x^2 - 36$
2. $f(x) = x^4 - 12x^2 + 27$
3. $f(x) = x^3 - 3x^2 + 2x$
4. $f(x) = x^3 - 3x^2 - 5x + 15$
5. $f(x) = x^3 - 4x^2 - 3x + 12$
6. $f(x) = x^4 + 2x^3 + x^2 + 8x - 12$
7. $f(x) = x^3 - 2x^2 - 2x + 1$
8. $f(x) = 12x^3 - 32x^2 - 145x + 25$

B. Which functions above have zeros that are **complex conjugates**?

C. How do you determine whether zeros of a polynomial are complex conjugates?

D. If you are asked to write a polynomial function with zeros of 4 and $3 - 2i$, what is the **least degree** it could have?

II. Polynomial Division $\left(4x^4 - 20x^3 + 23x^2 + 5x - 6\right) \div (x - 3)$

A. Find the quotient using long division. **Show work!**

B. Find the quotient using synthetic division. **Show work!**

C. Why do you think you need to subtract in the process of long division, but add when doing synthetic division?

D. Under what conditions can you use synthetic division to find a quotient?

E. What is the remainder to this problem and what information does it provide about $(x - 3)$?

F. List **all** of the zeros of the equation $f(x) = 4x^4 - 20x^3 + 23x^2 + 5x - 6$.

III. Polynomial Division $\left(4x^4 - x^2 - 2x + 1\right) \div (2x - 3)$

A. Find the quotient using long division. **Show work!**

B. Find the quotient using synthetic division. **Show work!**

C. Explain how it is still possible to use synthetic division when the leading coefficient of the divisor is not equal to 1. Be sure to clearly indicate what you must do to the quotient when you use synthetic division with a linear divisor that does not have its leading coefficient equal 1. (Hint: Compare your results in IIIB to your result in IIIA.)

IV. Grading

A. Write all answers and show work on the answer sheet provided to you.

B. There are 20 questions & each answer is worth a maximum of 5 points

Answer Sheet

I. Finding and Exploring Zeros of Polynomial Functions

A.

1. ______________________________
2. ______________________________
3. ______________________________
4. ______________________________
5. ______________________________
6. ______________________________
7. ______________________________
8. ______________________________

B. ______________________________

C. ______________________________

____

D. ______________________________

II. Polynomial Division

$(4x^4 - 20x^3 + 23x^2 + 5x - 6) \div (x - 3)$

A.

B.

C. ______________________________

____

____

D. ______________________________

____

E. ______________________________

____

F. ______________________________

III. Polynomial Division

$(4x^4 - x^2 - 2x + 1) \div (2x - 3)$

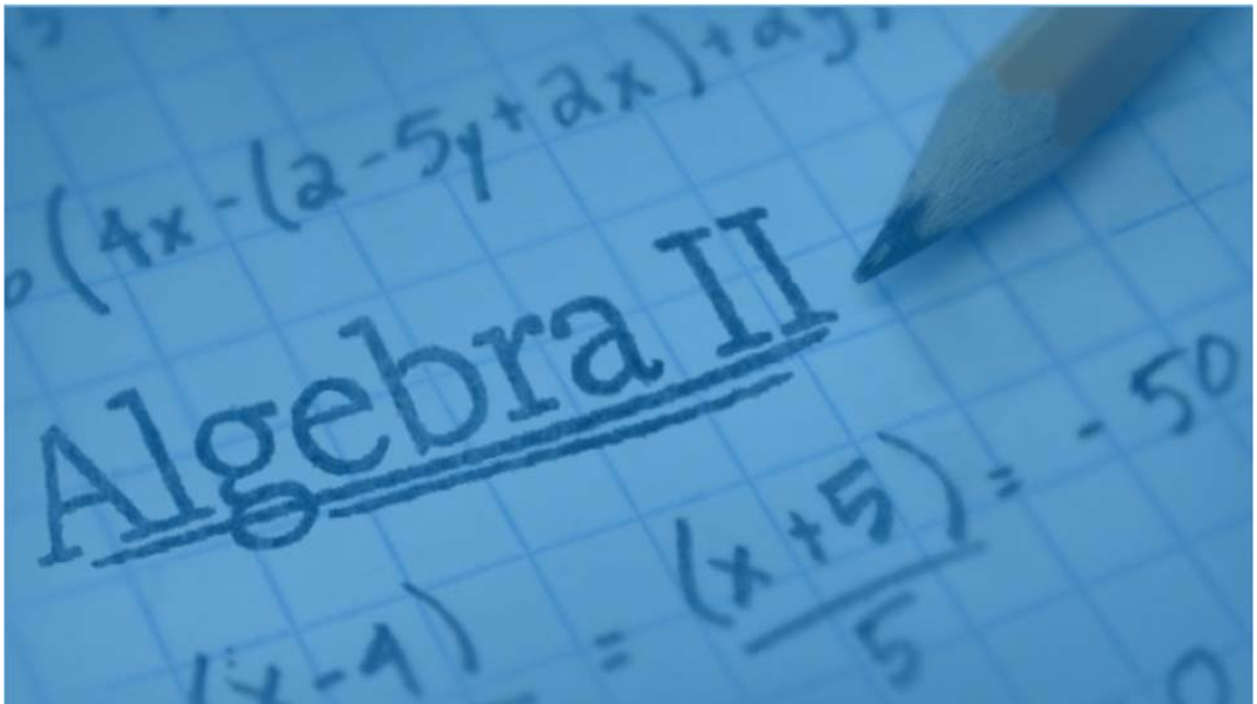
A.

B.

C. ______________________________

____

____



Unit 4A

Radical Functions

Algebra 2

Unit 4A: ~~Rational and~~ Radical Functions

Concept 1: Rational Exponents

Lesson A: Rational Exponents	(A2.U4A.C1.A.____.RationalExponents)
Lesson B: Simplify Expressions Rational Exp	(A2.U4A.C1.B.____.SimplifyRationalExpressions)

Concept 2: Operations with Radical Expressions

Lesson C: Simplify Add Subtract Radicals	(A2.U4A.C2.C.____.SimplifyAddSubtRadicals)
Lesson D: Multiply Divide Radicals	(A2.U4A.C2.D.____.MultDivRadicals

Concept 3: Solving Radical Equations

Lesson E: Solving Radical Equations	(A2.U4A.C1.E.____.SolveRadicals)

Concept 4: Graphing Radical Functions

Lesson F: Graph Radicals	(A2.U4A.C1.FI.____.GraphRadicals)

A2.U4A.C1.**A.01.Vocab.RationalExponents**

Term	Definition	Diagram/Visual
Rational exponent		
Radical Expression		
Radicand		
Index		

Rational Exponents

Remember that a number a is an nth root of a number b if $a^n = b$. You can use the definition of a root and properties of equality and exponents to explore how to express roots using rational exponents.

GUIDED NOTES

Sometimes ________________ ________________ are used to represent power of numbers or variables. The numerator of the fraction (m) represents the ________________, the denominator (n) represents the ________________.

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

The exponent in the denominator must always be ________________.

Example!

$x^{\frac{2}{5}} =$	$a^{\frac{7}{4}} =$
$\sqrt[3]{z^2} =$	$\sqrt[3]{\left(\frac{c}{d}\right)^2} =$

SELF CHECK Rewrite each rational exponents as a radical expression.

$x^{\frac{1}{2}} =$	$9^{\frac{3}{2}} =$	$\left(\frac{2x}{y}\right)^{\frac{5}{3}} =$

SELF CHECK Rewrite each radical expression using rational exponents.

$\sqrt[5]{6^4} =$	$\sqrt[3]{x^4} =$	$\sqrt{y} =$

Questions To Ponder

1. What does the numerator of the rational exponent indicate?
2. What does the denominator indicate?

Unit 4A Concept 1 Lesson A

Write each expression in radical form. Simplify numerical expressions when possible.

1. $64^{\frac{5}{6}}$

2. $(6x)^{\frac{3}{2}}$

3. $(-8)^{\frac{4}{3}}$

4. $(5r^3)^{\frac{1}{4}}$

5. $27^{\frac{2}{3}}$

6. $(100a)^{\frac{1}{2}}$

7. $10^{\frac{8}{5}}$

8. $(x^2)^{\frac{2}{5}}$

9. $(7x)^{-\frac{1}{3}}$

Write each expression by using rational exponents. Simplify numerical expressions when possible.

10. $\left(\sqrt[4]{2}\right)^7$

11. $\left(\sqrt{5x}\right)^3$

12. $\sqrt[5]{51^4}$

13. $\left(\sqrt{169}\right)^3$

14. $\left(\sqrt[4]{2v}\right)^3$

15. $\left(\sqrt[5]{n^2}\right)^2$

16. $\dfrac{1}{\left(\sqrt{3m}\right)^3}$

17. $\sqrt[7]{36^{14}}$

18. $\dfrac{1}{\left(\sqrt[4]{5p}\right)^7}$

Rational Exponents - Application

Part 1) Myca and John determined that each set of three expressions below are equivalent. In each set determine if their work is correct, then justify your answers.

Set 1) $\sqrt{x}$, x^2, $x^{1/2}$:	Set 2) $x^{4/3}$, $\left(x^{1/3}\right)^4$, $\sqrt[4]{x^3}$:
Set 3) $\left(\sqrt[5]{a^2}\right)^4$, $a^{\frac{8}{5}}$, $\left(a^{\frac{2}{5}}\right)^4$:	Set 4) ab^4 , $\left(\sqrt[3]{ab}\right)^{12}$, $\left((ab)^{\frac{1}{3}}\right)^{12}$:

Part 2) Rewrite each of the following expressions in two equivalent forms.

$\sqrt{8}$
9
$\sqrt{y^3}$
$\left(\sqrt[4]{xy}\right)^3$

Translate expressions with rational exponents into radical expressions. Simplify numerical expressions when possible. Assume all variables are positive.

1. $64^{\frac{5}{3}}$

2. $x^{\frac{p}{q}}$

3. $(-512)^{\frac{2}{3}}$

4. $3^{\frac{2}{7}}$

5. $-\left(\frac{729}{64}\right)^{\frac{5}{6}}$

6. $0.125^{\frac{4}{3}}$

7. $vw^{\frac{2}{3}}$

8. $(-32)^{0.6}$

Translate radical expressions into expressions with rational exponents. Simplify numerical expressions when possible. Assume all variables are positive.

9. $\sqrt[7]{y^5}$

10. $\sqrt[7]{(-6)^6}$

11. $\sqrt[3]{3^{15}}$

12. $\sqrt[4]{(\pi z)^3}$

13. $\sqrt[6]{(bcd)^4}$

14. $\sqrt{6^6}$

15. $\sqrt[5]{32^2}$

16. $\sqrt[3]{\left(\frac{4}{x}\right)^9}$

A2.U4A.C1.**B.01.Vocab.SimRatExp**

Term	Definition	Diagram/Visual
Rational exponent		
Radical Expression		
Radicand		
Index		

Simplifying Expressions with Rational Exponents

You have used properties of integer exponents to simplify and evaluate expressions. Rational exponents have the same properties as integer exponents.

Words	Numbers	Algebra
Product of Powers Property To multiply powers with the same base, add the exponents.		
Quotient of Powers Property To divide powers with the same base, subtract the exponents.		
Power of a Power Property To raise one power to another, multiply the exponents.		
Power of a Product Property To find the power of a product, distribute the exponent to the exponents inside the parenthesis.		
Power of a Quotient Property To find the power of a quotient, distribute the exponents to the exponents inside the parenthesis.		

GUIDED NOTES

Now that you are familiar with rational exponents, you can apply the properites of exponents to simplify them.

Example! Let $a = 64, b = 4, m = \frac{1}{3}$, and $n = \frac{3}{2}$. Evaluate each expression.

Expression	Substitute	Simplify	Result
$a^m \cdot a^n$	$64^{\frac{1}{3}} \cdot 64^{\frac{3}{2}}$	$4 \cdot 512$	2048
$(a \cdot b)^n$	$(64 \cdot 4)^{\frac{3}{2}}$	$256^{\frac{3}{2}}$	
$(a^m)^n$			
$\frac{a^n}{a^m}$			
$\left(\frac{a}{b}\right)^n$			

Example!

$\left(5x^{\frac{1}{2}}\right)\left(7x^{\frac{3}{4}}\right)$	$\left(2x^{\frac{5}{3}}y^{\frac{1}{6}}\right)^{6}$
$\dfrac{32x^{\frac{7}{3}}}{16x^{\frac{1}{4}}}$	$\left(\dfrac{x^{\frac{1}{4}}y^{\frac{-2}{3}}}{x^{\frac{3}{2}}y^{\frac{7}{4}}}\right)^{12}$

SELF CHECK Simplify each expression. Assume all variables are positive.

$\left(9a^{\frac{2}{3}}\right)^{\frac{5}{2}}$	$\left(\dfrac{x^{\frac{1}{2}}}{x^{\frac{5}{2}}}\right)^{3}$	$\left(\dfrac{z^{\frac{5}{6}}gz^{\frac{2}{3}}}{z^{\frac{1}{2}}}\right)^{\frac{1}{5}}$

1. How do you multiply powers with the same base when the exponents are rational?

2. How do you divide powers with the same base when the exponents are rational?

Unit 4A Concept 1 Lesson B

Simplify each expression. Assume all variables are positive.

1. $\left(12^{\frac{2}{3}} \cdot 12^{\frac{4}{3}}\right)^{\frac{3}{2}}$

2. $4^{\frac{3}{2}} \cdot 4^{\frac{5}{2}}$

3. $\frac{27^{\frac{4}{3}}}{27^{\frac{2}{3}}}$

4. $\frac{(a^2)^2 b}{a^{\frac{3}{2}} b^{\frac{1}{2}}}$

5. $(27 \cdot 64)^{\frac{2}{3}}$

6. $\left(\frac{1}{243}\right)^{\frac{1}{5}}$

7. $\frac{(25x)^{\frac{3}{2}}}{5x^{\frac{1}{2}}}$

8. $(4x)^{-\frac{1}{2}} \cdot (9x)^{\frac{1}{2}}$

9. $\frac{(6x^{\frac{1}{3}})^2}{x^{\frac{5}{3}} y}$

10. $\left(\frac{y^{\frac{4}{3}}}{16y^{\frac{2}{3}}}\right)^{\frac{3}{2}}$

Rational Exponents - Application

Check out this video to get a refresher of the lesson
Roots and Unit Fraction Exponents

Find Some Who Can…

Your mission is to find a different person among your classmates to simplify each expression in the table below. Your teammates must write the correct answer in the square and then sign the square. You may not have a person sign/answer more than one square.

Simplify $8^{\frac{2}{3}}$.	Simplify $81^{-\frac{1}{2}}$.	Simplify $64^{\frac{2}{3}}$.	Simplify $25^{\frac{3}{2}}$.
Rewrite $(37)^{\frac{3}{5}}$ as a radical.	Rewrite $(13)^{\frac{4}{3}}$ as a radical.	Rewrite $(9)^{\frac{1}{6}}$ as a radical.	Rewrite $(16)^{\frac{2}{6}}$ as a radical.
Rewrite $\sqrt[5]{17^3}$ using exponents.	Rewrite $\sqrt[3]{8^4}$ using exponents.	Rewrite $\sqrt[2]{2^{-1}}$ using exponents.	Rewrite $\sqrt[6]{48^4}$ using exponents.
Simplify $(16x^3y)^{\frac{1}{2}}$.	Simplify $(9ab^2y^8)^{\frac{1}{2}}$.	Simplify $(27m^3n^5)^{\frac{1}{3}}$.	Simplify $(49x^8z^2)^{-\frac{1}{2}}$.

Simplify the expression. Assume that all variables are positive. Exponents in simplified form should all be positive.

1. $\left(\left(\frac{1}{16}\right)^{-\frac{2}{3}}\right)^{\frac{3}{4}}$

2. $\dfrac{x^{\frac{1}{3}} \cdot x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$

3. $\dfrac{9^{\frac{3}{2}} \cdot 9^{\frac{1}{2}}}{9^{-2}}$

4. $\left(\dfrac{16^{\frac{5}{3}}}{16^{\frac{5}{6}}}\right)^{\frac{9}{5}}$

5. $\dfrac{2xy}{\left(x^{\frac{1}{3}}y^{\frac{2}{3}}\right)^{\frac{3}{2}}}$

6. $\dfrac{3y^{\frac{3}{4}}}{2xy^{\frac{3}{2}}}$

A2.U4A.C2.**C.01.Vocab.SimAddSubRad**

Term	Definition	Diagram/Visual
Radical Expression		
Radicand		
Index		
Prime Factor		

Simplifying Radical Expressions

A radical with ________________ n is in ______________ ____________ when these three conditions are met.

- ✓ No radicands have perfect nth powers as factors other than 1.
- ✓ No radicands contain fractions.
- ✓ No radicals appear in the denominator of a fraction.

GUIDED NOTES

Step 1. Find all prime factors of the radicand. (Factor Tree)

Step 2. Circle same prime factors that are as many as the ***index*** number.

Step 3. Take each circled factors out of the radical sign as a single factor.

Step 4. Multiply factors outside and inside the radical sign separately to simplify.

<u>Product Property of Square Roots</u>

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

- ➢ ***<u>Shortcut</u>****: Find the largest perfect square that is a factor of the radicand.*
- ➢ ***<u>HINT</u>****: If there are exponents with variables, divide the exponent by the index. The quotient is the new exponent of the variable outside the radical. If there is a remainder, that is the power of the variables under the radical.*

$\sqrt{28x^2}$ $3\sqrt{96x^7}$

Example! Simplify each expression.

$\sqrt{320x^3}$	$3\sqrt{324x^2yz^5}$	$\sqrt{32xy^{10}z^7}$

SELF CHECK Simplify each expression.

$\sqrt{90x^3y^4z^5}$	$4\sqrt{56x^2y^4z^3}$

Add and Subtract Radical Expressions

________________ ____________ with the same ____________ can be added, subtracted, multiplied or divided. Before you can add or subtract radical expressions, each radical must be simplified. You can then only add or subtract the coefficients if the expressions have the same ______________.

GUIDED NOTES

Like Radicals	$2\sqrt{5}$ and $4\sqrt{5}$	$6\sqrt{x}$ and $-2\sqrt{x}$	$3\sqrt{4t}$ and $\sqrt{4t}$
Unlike Radicals	2 and $\sqrt{15}$	$6\sqrt{x}$ and $\sqrt{6x}$	$3\sqrt{2}$ and $2\sqrt{3}$

$3\sqrt{7}+2\sqrt{7}$

$5\sqrt{10x}-3\sqrt{10x}+2\sqrt{10x}$

$2\sqrt{7}-\sqrt{63}$

$6\sqrt{27}+\sqrt{75}-4\sqrt{48}$

Example! Simplify each expression.

$9\sqrt{3}-4\sqrt{3}$	$\sqrt{45}+\sqrt{20}$
$\sqrt{12y}+\sqrt{27y}$	$\sqrt{36}-\sqrt{48}-4\sqrt{3}-\sqrt{9}$

SELF CHECK Simplify each expression.

$9\sqrt{75}+2\sqrt{50}$	$2\sqrt{32}-5\sqrt{2}+\sqrt{400}$

Questions To Ponder

1. How can you check that the simplified form of the expression is equivalent to the original expression?
2. When adding and subtracting radical expressions, what conditions have to be met?

Unit 4A Concept 2 Lesson A

Simplify each radical expression.

1. $-2\sqrt{48}$

2. $\frac{1}{2}\sqrt{20}$

3. $10\sqrt{32x^3y}$

4. $\sqrt{72x^4y^3z^2}$

5. $7\sqrt{6}+2\sqrt{6}$

6. $-8\sqrt{15}-9\sqrt{15}$

7. $3\sqrt{5}-2\sqrt{45}$

8. $\sqrt{200}+3\sqrt{2}$

9. $\sqrt{48}-\sqrt{27}$

10. $2\sqrt{80}+\sqrt{45}$

11. $-\sqrt{54}+\sqrt{49}+\sqrt{24}$

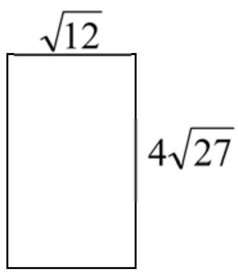
12. $2\sqrt{28}-5\sqrt{7}+6\sqrt{63}$

Simplify, Add, and Subtract Radical Expressions - Application

Apply what you know about simplifying, adding and subtracting radical expressions to find the perimeter of each quadrilateral.

Rectangle/Square: $Perimeter = l + l + w + w$

1.



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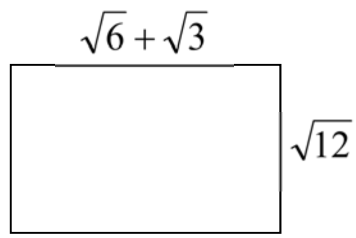

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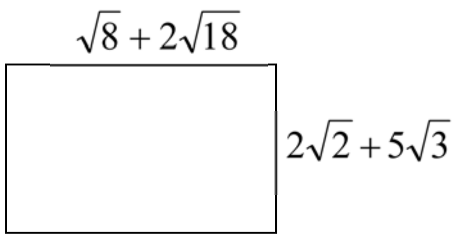

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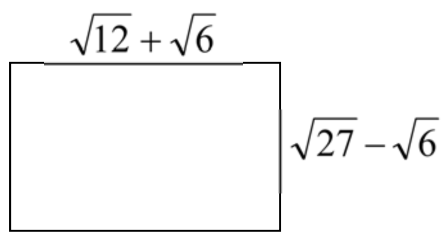

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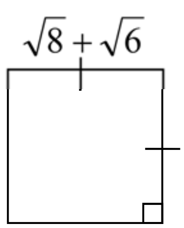

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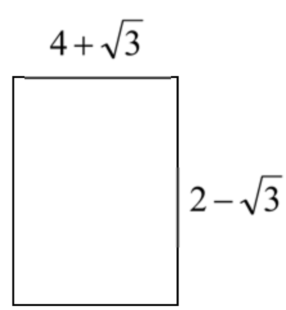


Unit 4A Concept 2 Lesson A

Simplify each expression.

1. $2\sqrt{18}$

2. $\sqrt{6x^3}$

3. $3\sqrt{100x^5y}$

4. $-2\sqrt{63x^4y}$

5. $\sqrt{80x^{100}y^{49}}$

6. $\sqrt{45a^7}$

7. $4\sqrt{36}$

8. $9\sqrt{m^{11}}$

9. $7\sqrt{2} - 10\sqrt{2}$

10. $8\sqrt{7} - 5\sqrt{7} + 12\sqrt{7}$

11. $\sqrt{13} + 3\sqrt{13} - 9\sqrt{13}$

12. $2\sqrt{27} + 5\sqrt{3}$

13. $\sqrt{24} - \sqrt{96} + \sqrt{6}$

14. $3\sqrt{24} - 2\sqrt{384} - \sqrt{96}$

15. $\sqrt{20} + \sqrt{40} + \sqrt{60}$

16. $3\sqrt{72} - 5\sqrt{32}$

17. $5\sqrt{36} + 4\sqrt{30}$

18. $2\sqrt{20} - \sqrt{500}$

19. $5\sqrt{16x^4} + 3\sqrt{25x^4}$

20. $\sqrt{45x^3} - \sqrt{20x^3}$

Term	Definition	Diagram/Visual
Radical Expression		
Radicand		
Index		
Prime Factor		
Rationalizing the Denominator		
Conjugate		

Multiplying Radical Expressions

When multiplying radical expressions, multiply the coefficients and then multiply the ________________.
If possible, you may want to simplify each expression before multiplying.

GUIDED NOTES

$2\sqrt{5}\cdot 3\sqrt{8}$ **OR** $2\sqrt{5}\cdot 3\sqrt{8}$

Example! Simplify each expression.

$5\sqrt{6}\cdot 4\sqrt{8}$	$-4\sqrt{3}\cdot 7\sqrt{15}$	$2\sqrt{5}\cdot 4\sqrt{10}$

SELF CHECK Simplify each expression.

$3\sqrt{2}\cdot 9\sqrt{20}$	$-\sqrt{12}\cdot 6\sqrt{5}$

GUIDED NOTES

Since we are multiplying we can also use the distributive property with radical expressions. You still can only multiply a _____________ by a __________, and a coefficient by a coefficient. ALWAYS simplify your final answer.

$\sqrt{2}(5+\sqrt{12})$ $5\sqrt{5}(-4+6\sqrt{5})$

Remember when you have binomials or trinomials you need distribute each term in the first polynomials to every term in the second polynomial.

$(4+\sqrt{5})(3-\sqrt{5})$ $(\sqrt{7}-5)^2$

Example! Simplify each expression.

$3\sqrt{12} \cdot \sqrt{6}$	$\sqrt{5} \cdot -4\sqrt{20}$
$2\sqrt{5}(\sqrt{6} + 2)$	$(-2\sqrt{3} + 2)(\sqrt{3} - 5)$

SELF CHECK Simplify each expression.

$\sqrt{3}(-5\sqrt{10} + \sqrt{6})$	$(5 - 4\sqrt{5})(-2 + \sqrt{5})$

Dividing Radical Expressions

A quotient with a square root in the denominator is **not** simplified. To simplify these expressions, multiply the fraction by a value of one that contains the radical in the denominator. This is called ________________

____________ ________________.

GUIDED NOTES

When the denominator is a monomial (one term), multiply the numerator and the denominator of the fraction by the radical in the denominator. Simplify the final answer if possible.

$$\frac{2}{\sqrt{7}} =$$

$$\frac{4 + \sqrt{6}}{\sqrt{2}}$$

GUIDED NOTES

When there is more than one term in the denominator, you will need to multiply the numerator and the denominator by the ________________ of the denominator. REMEMBER, the ___________________ is the same expression as the denominator but with the opposite sign in the middle, separating the terms.

$$\frac{4}{1+\sqrt{3}}$$

$$\frac{\sqrt{5}+3}{4-\sqrt{5}}$$

Example! Simplify each expression.

$\frac{4}{\sqrt{5}}$	$\frac{6-\sqrt{5}}{\sqrt{3}}$
$\frac{6}{2-\sqrt{3}}$	$\frac{2+\sqrt{8}}{2-\sqrt{8}}$

SELF CHECK Simplify each expression.

$\frac{2}{\sqrt{3}-4}$	$\frac{2+\sqrt{10}}{5-\sqrt{10}}$

Questions To Ponder

1. What happens when you multiply a binomial by its conjugate?

Unit 4A Concept 2 Lesson B

Simplify each radical expression.

1) $3\sqrt{5} \cdot -4\sqrt{16}$

2) $-5\sqrt{10} \cdot \sqrt{15}$

3) $\sqrt{12m} \cdot \sqrt{15m}$

4) $\sqrt{5x^3} \cdot -5\sqrt{10x^2}$

5) $\sqrt{6}(\sqrt{2} + 2)$

6) $\sqrt{10}(\sqrt{5} + \sqrt{2})$

7) $(2 + 2\sqrt{2})(-3 + \sqrt{2})$

8) $(-2 + \sqrt{3})(-5 + 2\sqrt{3})$

9) $(4 + \sqrt{5})(4 - \sqrt{5})$

10) $(6 - \sqrt{2})^2$

11) $\frac{\sqrt{2}}{\sqrt{3}}$

12) $\sqrt{\frac{7}{5}}$

13) $\frac{3}{1-\sqrt{5}}$

14) $\frac{\sqrt{5}}{5+\sqrt{2}}$

15) $\frac{2-\sqrt{3}}{2-\sqrt{5}}$

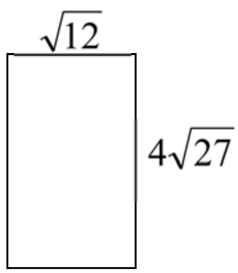
16) $\frac{5+\sqrt{7}}{4+\sqrt{3}}$

Multiplying Radical Expressions - Application

Apply what you know about simplifying, adding and subtracting radical expressions to find the area of each quadrilateral.

Rectangle/Square: $Area = l \cdot w$

1.



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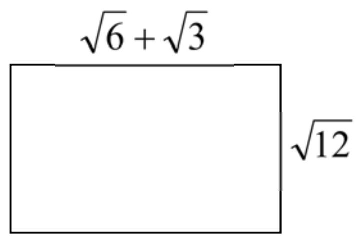

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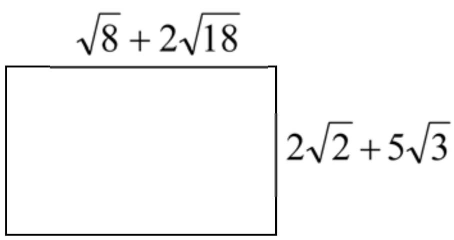

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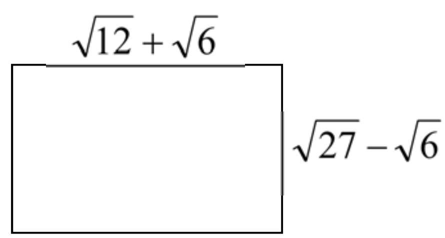

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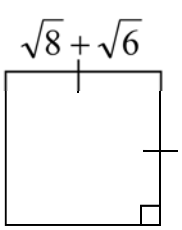

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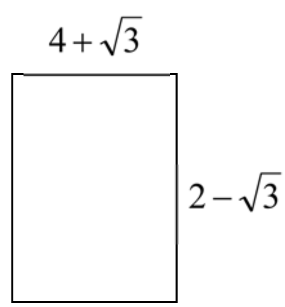


Unit 4A Concept 2 Lesson B

Multiply each pair of conjugates.

1. $\left(3\sqrt{2}-9\right)\left(3\sqrt{2}+9\right)$ **2.** $\left(1-\sqrt{7}\right)\left(1+\sqrt{7}\right)$ **3.** $\left(5\sqrt{3}+\sqrt{2}\right)\left(5\sqrt{3}-\sqrt{2}\right)$

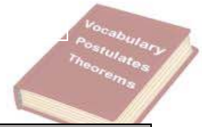
Add or subtract if possible.

4. $9\sqrt{3}+2\sqrt{3}$ **5.** $5\sqrt{2}+2\sqrt{3}$ **6.** $3\sqrt{7}-7\sqrt[3]{x}$ **7.** $14\sqrt[3]{xy}-3\sqrt[3]{xy}$

Rationalize each denominator. Simplify the answer.

8. $\dfrac{2}{2\sqrt{3}-4}$ **9.** $\dfrac{5}{2+\sqrt{3}}$ **10.** $\dfrac{1+\sqrt{5}}{1-\sqrt{5}}$ **11.** $\dfrac{2+\sqrt{12}}{5-\sqrt{12}}$

A2.U4A.C3.**E.01.Vocab.SolveRad**



Term	Definition	Diagram/Visual
Radical Expression		
Radicand		
Index		
Radical Equation		
Extraneous Solution		

Solving Radical Equations

GUIDED NOTES

- Isolate the radical expression/rational exponent on one side of the equation.
- Raise both sides of the equation to the same power. (If trying to undo a square root, square both sides. If trying to undo a cube root, cube both sides.)
- Check for _______________ _______________! Plug your answer(s) back into the original equation to make sure they work!

$6\sqrt{x+10} = 42$

CHECK

$\sqrt{2x+14} = x+3$

CHECK

$(5x+7)^{\frac{1}{2}} = 3$

CHECK

$\sqrt{8x+6} - 3\sqrt{x} = 0$

CHECK

Example!

Solve each equation.

$2+\sqrt{3x-2}=6$	$2(x-2)^{\frac{1}{2}}=50$
$\sqrt{3x+2}-\sqrt{2x+7}=0$	$\sqrt{-3x+33}=5-x$

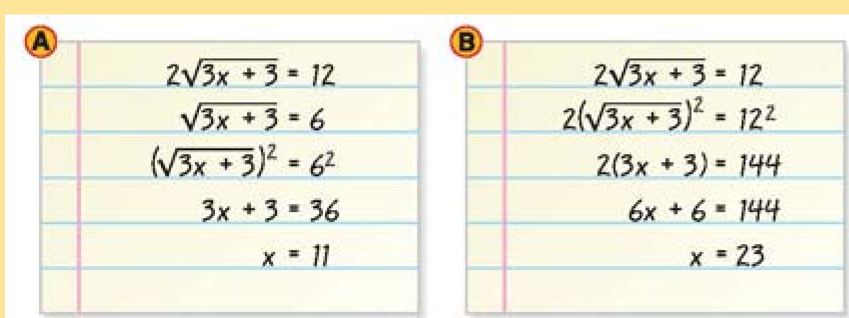
SELF CHECK

Solve each equation.

$(2x-5)^{\frac{1}{2}}=3$	$\sqrt{x+6}-x=4$

Questions To Ponder

Below are two solutions to the equation $2\sqrt{3x+3}=12$. Which is incorrect? Explain the error.



Unit 4A Concept 3

Tell whether the given value is a solution of the equation.

1. $4\sqrt{2x-3} = 12; 2$

2. $2\sqrt{9x-1} = 20; 7$

3. $\sqrt{4x+8} = \sqrt{6+2x};\ -1$

4. $\sqrt{7x-2} = \sqrt{8-3x};\ -1$

5. $x = \sqrt{4x-3}; 3$

6. $\sqrt{4x-3} = x-2; 7$

Solve each equation. Be sure to check for extraneous solutions.

7. $(45-9x)^{\frac{1}{2}} = x-5$

8. $2\sqrt{x+2} = 4$

9. $\sqrt{x+3} = x-3$

10. $(4x)^{\frac{1}{2}} = 6$

11. $4\sqrt{x-2} = \sqrt{x+13}$

12. $\sqrt{x+2} = 5$

Solving Radical Equations - Application

Peer to Peer Activity

Working in pairs, you are challenged to write a radical equation that meets the following conditions.

- *The solution of the equation is 6.*
- *The equation contains 2x on one side and 3x on the other.*
- *The radicand is the sum of a variable expression and a constant.*

Once you have written your equation, solve it and identify any extraneous solutions. You will share your work with the class.

A2.U4A.C3.**E.05.HW.SolveRad**

Solve each radical equation. Be sure to check for extraneous solutions.

1. $\sqrt{4x+9} = 5$

2. $(x+4)^{\frac{1}{2}} = 6$

3. $\sqrt{x+2} + 2 = 5$

4. $\sqrt{x+4} + 2 = -x$

5. $4(x-3)^{\frac{1}{2}} = 8$

6. $\sqrt{2x+5} = \sqrt{3x-1}$

7. $\sqrt{x+2} = 3$

8. $\sqrt{x+4} = 3\sqrt{x}$

9. $\sqrt{x+3} = 2x$

10. $\sqrt{x+7} = x-5$

A2.U4A.C4.**F.01.Vocab.GraphRad**

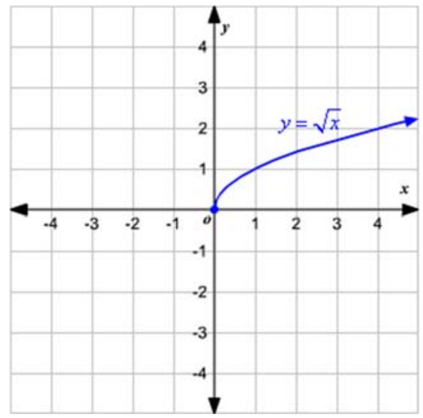
Term	Definition	Diagram/Visual
Radical Function		
Square Root Function		
Cube Root Function		
Transformations		
Domain		
Range		
End Behavior		
X-Intercept		
Y-Intercept		

Parent Radical Functions

In this lesson, we will graph and describe radical functions. To graph square root or cube root functions, create a table by plugging x-values into the function or just use the table function in your calculator.

The parent square root function is: ________________

Below are the characteristics of the parent square root function.



Domain: Range:

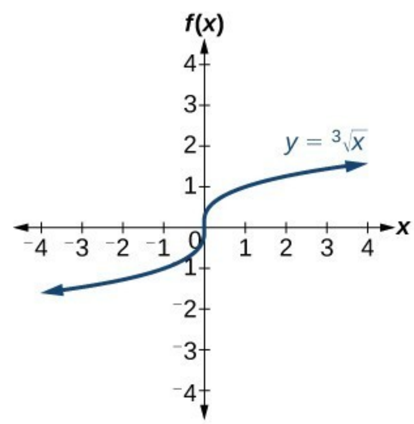
x-intercept:

y-intercept:

End Behavior:

The parent cube root function is: ________________ .

Below are the characteristics of the parent cube root function.



Domain:

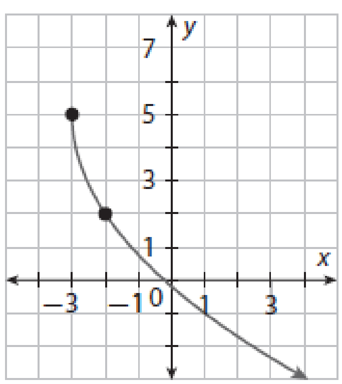
Range:

x-intercept: y-intercept:

End Behavior:

$f(x) = -3\sqrt{x+3} + 5$

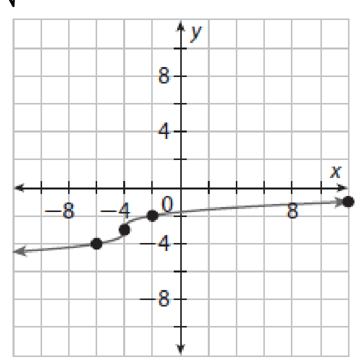
x	y
−3	
−2	
0	
1	



Domain:
Range:
x-intercept:
y-intercept:
End Behavior:

$f(x) = \sqrt[3]{\frac{1}{2}(x+4)} - 3$

x	y
−6	
−4	
−1	
12	



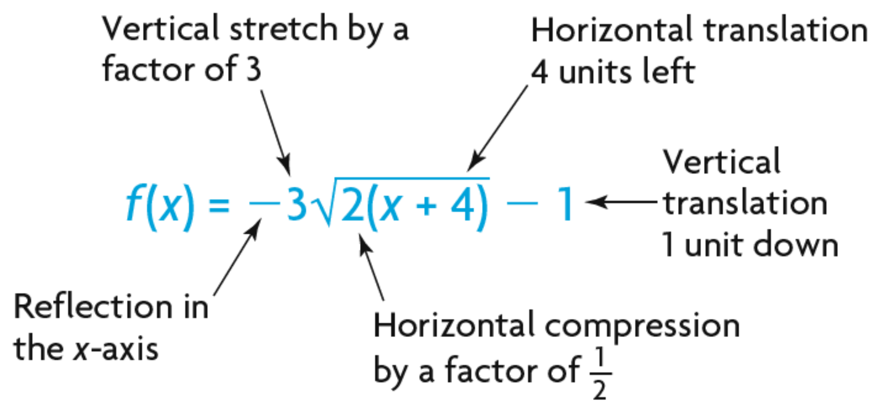
Domain: **Range**:
x-intercept: **y-intercept**:
End Behavior:

Transformation of Radical Functions

When listing the characteristics of each function, we must also describe the transformations from the parent function.

GUIDED NOTES

No matter which radical function is graph. The transformations are the same. Each part of the function helps you describe the transformations.



Transformation	**$f(x)$ Notation**	**Examples**	
Vertical shift	$f(x) + k$	$y = \sqrt{x} + 3$ $y = \sqrt[3]{x} - 4$	3 units up 4 units down
Horizontal shift	$f(x - h)$	$y = \sqrt{x - 2}$ $y = \sqrt[3]{x + 1}$	2 units right 1 units left
Vertical stretch/compression	$af(x)$	$y = 6\sqrt{x}$ $y = \frac{1}{2}\sqrt[3]{x}$	vertical stretch by 6 vertical compression by $\frac{1}{2}$
Horizontal stretch/compression	$f\left(\frac{1}{b}x\right)$	$y = \sqrt{\frac{1}{5}x}$ $y = \sqrt[3]{3x}$	Horizontal stretch by 5 Horizontal compression by $\frac{1}{3}$
Reflection	$-f(x)$ $f(-x)$	$y = -\sqrt[3]{x}$ $y = \sqrt{-x}$	across x-axis across y-axis

List the transformations for each function from its parent function.

$g(x) = \sqrt[3]{x+3} + 6$

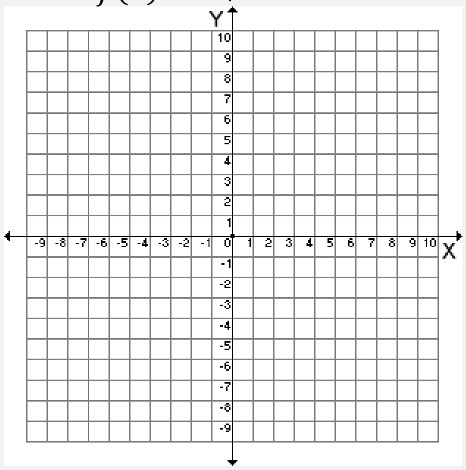
$k(x) = 3\sqrt{x-1}$

$h(x) = \sqrt{-\frac{1}{2}(x-2)} + 1$

$f(x) = -2\sqrt[3]{x+3} - 5$

Example! Graph and describe each function.

$f(x) = 2\sqrt{x-3} - 2$



Transformations:

Domain:

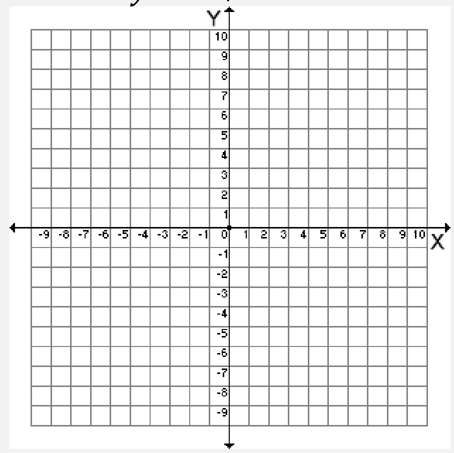
Range:

x-intercept:

y-intercept:

End Behavior: As $x \to$ _____, $y \to$ _____
As $x \to$ _____, $y \to$ _____

$y = -\sqrt[3]{2x} - 3$



Transformations:

Domain:

Range:

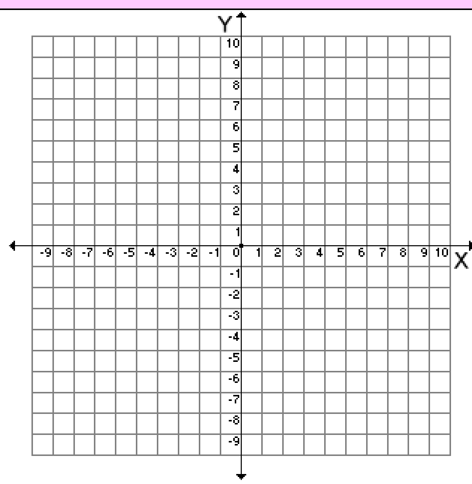
x-intercept:

y-intercept:

End Behavior: As $x \to$ _____, $y \to$ _____
As $x \to$ _____, $y \to$ _____

SELF CHECK Graph and describe each function.

$y = \sqrt[3]{x-1} + 4$



Transformations:

Domain:

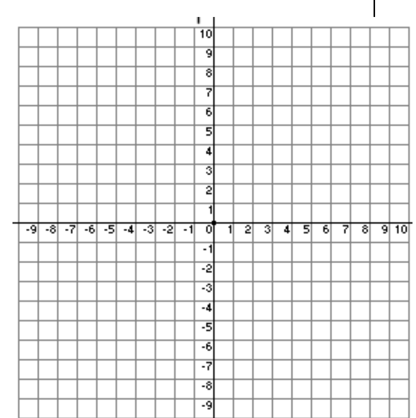
Range:

x-intercept:

y-intercept:

End Behavior: As $x \to$ _____, $y \to$ _____
As $x \to$ _____, $y \to$ _____

$y = -\frac{1}{2}\sqrt{x-2} + +1$



Transformations:

Domain:

Range:

x-intercept:

y-intercept:

End Behavior: As $x \to$ _____, $y \to$ _____
As $x \to$ _____, $y \to$ _____

Unit 4A Concept 4

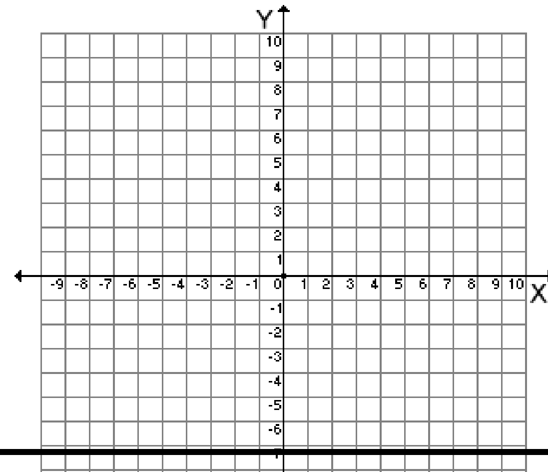
Graph and describe each radical function.

1.) $f(x) = 3\sqrt{x+5} - 3$

Transformation(s):

x-intercept:	y-intercept:
Domain:	Range:

End Behavior:

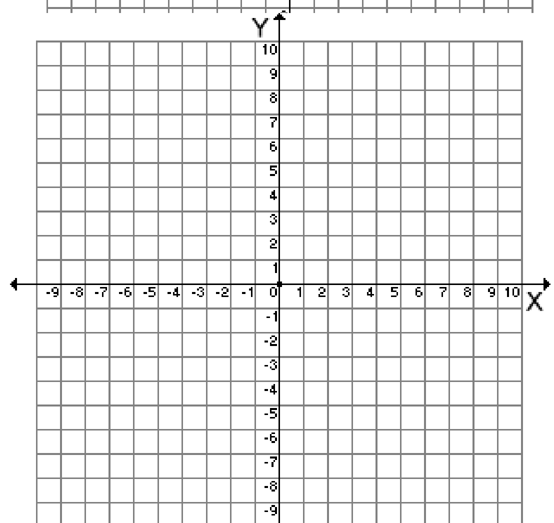


2.) $f(x) = -\sqrt{x-4} + 1$

Transformation(s):

x-intercept:	y-intercept:
Domain:	Range: Type equation here.

End Behavior:

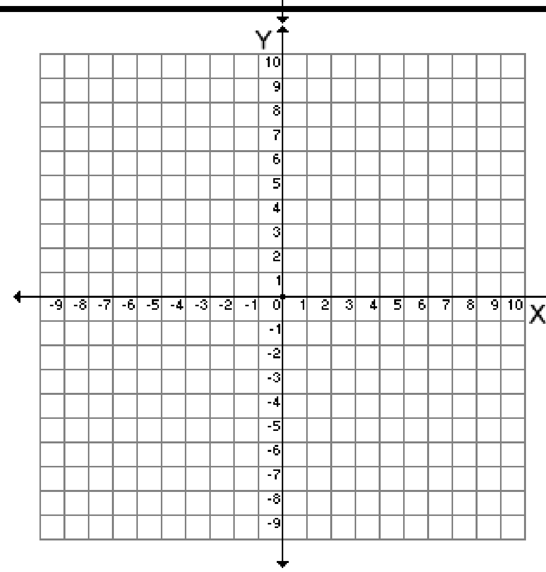


3.) $f(x) = 2\sqrt{-x} + 4$

Transformation(s):

x-intercept:	y-intercept:
Domain:	Range:

End Behavior:

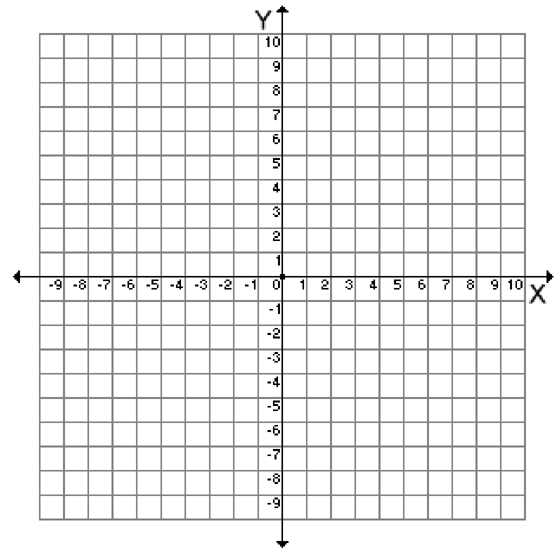


4.) $f(x) = 2\sqrt[3]{x-2} + 3$

Transformation(s):

x-intercept:	y-intercept:
Domain:	Range:

End Behavior:

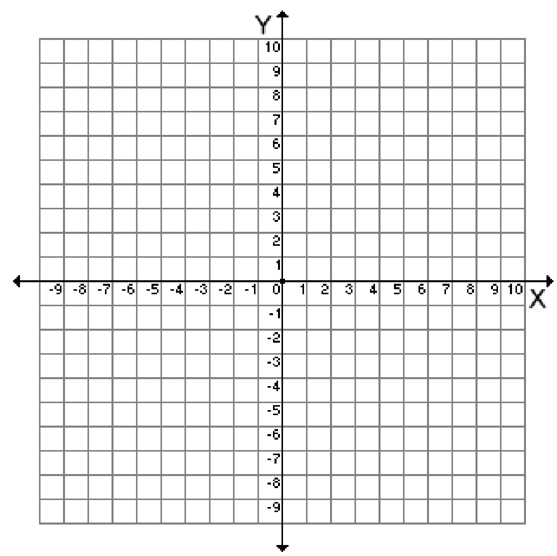


5.) $f(x) = -3\sqrt[3]{x+1}$

Transformation(s):

x-intercept:	y-intercept:
Domain:	Range:

End Behavior:

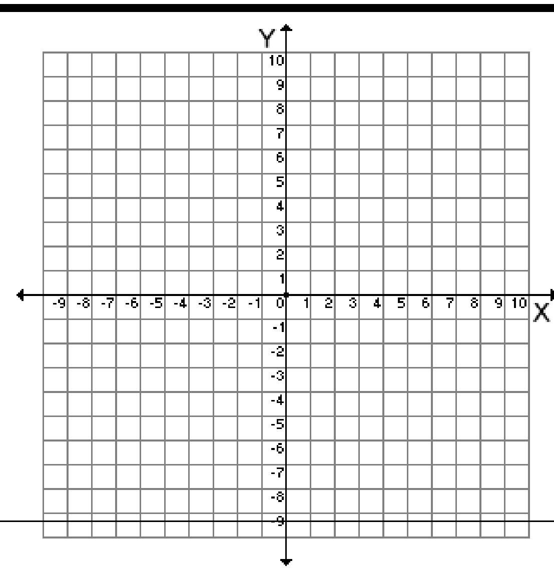


6.) $f(x) = \sqrt[3]{-x} + 5$

Transformation(s):

x-intercept	y-intercept:
Domain:	Range:

End Behavior:



Graphing Radical Functions - Application

Desmos Activity to Explore Radical Functions

Graphing Radical Functions - Application

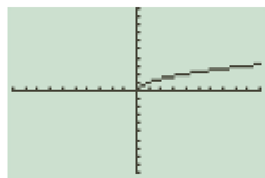
Let's explore radical functions. By definition, a radical function is one that contains any sort of radical. We are going to explore two of the more common radical functions, the square root and the cube root.

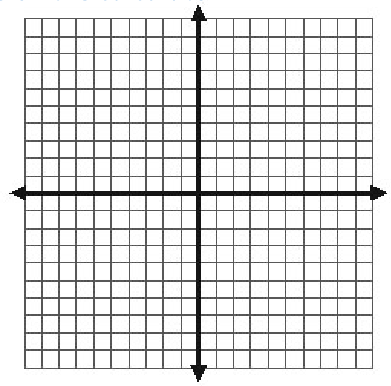
Complete the table of value for the function, $f(x) = \sqrt{x}$. This is the square root function.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	*Error*	*Error*	*Error*	*Error*	*0*	*1*	≈ 1.41	≈ 1.73	*2*

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why? ***Not every x-value will produce a y-value. You can't take the square root of negative numbers so some of the values will yield an error message on the calculator.***

Graph the function in the grid provided below.





What is the domain of this function?

D: $[0, \infty)$

What is the range of the function?

R: $[0, \infty)$

What is the end behavior?

$\boldsymbol{as\ x\ approaches\ +\infty, f(x)\ approaches\ +\infty;}$
$\boldsymbol{as\ x\ approaches\ -\infty, f(x) approaches\ 0}$

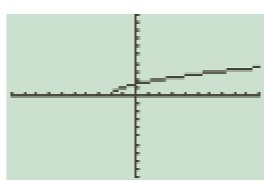
Complete the table of values for the function, $f(x) = \sqrt{x+2}$

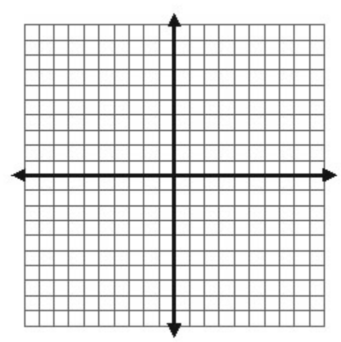
x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	*Error*	*Error*	*0*	*1*	≈ 1.41	≈ 1.73	*2*	≈ 2.24	≈ 2.45

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why? ***Not every x-value will produce a y-value. You can't take the square root of negative numbers so some of the values will yield an error message on the calculator. This time though, not all the negative x-values produced an error message.***

Graph the function in the grid provided below.

Solution:





What is the domain of this function?

D: $[-2, \infty)$

What is the range of the function?

R: $[0, \infty)$

What is the end behavior?

$\boldsymbol{as\ x\ approaches\ +\infty, f(x)\ approaches\ +\infty;}$

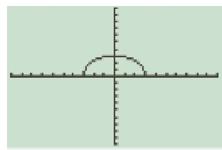
$\boldsymbol{as\ x\ approaches\ -\infty, f(x) approaches\ 0}$

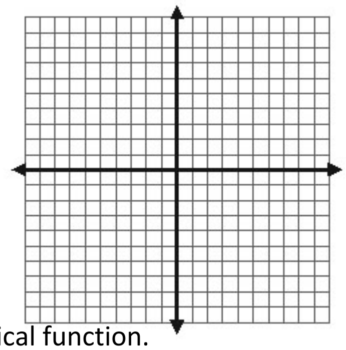
Complete the table of values for the function, $f(x) = \sqrt{9 - x^2}$.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	*Error*	*0*	≈ 2.24	≈ 2.83	*3*	≈ 2.83	≈ 2.24	*0*	*Error*

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why? ***Not every x-value will produce a y-value. You can't take the square root of negative numbers so some of the values will yield an error message on the calculator. This time though, even some of the positive numbers produced an error message.***

Graph the function in the grid provided below.





What is the domain of this function?

D: $[-3, 3)$

What is the range of the function?

R: $[0, 3)$

Using the three examples above, make a conjecture about the domain of a radical function.

Answers vary, but want students to understand radicand of radical function cannot be a number less than 0.

Use your conjecture to determine the domain of this function, $f(x) = \sqrt{2x + 5}$, without graphing it. Check your solution by graphing it on a graphing calculator. ***Students should set up the inequality $2x + 5 \geq 0$ to determine domain, so domain must be $x \geq \frac{-5}{2}$. The graph starts at (-2.5,0) and extends in positive direction.***

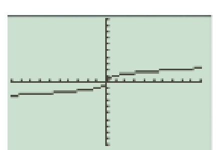
Now let's look at another common radical function, the cube root.

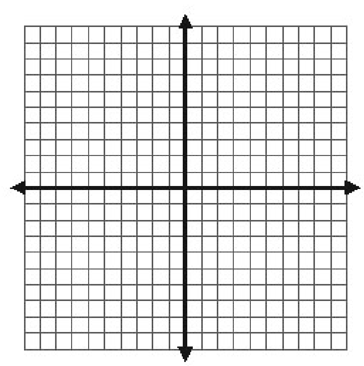
Complete the table of values for the function, $f(x) = \sqrt[3]{x}$.

x	-8	-6	-2	-1	0	1	2	6	8
$f(x)$	*-2*	≈ -1.82	≈ -1.26	*-1*	*0*	*1*	≈ 1.26	≈ 1.82	*2*

Do you get any of the same error messages for this function that you did in the table of values for the square root function? Why do you think that is so? ***There are no error messages this time. This is because you can take a cube root of a negative number so the domain of this function is all real numbers.***

Graph the function in the grid provided below.





What is the domain of this function?

D: $(-\infty, \infty)$

What is the range of the function?

R: $(-\infty, \infty)$

What is the end behavior?

$\text{as } x \text{ approaches } +\infty, f(x) \text{ approaches } +\infty$

$\text{as } x \text{ approaches } -\infty, f(x) \text{ approaches } -\infty$

Graphing Radical Functions - Application

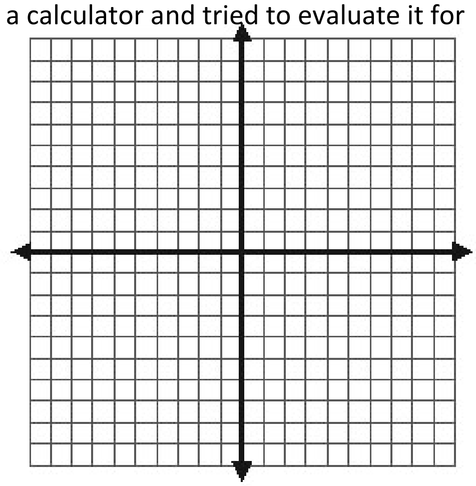
That's Radical Dude Task

Let's explore radical functions. By definition, a radical function is one that contains any sort of radical. We are going to explore two of the more common radical functions, the square root and the cube root.

Complete the table of value for the function, $f(x) = \sqrt{x}$. This is the square root function.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?



Graph the function in the grid provided.

What is the domain of this function?

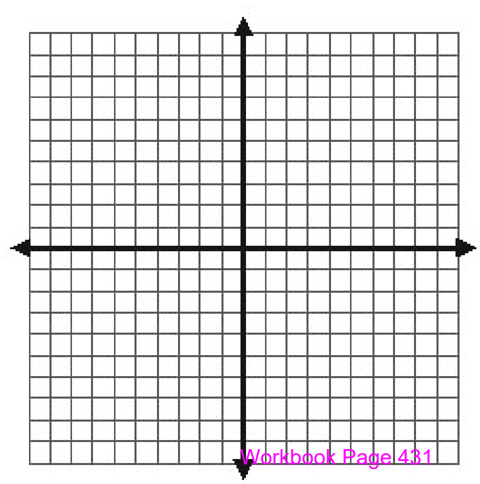
What is the range of the function?

What is the end behavior?

Complete the table of values for the function, $f(x) = \sqrt{x+2}$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?



Graph the function in the grid provided:

What is the domain of this function?

What is the range of the function?

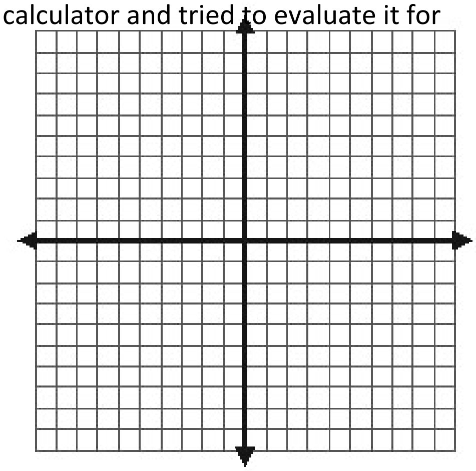
What is the end behavior?

A2.U4A.C4.**F.04.Task2.GraphRad**

Complete the table of values for the function, $f(x) = \sqrt{9 - x^2}$.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?



Graph the function in the grid provided.

What is the domain of this function?

What is the range of the function?

Using the three examples above, make a conjecture about the domain of a radical function.

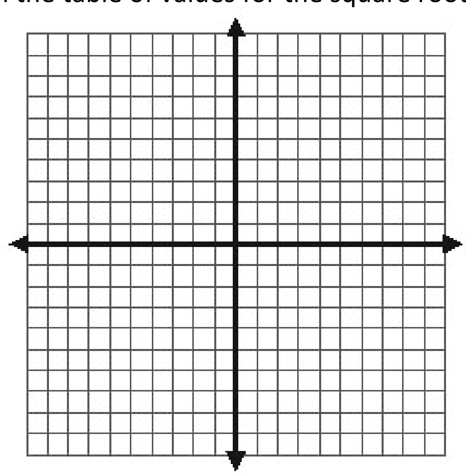
Use your conjecture to determine the domain of this function, $f(x) = \sqrt{2x + 5}$, without graphing it. Check your solution by graphing it on a graphing calculator.

Now let's look at another common radical function, the cube root.

Complete the table of values for the function, $f(x) = \sqrt[3]{x}$.

x	-8	-6	-2	-1	0	1	2	6	8
$f(x)$									

Do you get any of the same error messages for this function that you did in the table of values for the square root function? Why do you think that is so?



Graph the function in the grid provided.

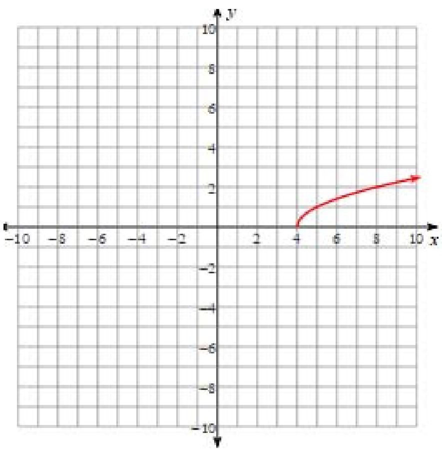
What is the domain of this function?

What is the range of the function?

What is the end behavior?

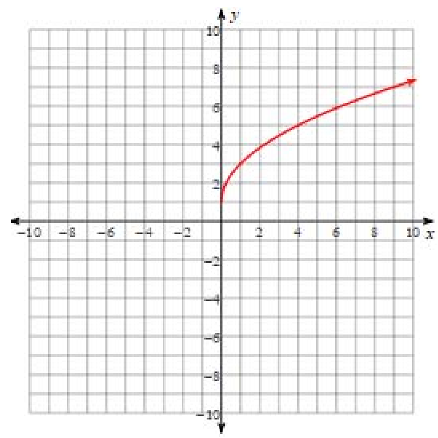
Graph each function, and identify the transformations, domain, range, x-intercept, y-intercept, and end behavior.

1. $f(x) = \sqrt{x-4}$



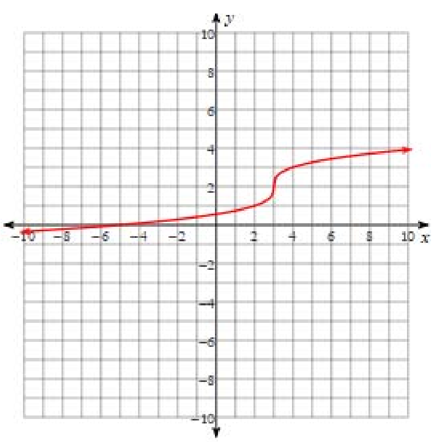
Transformations: right 4
Domain: $x \geq 4$
Range: $y \geq 0$
x-intercept: $(4,0)$
y-intercept: NONE
End Behavior: $As\ x \to -\infty, y \to 0$
$As\ x \to \infty, y \to \infty$

2. $f(x) = 2\sqrt{x} + 1$



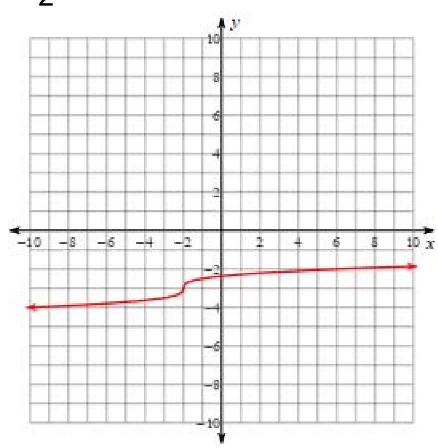
Transformations: vertical stretch, up1
Domain: $x \geq 0$
Range: $y \geq 1$
x-intercept: $NONE$
y-intercept: $(0,1)$
End Behavior: $As\ x \to -\infty, y \to 1$
$As\ x \to \infty, y \to \infty$

3. $g(x) = \sqrt[3]{x-3} + 2$



Transformations: right 3, up 2
Domain: All real numbers
Range: All real numbers
x-intercept: $(-5,0)$
y-intercept: $(0,0.558)$
End Behavior: $As\ x \to -\infty, y \to -\infty$
$As\ x \to \infty, y \to \infty$

4. $g(x) = \frac{1}{2}\sqrt[3]{x+2} - 3$



Transformations: reflection across the x-axis, horizontal shrink, down 3
Domain: All real numbers
Range: All real numbers
x-intercept: $(214,0)$
y-intercept: $(0,-2.37)$
End Behavior: $As\ x \to -\infty, y \to -\infty$
$As\ x \to \infty, y \to \infty$

Write an equation, $g(x)$, for the transformation equation described.

5. The parent function $f(x)=\sqrt{x}$ is reflected across the *y*-axis, vertically stretched by a factor of 7, and translated 3 units down. $\boldsymbol{f(x)=7\sqrt{-x}-3}$

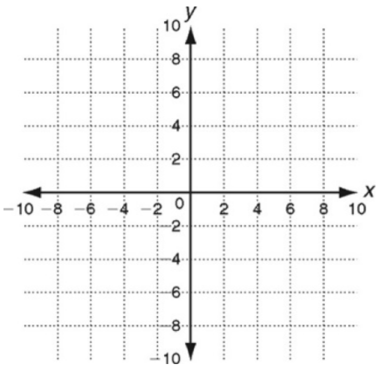
6. The parent function $f(x)=\sqrt{x}$ is translated 2 units right, compressed horizontally by a factor of $\frac{1}{2}$, and reflected across the *x*-axis. $\boldsymbol{f(x)=-\sqrt{2(x-2)}-3}$

7. The graph of $f(x)=\sqrt[3]{x}$ is reflected across the *y*-axis and then translated 4 units down and 12 units to the left. $\boldsymbol{f(x)=\sqrt[3]{-(x+12)}-4}$

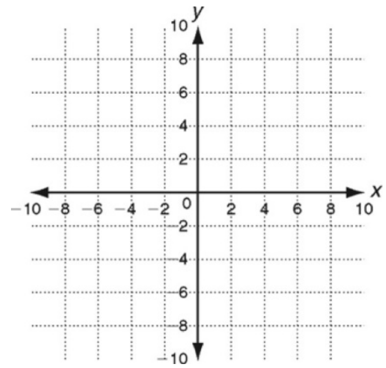
8. The graph of $f(x)=\sqrt[3]{x}$ is stretched vertically by a factor of 8, reflected across the *x*-axis, and then translated 11 units to the right. $\boldsymbol{f(x)=-8\sqrt[3]{x-11}}$

Graph each function, and identify the transformations, domain, range, x-intercept, y-intercept, and end behavior.

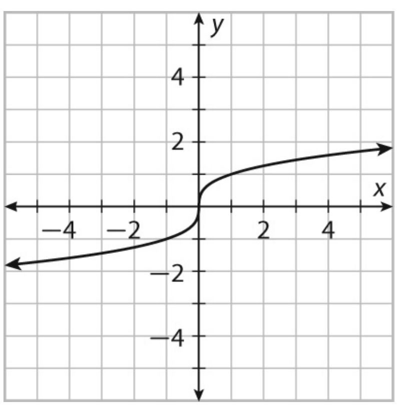
1. $f(x)=\sqrt{x-4}$



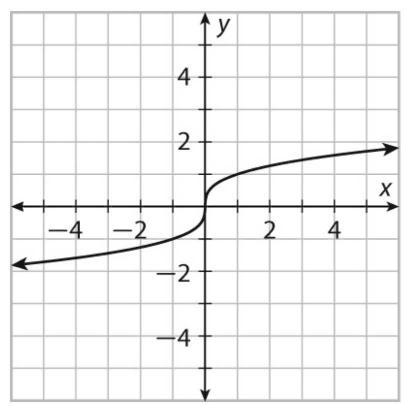
2. $f(x)=2\sqrt{x}+1$



3. $g(x)=\sqrt[3]{x-3}+2$



4. $g(x)=\frac{1}{2}\sqrt[3]{x+2}-3$



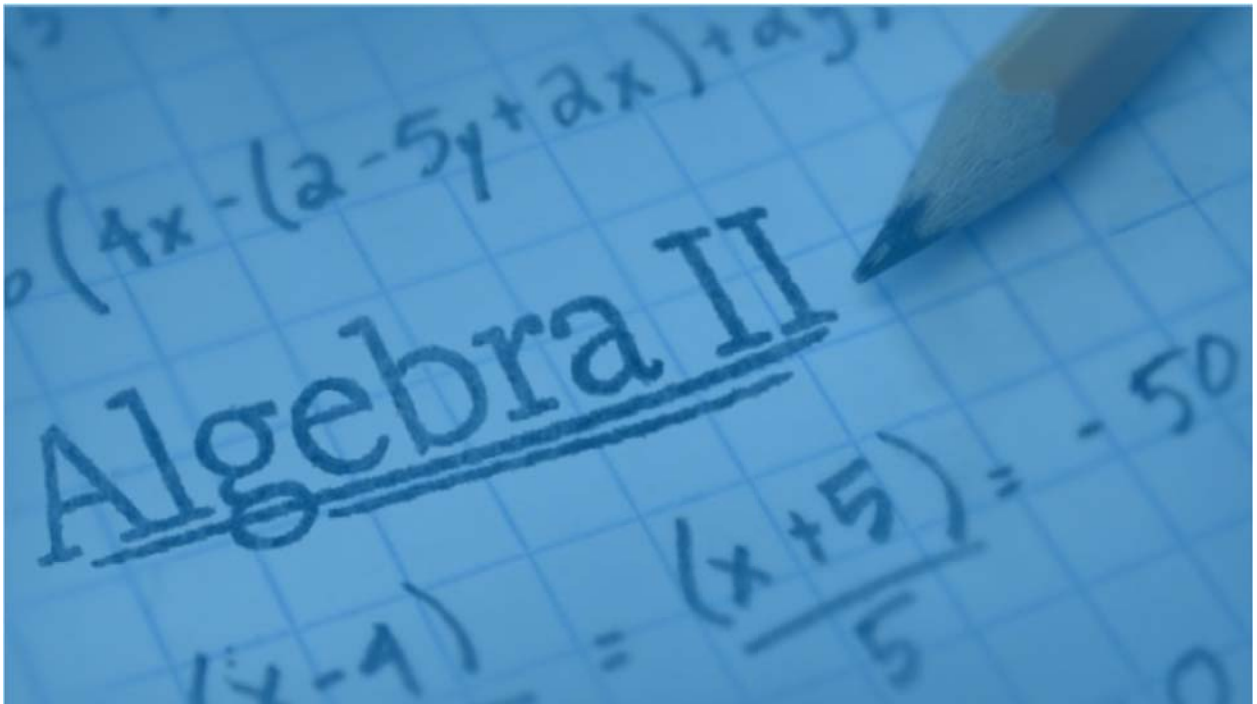
Write an equation, $g(x)$, for the transformation equation described.

5. The parent function $f(x)=\sqrt{x}$ is reflected across the *y*-axis, vertically stretched by a factor of 7, and translated 3 units down.

6. The parent function $f(x)=\sqrt{x}$ is translated 2 units right, compressed horizontally by a factor of $\frac{1}{2}$, and reflected across the *x*-axis.

7. The graph of $f(x)=\sqrt[3]{x}$ is reflected across the *y*-axis and then translated 4 units down and 12 units to the left.

8. The graph of $f(x)=\sqrt[3]{x}$ is stretched vertically by a factor of 8, reflected across the *x*-axis, and then translated 11 units to the right.



Unit 4B
Rational Functions

Unit 4B: Rational ~~and Radical~~ Functions

Concept 1: Operations with Rational Expressions

Lesson A: Simplify Multiply Divide Rational Expressions

Lesson B: Add, Subtract Rational Expressions

Concept 2: Solving Rational Equations

Lesson C: Solving Rational Equations

Concept 3: Graphing Rational Functions

Lesson D: Graphing Simple Rational Functions

Lesson E: Graphing Complex Rational Functions

A2.U4B.C1.**A.01.Vocab.SimplifyMultDivRat**

Term	Definition	Diagram/Visual
Rational Expression		
Excluded Value		
Simplest Form		

Simplifying Rational Expressions

To simplify _______________ ________________, factor the numerator and the denominator as much as possible and then cancel any expressions that appear in both the numerator and the denominator. Or you can use long division to simplify _____________________ _______________.

GUIDED NOTES

A rational expression is one where two polynomials are in the form of a fraction (dividing two polynomials) – one in the numerator and one in the denominator of a fraction.

The key to simplifying rational expressions is to be able to factor. Let's review factoring basics.

GCF
$3x^2 + 15x$

Difference of Squares
$x^2 - 4$

Product/Sum
$x^2 + 2x - 15$

a>1
$3x^2 - 7x - 6$

Combination
$2x^2 - 18$

Combination
$4x^3 - 28x^2 + 48x$

So if we have a ________________ ________________ that looks like these, we can simplify it by writing each part as factors. Then we can cancel out common factors (same factor on top and bottom).

$$\frac{x^2 - 4}{2x^3 - 4x^2 + 5x - 10}$$

$$\frac{3x^2 - 7x - 6}{x^2 + 2x - 15}$$

GUIDED NOTES

A number that makes a ________________ __________________ undefined is called an ______________ _______________. For example, $\frac{2}{x-3}$ is undefined when $x = 3$. So, 3 is an excluded value.

You find the excluded value by setting the original denominator equal to zero and solving for the variable. If there is not a variable in the denominator, then there will not be an ***excluded value***.

$$\frac{13}{2y}$$

$$\frac{n+8}{n^2 - 64} =$$

$$\frac{4x}{20}$$

Example! Simplify each expression. Find the excluded values if possible.

$\frac{6}{6x-24}=\frac{6}{6(x-4)}$	$\frac{x^2-5x-6}{x^2-1}$	$\frac{x^2+5x}{x^2}$

SELF CHECK Simplify each expression. Find the excluded values if possible.

$\frac{y^2+3y+2}{y^2-1}$	$\frac{12w+24}{48w}=\frac{12(w+2)}{48w}$

Multiplying Rational Expressions

Multiplying and dividing rational expressions is similar to multiplying and dividing numerical fractions.

GUIDED NOTES

To multiply rational expressions, multiply the _______________ and multiply the ____________________. Factor all expressions if possible. Then, simplify the product by canceling common factors. Note the excluded values of the product.

$$\frac{3x^2+3x}{4x^2-24x+36}\cdot\frac{x^2-4x+3}{x^2-x}$$

$$\frac{5x}{x^2 + 5x + 6} \cdot (x + 3)$$

GUIDED NOTES

To divide rational expressions, change the division problem to a ________________ problem by multiplying by the ________________ of the second expression. Then follow the steps for multiplying rational expressions.

A $\frac{4x^3}{9x^2y} \div \frac{16}{9y^5}$

B $\frac{x^5 - 4x^3}{x^2 - x - 2} \div \frac{x^5 - x^4 - 2x^3}{x^2 - 1}$

Example! Simplify each expression. Assume that all expressions are defined.

$\frac{x-3}{x+2} \cdot \frac{x^2+5x+6}{x^2-9}$	$\frac{2x^4y^5}{3x^2} \cdot \frac{15x^2}{8x^3y^2}$
$\frac{x^2}{4} \div \frac{x^4y}{12y^2}$	$\frac{2x^2-7x-4}{x^2-9} \div \frac{4x^2-1}{8x^2-28x+12}$

SELF CHECK Simplify each expression. Assume that are expressions are defined.

$\frac{x^2-16}{x^2-4x+4} \cdot \frac{x-2}{x^2+6x+8}$	$\frac{x+3}{x^2-2x+1} \div (x+3)$

Questions To Ponder

1. How do you find excluded values for rational expressions?
2. How can you use the steps for dividing rational numbers to divide rational expressions?

Unit 4B Concept 1 Lesson A

Answer each question.

1. What could make a rational expression undefined?

2. What value of *x* gives a denominator equal to 0 for the expression $\frac{x+4}{x-3}$?

Simplify. Identify any *x*-values for which the expression is undefined.

3. $\frac{5x}{x^3}$

4. $\frac{x^2+3x}{x^3}$

5. $\frac{4x+12}{6x+18}$

6. $\frac{2x+7}{x+3}$

Complete. Assume that all expressions are defined.

7. $\frac{4x+16}{2x+6} \cdot \frac{x^2+2x-3}{x+4}$

8. $\frac{3x}{2y} \cdot \frac{2}{y}$

9. $\frac{2x^5}{9y^4} \cdot \frac{3y^2}{x}$

10. $\frac{x+3}{x-1} \cdot \frac{x^2-2x+1}{x^2+5x+6}$

11. $\frac{5x^6}{x^2y} \div \frac{10x^2}{y}$

12. $\frac{x^2-2x-8}{x^2-2x-15} \div \frac{2x^2-8x}{2x^2-10x}$

Simplify, Multiply, Divide Rational Expressions - Application

Thinking about operations with rational numbers, or fractions, will help us perform addition, subtraction, multiplication, and division with rational expressions. We will use examples involving fractions to help us extend our thinking to dealing with fractions with variables, or rational expressions.

- **Simplifying Rational Expressions**

Think about the fraction $\frac{108}{210}$. What operation do we use to rewrite this fraction in simplest form?

What is the possible obstacle in using this operation to simplify fractions?

Let's try to simplify another way. Find the prime factorization of the numerator and denominator of the fraction above. Use this form of the numerator and denominator to quickly simplify the fraction.

Now let's think about the fraction: $\frac{x^2-9}{x^2+7x+12}$. How can we use the idea of prime factorization to help us simplify this rational expression?

Try this one: $\frac{4-x}{x-4}$ How can factoring help us simplify this rational expression?

- **Multiplying and Dividing Rational Expressions**

We now need to think about multiplying fractions. Take a minute to discuss with a partner how you would solve the following problem. Try to find more than one way and show your results below:

$$\frac{4}{14} \cdot \frac{24}{10}$$

Which method from above do you think would be easiest to extend to multiplication of rational expressions? Why?

Take a look at $\frac{5x^2}{x^2-4} \cdot \frac{x+2}{10x^3}$. Use your ideas from above to help you multiply these two fractions.

Now try this one: $\frac{3x+6}{x^2-9} \cdot \frac{4x+12}{6x^2+12x}$

What if we change the fraction multiplication problem that we started with to a division problem? Talk to your partner about how to solve the problem below:

$$\frac{4}{14} \div \frac{24}{10}$$

What is the one difference in solving a fraction division problem versus a fraction multiplication problem?

Apply that idea to this problem: $\frac{4x+8}{8x} \div \frac{x^2-4}{6x^2}$

Let's try one more: $\frac{x^2-2x-15}{3x^2+12x} \div \frac{x^2-9}{x^2+4x}$

Unit 4B Concept 1 Lesson A

Simplify each expression. Identify any excluded values.

1. $\dfrac{28x^3}{35x^5}$

2. $\dfrac{5x+40}{4x+32}$

3. $\dfrac{36y^2}{12y}$

4. $\dfrac{x^2+12x+20}{3x+6}$

5. $\dfrac{6x+30}{x^2+8x+15}$

6. $\dfrac{25a^3b^7}{-15a^8b^3}$

Multiply or Divide. Assume that all expressions are defined.

7. $\dfrac{x+3}{10x+20}\cdot\dfrac{x+2}{x^2+4x+3}$

8. $\dfrac{x^2-x-12}{x-4}\div\dfrac{2x+6}{x-5}$

9. $\dfrac{x^2-5x-6}{5x+15}\div\dfrac{x^2-3x-4}{7x+21}$

10. $\dfrac{24x^3}{25y^5}\cdot\dfrac{15y^2}{8x^2}$

11. $\dfrac{6x-18}{4x}\cdot\dfrac{x}{2x-6}$

12. $\dfrac{3x+12}{12x}\div\dfrac{x+4}{48x^3}$

A2.U4B.C1.**B.01.Vocab.SimplifyAddSubtRat**

Term	Definition	Diagram/Visual
Rational Expression		
Excluded Value		
Least Common Denominator		

Add and Subtract Rational Expressions

Adding and subtracting rational expressions is similar to adding and subtracting fractions. You must have a common denominator when added and subtracting rational expressions.

GUIDED NOTES

To add or subtract rational expressions with like denominators, add or subtract the numerators and use the same denominator.

$$\frac{3x-4}{x+3}+\frac{2x+5}{x+3}$$

$$\frac{2x-1}{x^2+2}-\frac{4x+4}{x^2+2}$$

Example! Add or subtract.

$$\frac{6x+5}{x^2-3}+\frac{3x-1}{x^2-3}$$

$$\frac{3x^2-5}{3x-1}-\frac{2x^2-3x-2}{3x-1}$$

SELF CHECK Simplify each expression.

$$\frac{3x-4}{4x+5}-\frac{5x+3}{4x+5}$$

$$\frac{2x-3}{4x-1}+\frac{3x+4}{4x-1}$$

GUIDED NOTES

To add or subtract rational expressions with unlike denominators, first find the least common denominator (LCD). The LCD is the lease common multiple of the polynomials in the denominator.

Know it! Note

Least Common Multiple (LCM) of Polynomials
To find the LCM of polynomials:
1. Factor each polynomial completely. Write any repeated factors as powers. For example, $x^3 + 6x^2 + 9x = x(x + 3)^2$.
2. List the different factors. If the polynomials have common factors, use the highest power of each common factor.

Find the least common multiple.

$2x^3y^4$ and $3x^5y^3$

Find the least common multiple.

$x^2 + 3x - 4$ and $x^2 - 3x + 2$

Example! Find the least common multiple for each pair.

$4x^3y^7$ and $3x^5y^4$	$x^2 - 4$ and $x^2 + 5x + 6$

GUIDED NOTES

To add or subtract rational expressions with unlike denominators, rewrite both expressions with the LCD. This process is similar to adding/subtracting fractions.

$$\frac{x}{x+3} + \frac{-18}{x^2-9}$$

$$\frac{2x^2-16}{x^2-4}-\frac{x+4}{x+2}$$

Example! Add or subtract. Identify any excluded values.

$\frac{3x}{2x-2}+\frac{3x-2}{3x-3}$	$\frac{x}{x+3}+\frac{2x+6}{x^2+6x+9}$
$\frac{3x-2}{2x+5}-\frac{2}{5x-2}$	$\frac{2x^2+64}{x^2-64}-\frac{x-4}{x+8}$

SELF CHECK — Simplify each expression. Identify any excluded values.

$\dfrac{3x-2}{x+6}+\dfrac{2x-2}{2x-1}$	$\dfrac{x+2}{x^2+4x+3}-\dfrac{x+1}{x+3}$

Questions To Ponder

1. Why does the rational expression $\frac{x^2+1}{x^2-1}$ have tow excluded values, but the expression $\frac{x^2-1}{x^2+1}$ have none?
2. How do you subtract rational expressions?

Find the least common multiple for each pair.

1. $3x^2y^6$ and $5x^3y^2$

2. $x^2 + x - 2$ and $x^2 - x - 6$

Add or subtract. Identify any *x*-values for which the expression is undefined.

3. $\dfrac{2x-3}{x+4} + \dfrac{4x-5}{x+4}$

4. $\dfrac{x+12}{2x-5} - \dfrac{3x-2}{2x-5}$

5. $\dfrac{x+4}{x^2-x-12} + \dfrac{2x}{x-4}$

6. $\dfrac{3x^2-1}{x^2-3x-18} - \dfrac{x+2}{x-6}$

7. $\dfrac{x+2}{x^2-2x-15} + \dfrac{x}{x+3}$

8. $\dfrac{x+6}{x^2-7x-18} - \dfrac{2x}{x-9}$

9. A messenger is required to deliver 10 packages per day. Each day, the messenger works only for as long as it takes to deliver the daily quota of 10 packages. On average, the messenger is able to deliver 2 packages per hour on Saturday and 4 packages per hour on Sunday. What is the messenger's average delivery rate on the weekend?

Add and Subtract Rational Expressions - Application

- Adding and Subtracting Rational Expressions

The idea of using the processes for operations with fractions to guide us as we operated with rational functions continues, but addition and subtraction may seem a little more involved. Just like with fractions it is necessary to have common denominators in both rational expressions before you can add or subtract. Think about the fraction addition problem $\frac{3}{10}+\frac{1}{6}$. What is the least common denominator (LCD)?

You might be able to quickly realize that 30 is the LCD, but why is it 30? Turn to your partner and explain a couple of ways of finding a common denominator.

When thinking about denominators like *x + 2* or *x – 3* it becomes important to understand what makes a LCD. In the fraction problem above, you might have been able to say that 30 is the LCD because it is the smallest number that both 10 and 6 divide into, but how do you create that number if it isn't obvious? (Hint: Think about prime factorization.)

When dealing with rational expressions, factoring is key. You must find all of the factors of each denominator to know what the LCD should be. Let's try some. Find the LCD for the following problems:

a. $\frac{3}{5a},\frac{b}{4a^2}$

b. $\frac{4}{x+5},\frac{3}{x-5}$

c. $\frac{2x}{x+2},\frac{x+1}{x^2-3x-10}$

d. $\frac{7}{x},\frac{5}{2x^2+3x}$

e. $\frac{x-9}{x^2+8x+16},\frac{x}{x^2+7x+12}$

Once you find the LCD, you complete the operation just like you would with fractions. Try these problems:

f. $\frac{3}{x+3}+\frac{2}{x-3}$

g. $\frac{6x+7}{x^2-4}+\frac{2}{x-2}$

h. $\frac{10}{6m^2}-\frac{2n}{5m^3}$

i. $\frac{6}{8a+4}+\frac{3a}{8}$

j. $\frac{x}{x^2+5x+6}-\frac{2}{x^2+3x+2}$

Unit 4B Concept 1 Lesson B

Simplify each expression.

1. $\frac{2x-4}{3x-6}$

2. $\frac{4x+8}{8x+4}$

3. $\frac{2x-10}{x^2-3x-10}$

Find each sum or difference.

4. $\frac{x-5}{x+3} + \frac{x+4}{x-2}$

5. $\frac{3x}{2x+6} + \frac{x}{x^2+7x+12}$

6. $\frac{1}{x^2+2x+1} + \frac{1}{x^2-1}$

7. $\frac{7}{x^2-4x+4} - \frac{3x}{x-2}$

8. $\frac{x+2}{x-1} - \frac{x+8}{3x^2+3x-6}$

9. $\frac{x}{x^2-2x+1} - \frac{x}{x^2-1}$

10. Anita exercises by running and walking. When she runs, she burns c Calories per minute for a total of 500 Calories. When she walks, she burns $(c-8)$ Calories per minute for a total of 100 Calories.

a. Write an expression for the time that she spends running and another expression for the time that she spends walking.

b. Write two equivalent expressions for her total time.

11. a. In the second row of the table, write the result after performing the operation given in the first row of the table.

$\frac{x}{x+2} + \frac{1}{x^2-4}$	$\frac{x}{x+2} - \frac{1}{x^2-4}$	$\frac{x}{x+2} \cdot \frac{1}{x^2-4}$	$\frac{x}{x+2} \div \frac{1}{x^2-4}$

A2.U4B.C2.**C.01.Vocab.SolveRationals**

Term	Definition	Diagram/Visual
Rational Expression		
Excluded Value		
Least Common Denominator		
Rational Equation		
Extraneous Solution		

Solving Rational Equations

GUIDED NOTES

Solve rational equations by first multiplying each term of the equation by ____________________ ______________________________ of all of the expressions in the equation. This step eliminates the denominators of the rational expressions and results in an equation you can solve by using algebra.

Solve the equation $x + \frac{8}{x} = 6$.

Solve the equation $\frac{1}{x^2+4x+4} - \frac{1}{x+2} = \frac{2}{x^2+4x+4}$

$$\frac{3x}{x-3}=\frac{2x+3}{x-3}$$

Example! Solve each equation. Be sure to check for extraneous solutions.

$\frac{10}{3}=\frac{4}{x}+2$	$\frac{16}{x^2-16}=\frac{2}{x-4}$
$\frac{x^2-29}{x^2-10x+21}=\frac{6}{x-7}+\frac{5}{x-3}$	$\frac{5}{x^2-3x+2}-\frac{1}{x-2}=\frac{x+6}{3x-3}$

SELF CHECK Solve each equation. Check for extraneous solutions.

$\frac{4x}{x-4}=\frac{2x+8}{x-4}$	$\frac{x}{x-2}+\frac{1}{x-4}=\frac{2}{x^2-6x+8}$

Real World Application of Solving Rational Equations

Rational equations are used to model real-world situations. These equations can be solved algebraically.

GUIDED NOTES

Kelsey is kayaking on a river. She travels 5 miles upstream and 5 miles downstream in a total of 6 hours. In still water, Kelsey can travel at an average speed of 3 miles per hour. What is the average speed of the river's current?

Identify the important information

- The answer will be the average speed of ______________________.
- Kelsey spends ______________________ kayaking.
- She travels __________upstream and ___________downstream.
- Her average speed in still water is _____________ .

Formulate a Plan

Let *c* represent the speed of the current in miles per hour. When Kelsey is going upstream, her speed is equal to her speed in still water _____________*c*. When Kelsey is going downstream, her speed is equal to her speed in still water __________ *c*. The variable *c* is restricted to ___________________ .

Complete the table.

	Distance (mi)	Average speed (mi/h)	Time (h)
Upstream	5		
Downstream	5		

Use the results from the table to write an equation.

total time = time upstream + time downstream

$$6 = \boxed{} + \boxed{}$$

Solve.

Use a rational equation to solve.

Kevin can clean a large aquarium tank in about 7 hours. When Kevin and Lara work together, they can clean the tank in 4 hours. Write and solve a rational equation to determine how long, to the nearest tenth of an hour, it would take Lara to clean the tank if she works by herself. Explain whether the answer is reasonable.

SELF CHECK

Solve each equation. Check for extraneous solutions.

A river barge travels at an average of 8 mi/h in still water. The barge travels 60 miles up the Mississippi River and 60 miles down the river in a total of 16.5 hours. What is the average speed of the current in this section of the Mississippi River? Round your answer to the nearest tenth. Is your answer reasonable?

Each month Leo must make copies of a budget report. When he uses both the large and the small copier, the job takes 30 minutes. If the small copier is broken, the job take him 50 minutes. How long will the job take if the large copier is broken?

1. How is the LCD of two rational expressions related to the denominator?
2. When is the LCD of two rational expressions the product of the denominators?

Unit 4B Concept 2

Solve each equation algebraically. Be sure to check for extraneous solutions.

1. $\frac{x}{2x-10} = 3$

2. $\frac{5}{2x+6} - \frac{1}{6} = \frac{2}{x+4}$

3. $\frac{9}{4x} - \frac{5}{6} = -\frac{13}{12x}$

4. $\frac{2x}{x-5} = \frac{3x^2-15x}{x^2-9x+20}$

5. $\frac{56}{x^2-2x-15} - \frac{6}{x+3} = \frac{7}{x-5}$

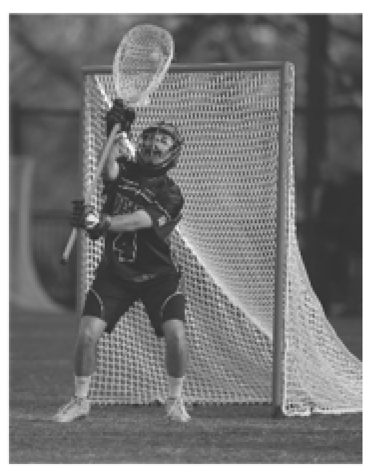
6. $\frac{3}{x^2+5x+4} = \frac{4x+3}{x+1} + \frac{1}{x+4}$

7. $\frac{x}{x+2} = \frac{6-x}{2x-1}$

8. $\frac{3}{x+1} + \frac{2}{7} = 2$

Write a rational equation for each real-world application. Do not solve.

9. A save percentage in lacrosse is found by dividing the number of saves by the number of shots faced. A lacrosse goalie saved 9 of 12 shots. How many additional consecutive saves *s* must the goalie make to raise his save percentage to 0.850?



10. Jake can mulch a garden in 30 minutes. Together, Jake and Ross can mulch the same garden in 16 minutes. How much time *t*, in minutes, will it take Ross to mulch the garden when working alone?

11. Geometry A new ice skating rink will be approximately rectangular in shape and will have an area of 18,000 square feet. Using an equation for the perimeter P, of the skating rink in terms of its width W, what are the dimensions of the skating rink if the perimeter is 580 feet?

12. Water flowing through both a small pipe and a large pipe can fill a water tank in 9 hours. Water flowing through the large pipe alone can fill the tank in 17 hours. Write an equation that can be used to find the amount of time t, in hours, it would take to fill the tank using only the small pipe.

13. A riverboat travels at an average of 14 km per hour in still water. The riverboat travels 110 km east up the Ohio River and 110 km west down the same river in a total of 17.5 hours. To the nearest tenth of a kilometer per hour, what was the speed of the current of the river?

Solving Rational Equations - Application

Rational Equations can be used to model some interesting real life phenomena. Distance, rate, and time problems as well as multi-person work problems are particularly suited to be modeled with a rational equation.

"Work" Problems

Two are better than one when it comes to completing a job. We can use rational equations to help us figure just how much better two can do a job than one person acting alone.

Jonah can paint a house by himself in 12 hours. Steve can do the same job in eight hours. How long will it take them to complete the job together? To help us answer this question we first need to think about how much of the job Jonah and Steve can do in one hour.

Let's let *t* = hours of work it takes to do job together. So to find out how much of the job they complete together in one hour, we will use this equation: $\frac{1}{12}+\frac{1}{8}=\frac{1}{t}$

Now we have to think about how to solve this equation. There are actually a few different ways to approach it. Talk to your partner to see what ways you can come up with together.

Equations are balanced statements and can be easily changed as long as you make sure to perform the same operation to every piece of the equation. This idea lets us multiply the equation by the LCD in order to create a simpler equation to solve. Try it with the equation above to see how long it will take Jonah and Steve to paint the house together.

Let's try another problem. Paul can paint a room two times as fast as Jamie. Working together they can paint the room in three hours. How long would it take each of them to paint the room alone?

Average Cost

Have you ever wondered why large retailers such as Wal-Mart can offer products as such low costs? The secret is in the quantity that they purchase. To understand this idea we are going to explore average cost.

At the Stir Mix-A-Lot blender company, the weekly cost to run the factory is $1400 and the cost of producing each blender is an additional $4 per blender.

a. Write a function rule representing the weekly cost in dollars, *C(x)*, of producing x blenders.

b. What is the total cost of producing 100 blenders in one week?

c. If you produce 100 blenders in one week, what is the total production cost per blender?

d. Will the total production cost per blender always be the same? Justify your answer.

e. Write a function rule representing the total production cost per blender *P(x)* for producing *x* blenders.

f. Using your graphing calculator, create a graph of your function rule from part e. Does the entire graph make sense for this situation? If not, what part does?

g. What is the production cost per blender if 300 blenders are produced in one week? If 500 blenders are produced in one week?

h. What happens to the total production cost per blenders as the number of blenders produced increases? Explain your answer.

i. How many blenders must be produced to have a total production cost per blender of $8?

j. The function for the production cost of the blenders is a rational function. What other information can we gather about this situation based on the characteristics of rational functions?

Pendulum

Tommy visited the Museum of History and Technology with his class. They saw Focault's Pendulum in Pendulum Hall and it was fascinating to Tommy. He knew from science class that the time it takes a pendulum to complete a full cycle or swing depends upon the length of the pendulum. The formula is given by $T = 2\pi\sqrt{\frac{L}{32}}$ where T represents the time in seconds and L represents the length of the pendulum in feet. He timed the swing of the pendulum with his watch and found that it took about 8 seconds for the pendulum to complete a full cycle. Help him figure out the length of the pendulum in feet.

Tommy thought that a pendulum that took a full 20 seconds to complete a full cycle would be very dramatic for a museum. How long must that pendulum be? If ceilings in the museum are about 20 feet high, would this pendulum be possible?

Solve each equation algebraically.

1. $\frac{1}{x}-\frac{x-2}{3x}=\frac{4}{3x}$

2. $\frac{5x-5}{x^2-4x}-\frac{5}{x^2-4x}=\frac{1}{x}$

3. $\frac{x^2-7x+10}{x}+\frac{1}{x}=x+4$

4. $\frac{4}{x^2-4}=\frac{1}{x-2}$

Solve.

5. A glassblower can produce a set of simple glasses in about 2 hours. When the glassblower works with an apprentice, the job takes about 1.5 hours. How long would it take the apprentice to make a set of glasses when working alone?

A2.U4B.C3.**D.01.Vocab.GraphSimpleRationals**

Term	Definition	Diagram/Visual
Rational Function		
Transformations		
Domain		
Range		
End Behavior		
X-Intercept		
Y-Intercept		
Asymptotes		

A2.U4B.C3.**D.02.Notes.GraphSimpleRationals**

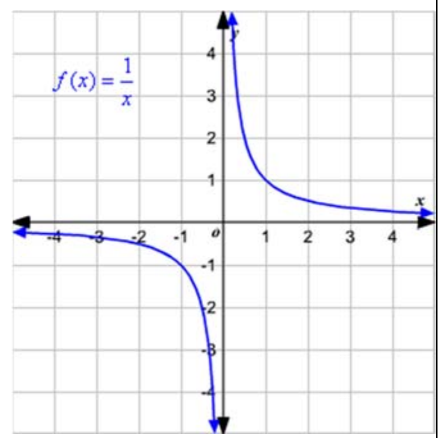
Parent Rational Functions

In this lesson, we will graph and describe simple rational functions.

GUIDED NOTES

The parent square root function is: ______________. The graph of a rational function is a

_____________________, which has two separate branches.



Below are the characteristics of the parent rational function.

Domain: **Range**:

***x*-intercept:** ***y*-intercept**:

End Behavior:

Vertical Asymptote: **Horizontal Asymptote:**

Transformations of Rational Functions

When listing the characteristics of each function, we must also describe the transformations from the parent function.

GUIDED NOTES

Each part of the function helps you describe the transformations. If the value of a is multiplied in, it becomes the numerator of the fraction. You will also see this form.

List the transformations for each function from the parent rational function.

$g(x) = 0.5\left(\frac{1}{x+2}\right) + 4$ $h(x) = \frac{1}{4x} + 1$ $f(x) = \frac{2}{-(x+5)}$ $k(x) = -\frac{1}{x-4} - 2$

Graphing of Simple Rational Functions

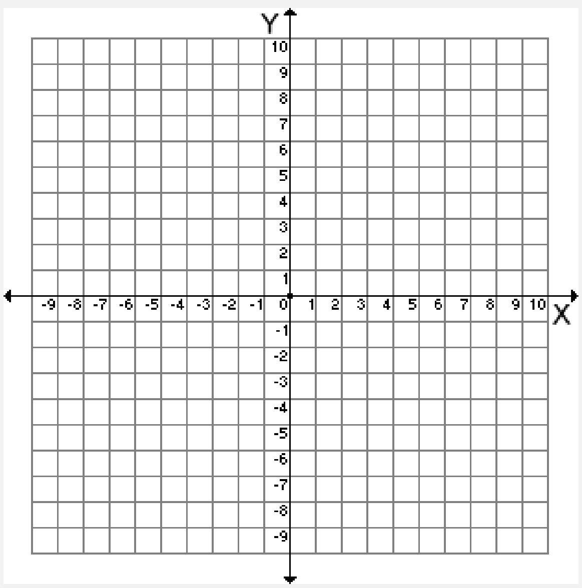
GUIDED NOTES

In order to sketch the graph of simple rational functions, you need the asymptotes, and to reference points.

Feature	$f(x) = \frac{1}{x}$	$g(x) = a\left(\frac{1}{\frac{1}{b}(x-h)}\right) + k$
Vertical asymptote	$x = 0$	$x = h$
Horizontal asymptote	$y = 0$	$y = k$
Reference point	$(-1, -1)$	$(-b + h, -a + k)$
Reference point	$(1, 1)$	$(b + h, a + k)$

Example! Graph and describe each function.

$$f(x) = 3\left(\frac{1}{x+1}\right) - 2$$



Transformations:

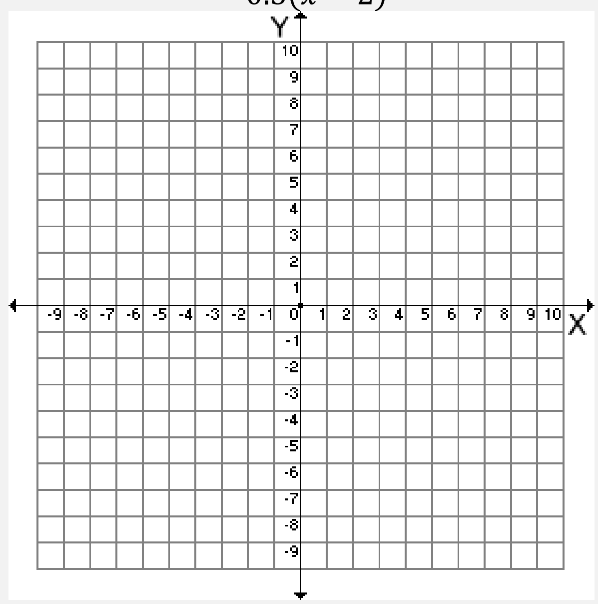
Domain: **Range**:

x-intercept: **y-intercept**:

Vert. Asymp: **Horiz. Asymp**:

End Behavior: $As\ x \to$ ____, $y \to$ ____
$As\ x \to$ ____, $y \to$ ____

$$y = \frac{1}{-0.5(x-2)} + 1$$



Transformations:

Domain: **Range**:

x-intercept: **y-intercept**:

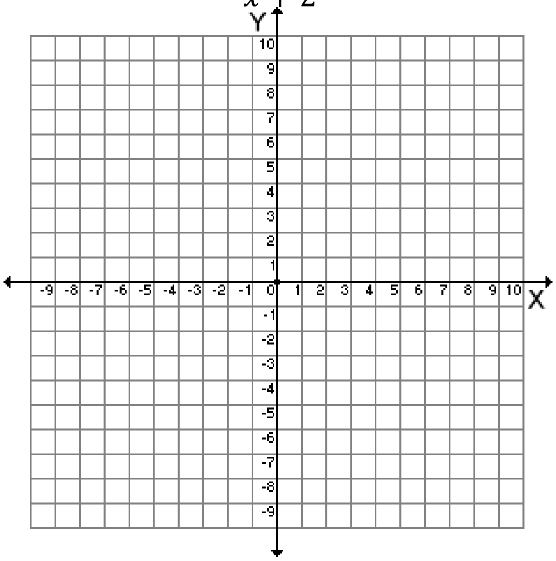
Vert. Asymp: **Horiz. Asymp**:

End Behavior: $As\ x \to$ ____, $y \to$ ____
$As\ x \to$ ____, $y \to$ ____

SELF CHECK

Graph and describe each function.

$$y = \frac{-2}{x+2} - 4$$



Transformations:

Domain:

Range:

x-intercept:

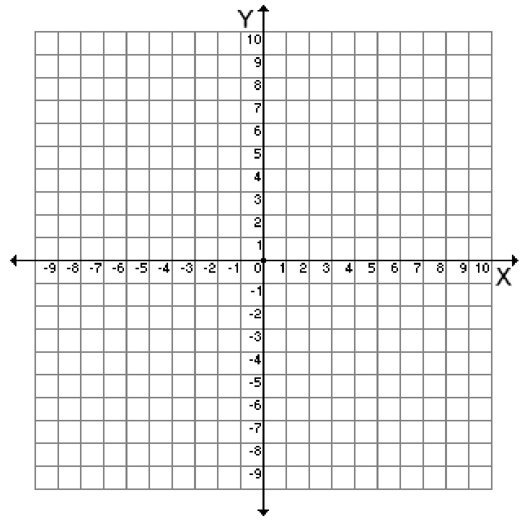
y-intercept:

Vert. Asymp:

Horiz. Asymp:

End Behavior: $As\ x \to$ _____, $y \to$ _____
$As\ x \to$ _____, $y \to$ _____

$$y = \frac{1}{2(x-3)} - 1$$



Transformations:

Domain:

Range:

x-intercept:

y-intercept:

Vert. Asymp:

Horiz. Asymp:

End Behavior: $As\ x \to$ _____, $y \to$ _____
$As\ x \to$ _____, $y \to$ _____

Questions To Ponder

1. Why are the domain and range of the graph of rational functions disjunctions or inequalities, and not conjunctions?

2. Why are there two parts to graph of the rational function?

Unit 4B Concept 3 Lesson A

Describe how the graph of $g(x)$ is related to the graph of $f(x) = \frac{1}{x}$.

1. $g(x) = \frac{1}{x} + 4$

2. $g(x) = 5\left(\frac{1}{x}\right)$

3. $g(x) = \frac{1}{x+3}$

4. $g(x) = \frac{1}{0.1x}$

5. $g(x) = \frac{1}{x} - 7$

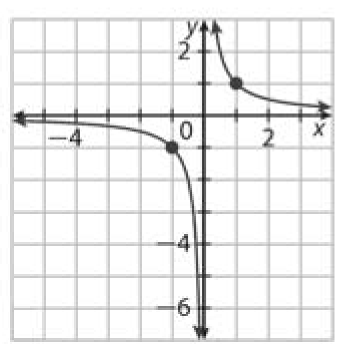
6. $g(x) = \frac{1}{x-8}$

7. $g(x) = -0.1\left(\frac{1}{x}\right)$

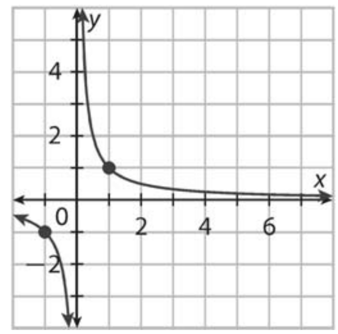
8. $g(x) = \frac{1}{-3x}$

Identify the transformations, domain, range, intercepts, asymptotes, and end behavior of each function. Then graph the function on the graph with the parent rational function.

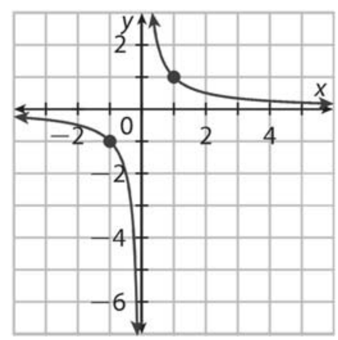
9. $g(x) = 3\left(\frac{1}{x+1}\right) - 2$



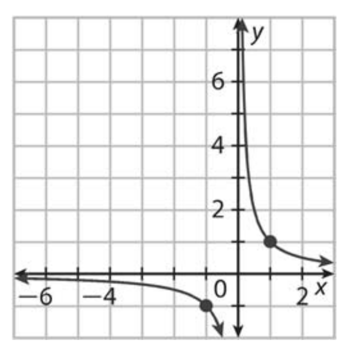
10. $g(x) = \dfrac{1}{-0.5(x-3)} + 1$



11. $g(x) = -0.5\left(\dfrac{1}{x-1}\right) - 2$



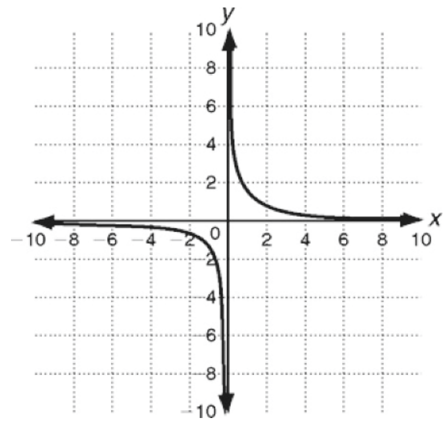
12. $g(x) = \dfrac{1}{2(x+2)} + 3$



Graphing Simple Rational Functions - Application

Desmos Activity to Explore Rational Functions

Using the graph of $f(x)=\frac{1}{x}$ as a guide, describe the transformation and graph the function.



1. $g(x)=\frac{2}{x+4}$

2. $y=\frac{2}{x-5}-3$

3. $y=\frac{5}{x+3}-1$

4. $y=\frac{-2}{x-4}+6$

5. $y=\frac{-1}{x}+7$

Identify the asymptotes, domain, and range of each function.

6. $g(x)=\frac{1}{x-3}+5$

7. $g(x)=\frac{1}{x+8}-1$

8. $g(x)=\frac{1}{x+5}+7$

9. $g(x)=\frac{4}{x-9}-\frac{1}{4}$

10. $g(x)=\frac{1}{x+\frac{2}{3}}-12$

A2.U4B.C3.**E.O1.Vocab.GraphComplexRationals**

Term	Definition	Diagram/Visual
Rational Function		
Transformations		
Domain		
Range		
End Behavior		
X-Intercept		
Y-Intercept		
Asymptotes		
Discontinuous Function		
Holes		

Describing Complex Rational Functions

The graphs of some rational functions are not hyperbolas. Also, some rational functions do not have horizontal asymptotes. Complex rational functions may have slant asymptotes.

GUIDED NOTES

Complex rational functions only one numerator and the denominator.

It is possible to have ______________ in the graph of a ________________ ________________.

Before putting the rational function into lowest terms, factor the numerator and denominator. If there is the same factor in the numerator and denominator, there is a ______________. Set this factor equal to zero and solve. The solution is the x-value of the hole. Now simplify the rational function (cross out the factor that is the numerator and denominator). Put the x-value of the hole into the simplified rational function. This will give the y-value of the hole. (Holes may not be visible on your calculator.)

When finding __________________ always write the __________________ __________________ in lowest terms. It is best not to have the function in factored form

To find the __________________ ________________, set the denominator equation to zero and solve for x. The equation for a vertical asymptote is written $x = k$, where k is the solution from setting the denominator to zero.

If you substitute k into the rational function and it equals zero in the numerator and denominator, then you have not written the function in lowest terms.

To find the ______________________ __________________, there are 3 cases. Let N be the degree of the numerator, and D be the degree of the denominator

1) If $N < D$, then the horizontal asymptotes is $y = 0$
2) If $N = D$, then the horizontal asymptote is $y = \frac{a}{b}$ (where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator).
3) If $N > D$, then there is no horizontal asymptote.

Important Note: The graph of a rational function can sometimes cross a horizontal asymptote.

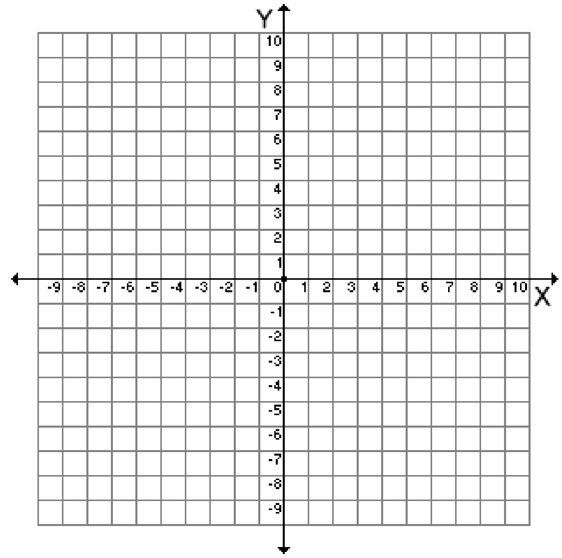
If the degree of the numerator is 1 more than the degree of the denominator, then there is an ______________ ____________________. To find the _________________ ______________, divide the numerator by the denominator. The quotient is the equation for the slant asymptote. Just ignore the remainder.

***Important Note:* A rational function will either have a horizontal or oblique asymptote but not both.**

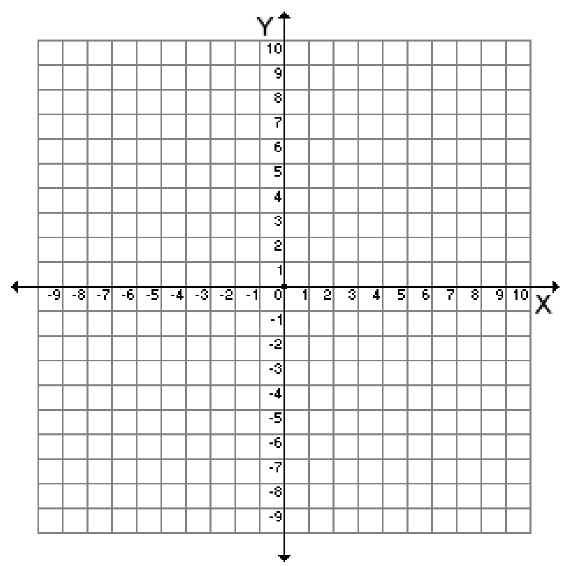
Graphing Complex Rational Functions

GUIDED NOTES

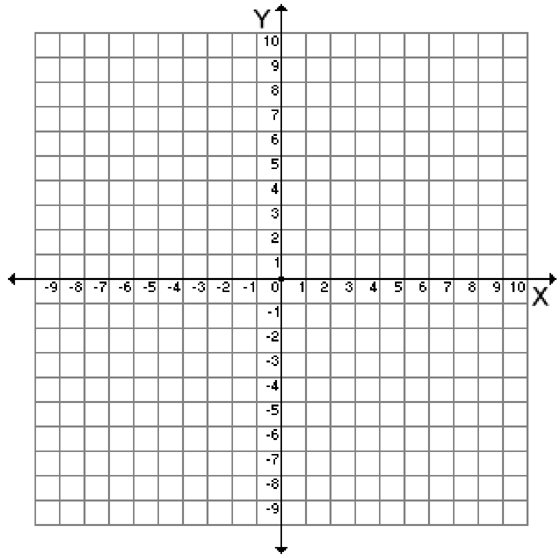
Identify the zeros and vertical asymptotes of $f(x) = \dfrac{x^2 - 2x - 3}{x - 2}$. Then graph.



$f(x) = \dfrac{2x^2 - 2}{x^2 - 4}$

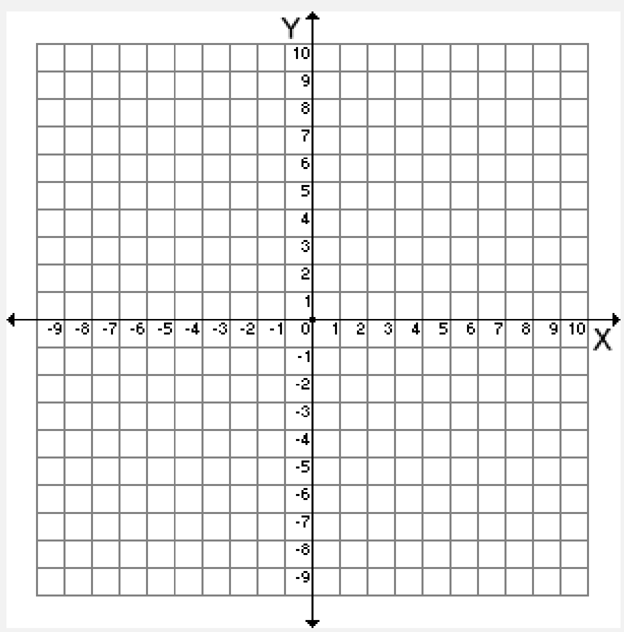


Identify holes in the graph of $f(x) = \frac{x^2 - 4}{x + 2}$. Then graph.



Example! Graph and describe each function.

$$f(x) = \frac{x^2 + 7x + 12}{x^2 + 2x - 8}$$



Hole:

Domain:

Range:

x-intercept:

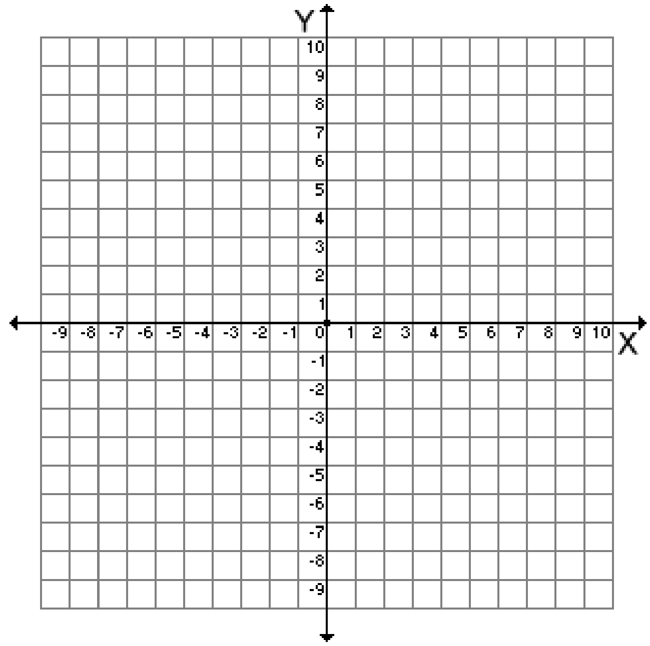
y-intercept:

Vert. Asymp:

Horiz./Slant Asymp:

SELF CHECK Graph and describe each function.

$$y = \frac{x^2 - 16}{3x - 9}$$



Hole:

Domain: **Range**:

x-intercept: **y-intercept**:

Vert. Asymp:

Horiz./Slant Asymp:

Questions To Ponder

1. What features of the graph of a rational function should you identify in order to sketch the graph? How do you identify these features?

2. How do you find the holes of a rational functions?

Unit 4B Concept 3

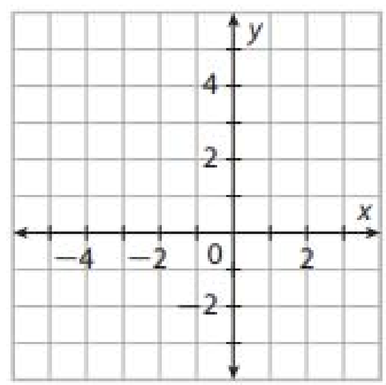
State the domain for each function. For any x-value excluded from the domain, state whether the graph has a vertical asymptote or a "hole" at that x-value. Use a graphing calculator to check your answer.

1. $f(x) = \frac{x+5}{x+1}$

2. $f(x) = \frac{x^2+2x-3}{x^2-4x+3}$

Sketch the graph of the given rational function. State the asymptotes, domain, and range.

3. $f(x) = \frac{x-1}{x+1}$



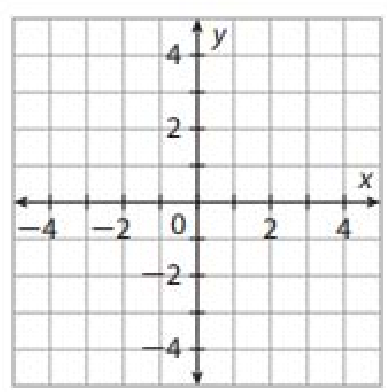
Vertical Asymptote:

Horizontal/Oblique Asymptote:

Domain:

Range:

4. $f(x) = \frac{x-1}{(x-2)(x+3)}$



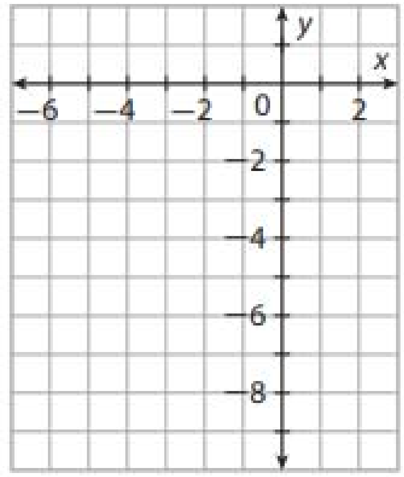
Vertical Asymptote:

Horizontal/Oblique Asymptote:

Domain:

Range:

5. $f(x) = \frac{(x+1)(x-1)}{x+2}$



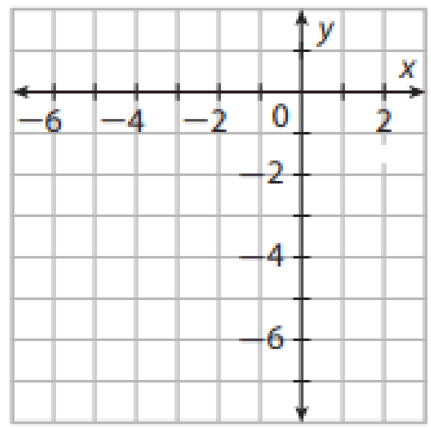
Vertical Asymptote:

Horizontal/Oblique Asymptote:

Domain:

Range:

6. $f(x) = \frac{-3x(x-2)}{(x-2)(x+2)}$



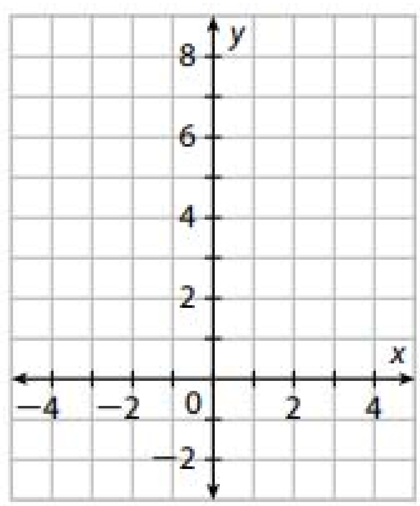
Vertical Asymptote:

Horizontal/Oblique Asymptote:

Domain:

Range:

7. $f(x) = \frac{x^2+2x-8}{x-1}$



Vertical Asymptote:

Horizontal/Oblique Asymptote:

Domain:

Range:

Graphing Complex Rational Functions - Application

Characteristics of Rational Functions

Now that we have worked with rational expressions, it is time to look at rational functions themselves. Since a rational function is the quotient of two polynomial functions it is important to first look at the characteristics of the individual polynomials.

Let's invistigate $g(x) = x^2 + 3x - 10$. What facts can you write about g(x)?

What is the Domain? How do you determine the Domain?

What is the Range? How do you determine the Range?

Where are the Roots or Zeros found? What are some different ways you know to find them?

What is the End Behavior? How do you know?

Let's investigate $f(x) = x + 1$. What facts can you write about f(x)?

What is the Domain?

What is the Range?

What are the Roots or Zeros?

What is the End Behavior? How do you know?

Now let's consider the case of the rational function $r(x) = \frac{f(x)}{g(x)}$ where f and g are the polynomial functions above. Write the expression for the function f(x).

What is the domain of $r(x)$? Which function, f or g, affects the domain the most? Why?

What do you think the range of $r(x)$ will be? Why is this so difficult to determine?

What are the roots or zeros of $r(x)$? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior?

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

Based on the graph from your calculator, what is the range of $r(x)$?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Let's try a few more problems and see if we can discover any patterns…

1. Let $f(x) = 5$ and $g(x) = x^2 - 6x + 8$. Let $r(x) = \dfrac{f(x)}{g(x)}$.

What is the domain of $r(x)$? Which function, f or g, affects the domain the most? Why?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

2. Let $r(x) = \dfrac{2x^2 + 7x - 4}{x^3 - 1}$.

What is the domain of $r(x)$?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

3. Let $r(x) = \dfrac{4x+1}{4-x}$.

What is the domain of $r(x)$?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

Now let's summarize our findings and conclusions.

When is the domain of a rational function not $(-\infty, \infty)$? So what is your advice on how to determine the domain of a rational function?

Is the range of a rational function difficult to find? Why or why not?

How do you find the zeros or roots of a rational function?

How do you know where to find vertical asymptotes?

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

.

How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

What possible things could occur at the x-values where a rational function changes signs?

Do you think it would be possible to use all of our knowledge of rational functions to create sketch without using the graphing calculator? Can you explain how this would work to another classmate?

Graphing Complex Rational Functions - Application

Let $r(x) = \dfrac{3x^2 + 27}{2x^2 - 6x - 8}$.

What is the domain of $r(x)$?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

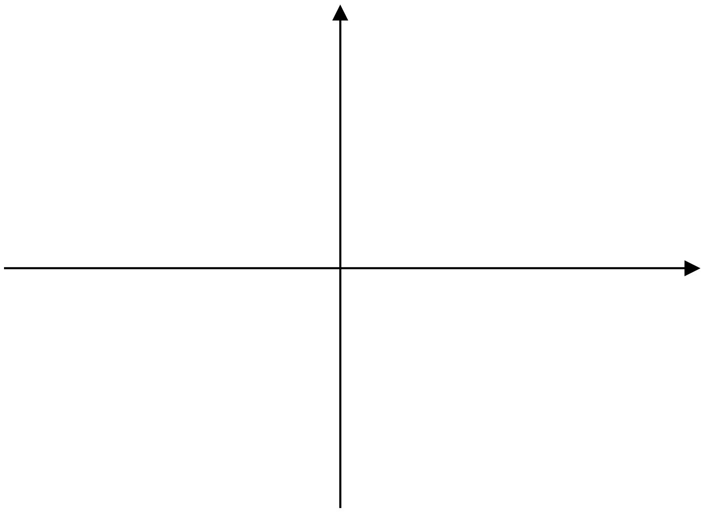
Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

Where is $r(x) > 0$? Where is $r(x) < 0$? (*Hint: use a sign chart)*

Now let's try to sketch the graph of $r(x)$ without using your calculator.



Based on your sketch, what do you think the range of $r(x)$ will be?

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

Try this one next. Let $r(x) = \dfrac{x^3 + 1}{3x^3 - 27x}$

What is the domain of $r(x)$?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

Where is $r(x) > 0$? Where is $r(x) < 0$?

Now let's try to sketch the graph of $r(x)$ without using your calculator.

Based on your sketch, what do you think the range of $r(x)$ will be?

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

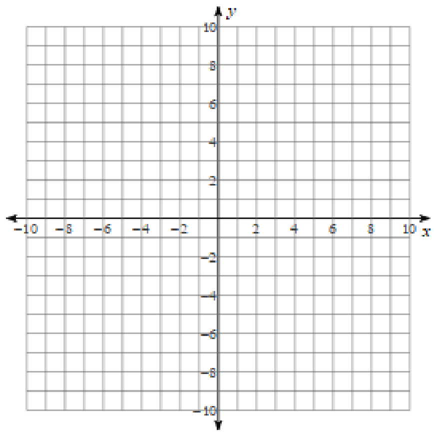
Based on the graph from your calculator, what is the range of $r(x)$?

In your own words, describe the process that you would go through in order to create a sketch of any rational function.

.

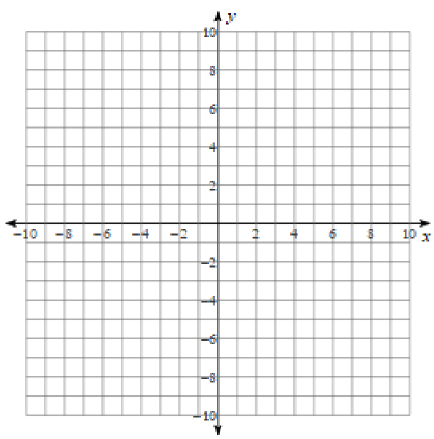
<u>Graph the function and label the following information. Horizontal Asymptotes can include slant asymptotes.</u>

1. $y = \dfrac{x^2 + 4x - 5}{x + 1}$



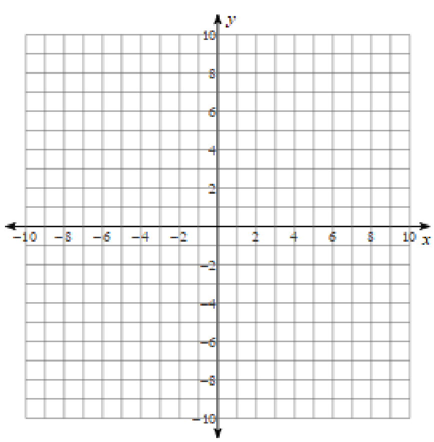
Zeros:	
Vertical Asymptotes:	
Horizontal Asymptotes:	
Holes:	
Y-Intercept(s):	
Domain:	
range	

2. $y = \dfrac{x^2 + 5x + 6}{x^2 - 9}$



Zeros:	
Vertical Asymptotes:	
Horizontal Asymptotes:	
Holes:	
Y-Intercept(s):	
Domain:	
range	

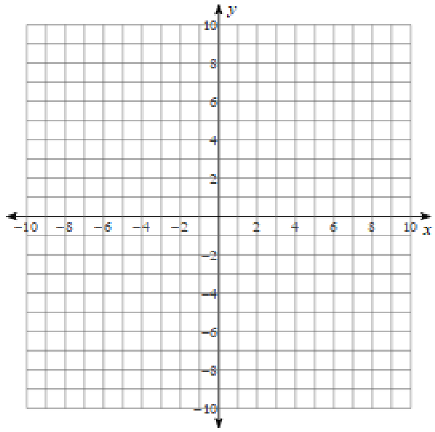
3. $y = \dfrac{x^2 - 4}{3x^2 - 15x + 18}$



Zeros:	
Vertical Asymptotes:	
Horizontal Asymptotes:	
Holes:	
Y-Intercept(s):	
Domain:	
range	

A2.U4B.C3.**E.O5.HW.GraphComplexRationals**

4. $f(x)=\dfrac{1}{x+4}-3$



Zeros:	
Vertical Asymptotes:	
Horizontal Asymptotes:	
Holes:	
Y-Intercept(s):	
Domain:	
range	

Match the following graphs with the equation

5. $f(x)=\dfrac{x^3-9x}{3x^2-6x-9}$ **6.** $f(x)=\dfrac{x^2-x-12}{x^2-2x-8}$ **7.** $f(x)=\dfrac{x^3+1}{x^2-1}$

factor ________________ ________________ ________________

HA: ________________ ________________ ________________

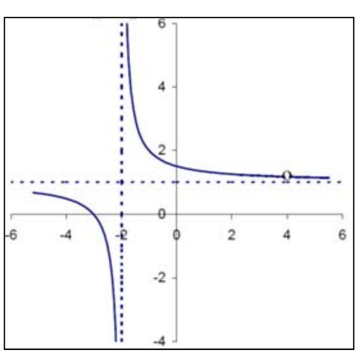
VA: ________________ ________________ ________________

roots: ________________ ________________ ________________

holes: ________________ ________________ ________________

graph: A B C graph: A B C graph: A B C

A.



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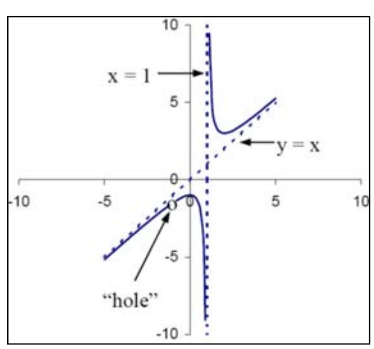
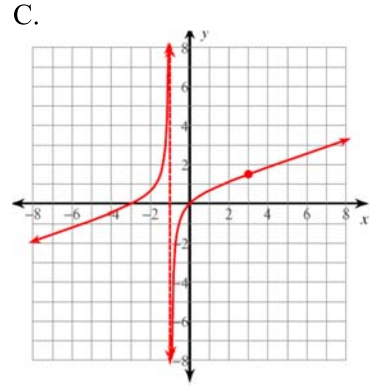


C.



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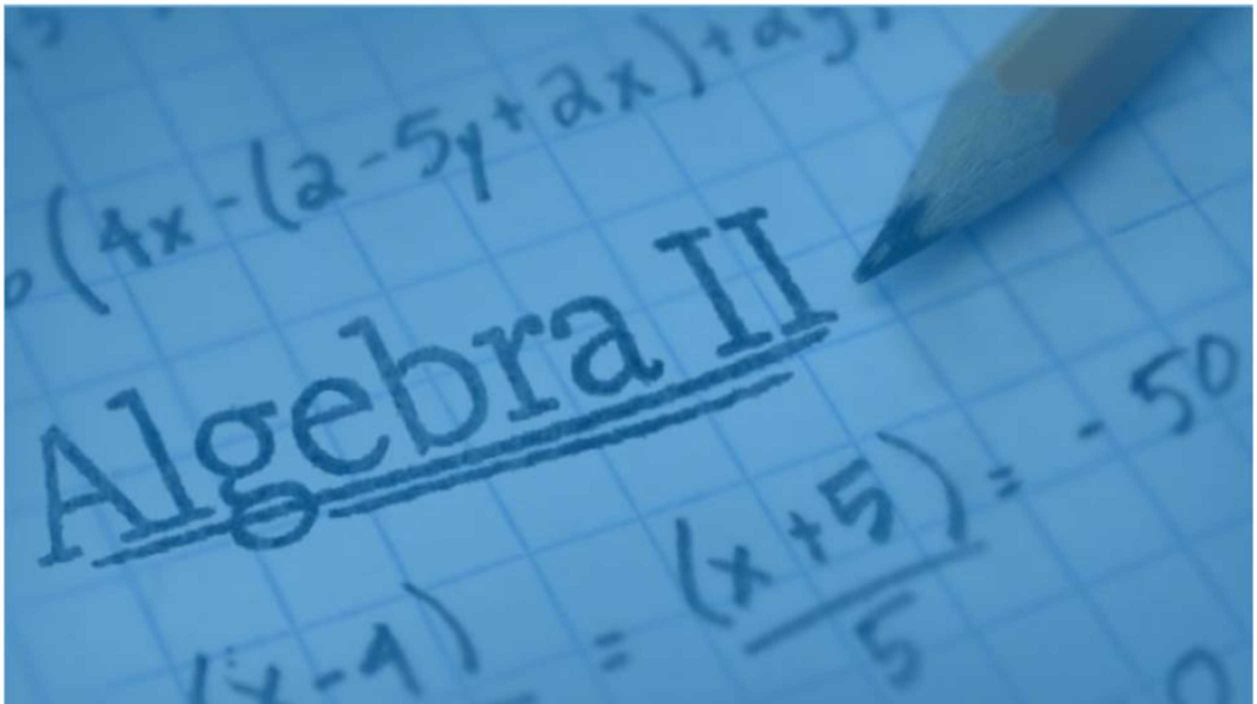


Unit 5
Exponential and Logarithmic Functions

Algebra 2

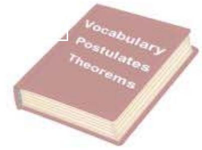
Unit 5: Exponential and Logarithmic Functions

Concept 1: Properties and Equations of Exponentials and Logarithms

Lesson A: Exponent Properties Review	(A2.U5.C1.A.____.ExponentProperties)
Lesson B: Solving Exponential Equations	(A2.U5.C1.B.____.SolvingExpEq)
Lesson C: Introduction to Logarithms	(A2.U5.C1.C.____.IntroLogs)
Lesson D: Logarithms as Inverses	(A2.U5.C1.D.____.LogAsInverse)
Lesson E: Properties of Logarithms	(A2.U5.C1.E.____.LogProperties)
Lesson F: Solving Logarithmic Equations	(A2.U5.C1.F.____.SolveLogEquations)

Concept 2: Graphing Exponential and Logarithmic Functions

Lesson G: Graph and Describe Exponential Functions	(A2.U5.C2.G.___.GraphExponential)
Lesson H: Applications of Exponential Functions	(A2.U5.C2.H.___.ApplicationExp)
Lesson I: Graph and Describe Logarithmic Functions	(A2.U5.C2.I.___.GraphLogs)



Vocabulary

Term	Definition	Notation	Diagram/Visual
Asymptote			
Common Logarithm			
Exponential Functions			
Logarithm Functions			
Logarithm			
Natural Exponential			
Natural Logarithm			

Exponent Properties Review

To help us solve exponential equations in this unit, we must first review our exponent properties.

GUIDED NOTES

Exponent Properties

Product of Powers	
Power of a Power	
Power of a Product	
Negative Exponent	
Zero Exponent	
Quotient of Powers	
Power of a Quotient	

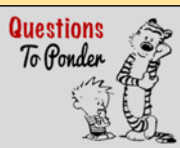
Example!

Exponent Properties	Example
Product of Powers: $a^m \bullet a^n = a^{m+n}$	$4^2 \cdot 4^3 = 4^{2+3} = 4^5$
Power of a Power: $(a^m)^n = a^{mn}$	$(4^2)^3 = 4^{2\cdot 3} = 4^6$
Power of a Product: $(ab)^m = a^m b^m$	$(2x^2)^3 = 2^3\,(x^2)^3 = 2^3x^6 = 8x^6$
Negative Exponent: $a^{-m} = \frac{1}{a^m}$ where $a \neq 0$ and $\frac{1}{a^{-m}} = a^m$ where $a \neq 0$	$x^{-5} = \frac{1}{x^5}$ $\frac{1}{x^{-3}} = x^3$
Zero Exponent: $a^0 = 1$ where $a \neq 0$	$(5)^0 = 1$

Example!

Exponent Properties	Examples
Product of Powers: $a^m \bullet a^n = a^{m+n}$	$2^8 \cdot 2^{11} =$ $2x^8 \cdot 5x^{10} \cdot 4x^3 =$
Power of a Power: $(a^m)^n = a^{mn}$	$(2^3)^4 =$

	$(c^4)^7 =$
Power of a Product: $(ab)^m = a^m b^m$	$(3x^3)^4 =$ $(2x^6y^3)^4 =$
Negative Exponent: $a^{-m} = \frac{1}{a^m}$ where $a \neq 0$ and $\frac{1}{a^{-m}} = a^m$	$z^{-8} =$ $2x^{-4} =$
Zero Exponent: $a^0 = 1$ where $a \neq 0$	$2y^0 =$ $(4x^3y^6)^0 =$



Which example problems use multiple exponent properties? What were the properties needed to simplify?

Simplify. Your answers should only contain positive exponents.

1) $2v^2 \cdot 3v$

2) $2x \cdot 2x^3$

3) $3x \cdot 3x^2$

4) $2a^3 \cdot 2a^0$

5) $\left(xy^2 \cdot 2yx^4\right)^{-1}$

6) $\left(-m^0n^4\right)^5\left(-n^{-5}\right)^{-2}$

7) $-2x^{-1}y^{-1} \cdot \left(2xy^{-4}\right)^4$

8) $m^{-1}n^0 \cdot \left(-2m^{-4}n^{-1}\right)^3$

A2.U5.C1.**A.05.HWK.ExponentProperties**

Simplify. Your answer should only contain positive exponents.

1) $2m^2 \cdot 2m^3$

2) $m^4 \cdot 2m^{-3}$

3) $4r^{-3} \cdot 2r^2$

4) $4n^4 \cdot 2n^{-3}$

5) $2k^4 \cdot 4k$

6) $2x^3y^{-3} \cdot 2x^{-1}y^3$

7) $2y^2 \cdot 3x$

8) $4v^3 \cdot vu^2$

9) $4a^3b^2 \cdot 3a^{-4}b^{-3}$

10) $x^2y^{-4} \cdot x^3y^2$

11) $(x^2)^0$

12) $(2x^2)^{-4}$

13) $(4r^0)^4$

14) $(4a^3)^2$

15) $\left(3k^4\right)^4$

16) $(4xy)^{-1}$

17) $\left(2b^4\right)^{-1}$

18) $\left(x^2y^{-1}\right)^2$

19) $\left(2x^4y^{-3}\right)^{-1}$

20) $(3m)^{-2}$

21) $\dfrac{r^2}{2r^3}$

22) $\dfrac{x^{-1}}{4x^4}$

23) $\dfrac{3n^4}{3n^3}$

24) $\dfrac{m^4}{2m^4}$

25) $\dfrac{3m^{-4}}{m^3}$

26) $\dfrac{2x^4y^{-4}z^{-3}}{3x^2y^{-3}z^4}$

Solving Exponential Equations

Previously, we learned how to simplify expressions with exponents using exponent properties. Today, we will use exponent properties to help us solve exponential equations.

How to Solve Exponential Functions

1. _____________ the exponential expressions on either side of the _______. We then rewrite the expression so that the _________ are the __________.
2. Set the ______________ equal to each other (___________________) and ________ the resulting equation.

Example!

Solve for the variable.

$$2^{8x-3} = 2^{5x}$$

Step 1 - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

This step has already been completed, since there is an exponential expression on either side of the equal sign. In addition, the bases are the same (the base equals 2).

Step 2 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

$$2^{8x-3} = 2^{5x}$$

$$8x - 3 = 5x$$

$$3x - 3 = 0$$

$$3x = 3$$

$$\boxed{x = 1}$$

Example!

Solve for the variable.

$$8^x = 2^{x+6}$$

<u>Step 1</u> - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

There is an exponential expression on either side of the equal sign. HOWEVER, the bases are NOT the same.

Therefore, rewrite the expression such that the bases are on the same on both sides of the equal sign. We typically use the smallest base.

$$8^x = 2^{x+6}$$

$$(2^3)^x = 2^{x+6}$$

$$2^{3x} = 2^{x+6}$$

<u>Step 2</u> - **Set the exponents equal to each other (drop the bases) and solve the resulting equation.**

$$3x = x + 6$$

$$2x = 6$$

$$\boxed{x = 3}$$

Example!

Solve for the variable.

$$8^{-3n} = 4^{3-2n}$$

<u>Step 1</u> - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

There is an exponential expression on either side of the equal sign. HOWEVER, the bases are NOT the same.

Therefore, rewrite the expression such that the bases are on the same on both sides of the equal sign. We typically use the smallest base. In this case, we cannot use 4 as our base. This is because it is not possible to raise 4 to a power and get 8.

Thus, we use an even smaller base (base 2) on both sides of the equation to solve.

$$8^{-3n} = 4^{3-2n}$$

$$(2^3)^{-3n} = (2^2)^{3-2n}$$

$$2^{-9n} = 2^{6-4n}$$

<u>Step 2</u> - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

$$-9n = 6 - 4n$$

$$-5n = 6$$

$$\boxed{n = -\frac{6}{5}}$$

Solve for the variable.

$$4^{-3b-2} = 1$$

<u>Step 1</u> - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

There is an exponential expression on either side of the equal sign. HOWEVER, the bases are NOT the same.

Therefore, rewrite the expression such that the bases are on the same on both sides of the equal sign. We typically use the smallest base. In this case, we cannot use 1 as our base. This is because it is not possible to raise 1 to a power and get 4.

Thus, we will use the larger of the two bases and apply the zero exponent property.

$$4^{-3b-2} = 1$$

$$4^{-3b-2} = 4^0$$

<u>Step 2</u> - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

$$-3b - 2 = 0$$

$$-3b = 2$$

$$\boxed{b = -\frac{2}{3}}$$

Example!

Solve for the variable.

$$3^{2x+1} - 5 = 4$$

Step 1 - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

To isolate the exponential expression, we need to add 5 to the other side.

$$3^{2x+1} - 5 = 4$$

$$3^{2x+1} = 9$$

Now, we need to rewrite the expression so that the bases are the same.

$$3^{2x+1} = 3^2$$

Step 2 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

$$2x + 1 = 2$$

$$2x = 1$$

$$\boxed{x = \frac{1}{2}}$$

Example!

Solve for the variable.

$$10^{1-x} = 10^4$$

Step 1 - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

Step 2 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

Example!

Solve for the variable.

$$3^{2x-1} = 27$$

Step 1 - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

Step 2 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

Questions To Ponder

How can you solve an exponential equation with a fraction? Which exponent property allows us to eliminate the fraction?

Example!

Solve for the variable.

$$\mathbf{4^{x+1} = \frac{1}{64}}$$

<u>Step 1</u> - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

<u>Step 2</u> - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

Example!

Solve for the variable.

$$\mathbf{6^{3k+2} = 1}$$

<u>Step 1</u> - Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.

<u>Step 2</u> - Set the exponents equal to each other (drop the bases) and solve the resulting equation.

Solve each equation.

1. $4^{x+1} = 4^9$

2. $(\frac{1}{2})^x = 4$

3. $4^{2x} + 1 = 65$

4. $4^3 = 2^x$

Zombie (Zom-bean) Apocalypse Simulation

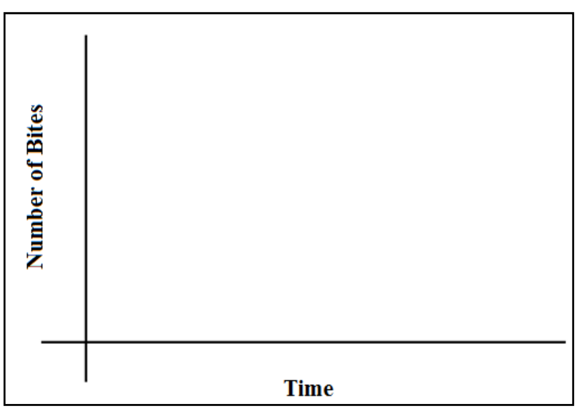
The situation:

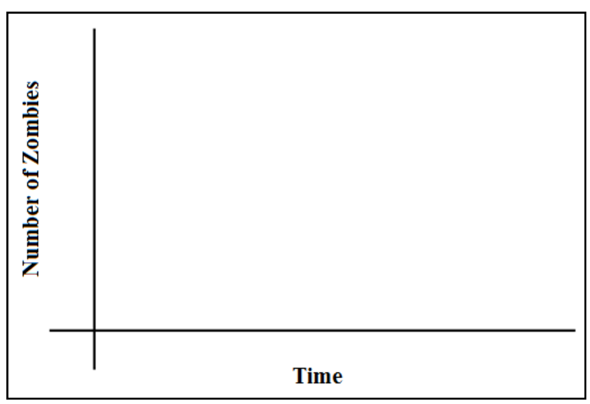
At exactly 8am, a Zombie staggers into a school building of healthy, disease free people. It takes a normal, disease-free person four hours to turn into a Zombie once bitten. In this exploration, you will use a model to examine the spread of the Zombie pathogen. You will need a cup of pinto beans which represents people and a cup of another type of beans which represents zombies.

TABLE 1: Simulated Spread of a Disease

Shake Number (x)	Number of Bites (*b(x)*)	Number of Zombies (*z(x)*)
0		
1		
2		
3		
4		
5		
6		

1. What times correlate to each Shake Number? Write those in the space to the left of the Shake Numbers in the chart.
2. Graph *b(x)* and z(x) on their respective axes, below, being sure to label completely.





3. What is the difference between the meanings of *b(x)* and *z(x)* at any given time?

4. About how many more shakes do you think it would take for your entire population of pinto bean people to be Zombies?

5. What does the y-intercept on your graph represent, as far as diseased/healthy people?

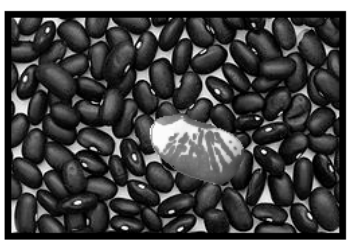
6. About how many healthy people did your diseased person infect in a single interaction?

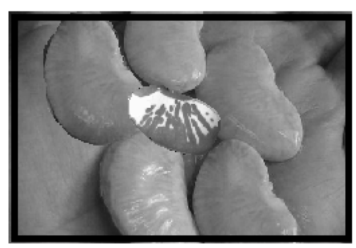
7. How does the size of the pinto bean vs. the Zombie bean affect the number of bites?

8. Now come together with another group whose "diseased person" was a different type of bean than the one that you measured, preferably one with a different size, and compare your answers to 1-4 with theirs. Do you notice any differences? If so, why do you think one "diseased person" affects the healthy population differently?

9. For each shake for your group and your neighbors, describe how fast the number of zombies changed mathematically compared with the last shake. What is the pattern or relationship? (It won't be exact because this is a simulation, but you should see a relationship between a shake and the one before it that is similar to the relationship between a shake and the one after).
10. Noting the relationships that you and your neighbors noticed #12, what do you think will happen from one shake of a box to another in the photos below, if the speckled bean is the "Zombie?"


a.


b.


c.


d.

Solve each equation.

1) $5^{3b+2} = 5^{-b-2}$

2) $2^{-2m} = 2^{3-2m}$

3) $4^{-3n} = 16$

4) $4^{-3b-3} = 4^{2b}$

5) $3^{-3m-2} = 3^{-m}$

6) $2^{-2n} = 2^{-n-2}$

7) $3^{-2n} = 3^{-2n}$

8) $3^{3n+3} = 3^{3n}$

9) $2^{2b-1} = 2^{-3b}$

10) $2^{3b+1} = 2^{2b}$

11) $3^{2m+3} = 3^{2m}$

12) $4^{-3b-2} = 1$

13) $4^{v-2} = 16$

14) $25^{2v+3} = \frac{1}{125}$

15) $6^{2n+3} = 36$

16) $4^{-n} = 4^{-2n}$

17) $36^{2n} = \frac{1}{6}$

18) $3^{2n-1} = \frac{1}{9}$

19) $5^{-2b} = 125$

20) $\left(\frac{1}{3}\right)^{3n} = 27^{2n}$

21) $5^{-2n} = 5^{-2n}$

22) $32^{n} = 4$

23) $36^{-x} = 216^{-2x+2}$

24) $\left(\frac{1}{64}\right)^{-3p} = \frac{1}{16}$

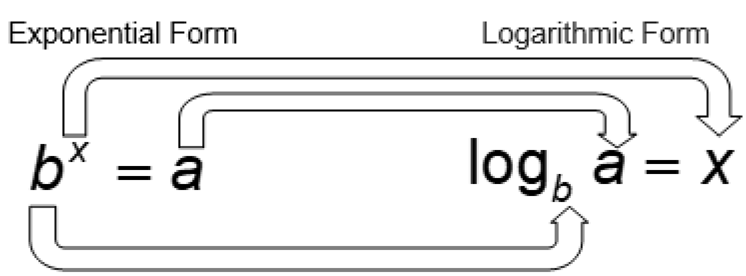
Introduction to Logarithms

Today, we will be learning about logarithms, their relationship with exponential functions, and how to evaluate them.

GUIDED NOTES

Facts about Logarithms

- A ______________ is the exponent that a specified base must be raised to in order to get a certain value.
- In simplest terms, a logarithm tells us how __________ of a certain number we need to ________ to get another number.
- Ex: How many 2s do we need to multiply to get 8? Since 2 · 2 · 2 = 8, note that we multiplied a total of 3 2s together to get 8. Therefore the answer, or logarithm, is 3.
- A logarithm function is written as $log_{base}number =$ ___________. Ex: $log_2 8 = 3$.
- An exponential function is written as $base^{exponent}$ = ____________. Ex: $2^3 = 8$
- Logarithms are also the ________________ of exponential functions. And vice versa.
- Written as a formula, $\log_b x = y$ if and only if $b^y = x$ in which b>0 and b≠1.



- A logarithm is understood is to have a base ______ , unless stated otherwise.

Example!

Write the following equation in logarithmic form.

$$7^2 = 49$$

Step 1 – Write log	log
Step 2 – Make the subscript the base. In this case, the base is 7.	log_7
Step 3 – Write the number that is on the other side of the equal sign.	$log_7 49$
Step 4 – Lastly, set your expression equal to the exponent.	$log_7 49 = 2$

Example!

Write the following equation in logarithmic form.

$$4^2 = 16$$

Step 1 – Write log	
Step 2 – Make the subscript the base. In this case, the base is ______.	
Step 3 – Write the number that is on the other side of the equal sign.	
Step 4 – Lastly, set your expression equal to the exponent.	

Knowing how to write exponential equations into logarithmic form, what steps do you think we need to take in order to write logarithm functions into exponential form?

Write the following equation in exponential form.

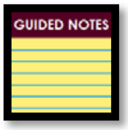
$$log_3 81 = 4$$

<u>Step 1</u> – Write the subscript as the base.	3
<u>Step 2</u> – Write the number on the other side of the equal sign as the exponent.	3^4
<u>Step 3</u> – Going counterclockwise, set your expression equal to the number on the other side of the equal sign. In this case, it is 81.	$3^4 = 81$

Write the following equation in exponential form.

$$log_{27} 3 = \frac{1}{3}$$

<u>Step 1</u> – Write the subscript as the base.	
<u>Step 2</u> – Write the number on the other side of the equal sign as the exponent.	
<u>Step 3</u> – Going counterclockwise, set your expression equal to the number on the other side of the equal sign. In this case, it is ______.	



How to Evaluate a Logarithm

Step 1 - Set the logarithm equal to a _______________.

Step 2 – Rewrite the expression to be in _______________ form.

Step 3 – Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the ___________ are the _______.

Step 4 - Set the ______________ equal to each other (______ the bases) and solve the resulting equation.

Evaluate the logarithm.

$$\boldsymbol{log_3 27}$$

Step 1 - Set the logarithm equal to a variable.	$log_3 27 = x$
Step 2 – Rewrite the expression to be in exponential form.	$3^x = 27$
Step 3 – Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.	$3^x = 3^3$
Step 4 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.	$x = 3$

Evaluate the logarithm.

$$\boldsymbol{log_4 32}$$

Step 1 - Set the logarithm equal to a variable.	$log_4 32 = x$
Step 2 – Rewrite the expression to be in exponential form.	$4^x = 32$
Step 3 – Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.	$(2^2)^x = 2^5$
Step 4 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.	$2x = 5$ $x = \frac{5}{2}$

Evaluate the logarithm.

$$\boldsymbol{log_5 \frac{1}{25}}$$

Step 1 - Set the logarithm equal to a variable.	$log_5 \frac{1}{25} = x$
Step 2 – Rewrite the expression to be in exponential form.	$5^x = \frac{1}{25}$
Step 3 – Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.	$5^x = 5^{-2}$
Step 4 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.	$x = -2$

Example!

Evaluate the logarithm.

$$log_5 125$$

<u>Step 1</u> - Set the logarithm equal to a variable.	
<u>Step 2</u> – Rewrite the expression to be in exponential form.	
<u>Step 3</u> – Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.	
<u>Step 4</u> - Set the exponents equal to each other (drop the bases) and solve the resulting equation.	

Example!

Evaluate the logarithm.

$$log_{49} 7$$

<u>Step 1</u> - Set the logarithm equal to a variable.	
<u>Step 2</u> – Rewrite the expression to be in exponential form.	
<u>Step 3</u> – Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.	
<u>Step 4</u> - Set the exponents equal to each other (drop the bases) and solve the resulting equation.	

Evaluate the logarithm.

$$log_4 \frac{1}{64}$$

Step 1 - Set the logarithm equal to a variable.	
Step 2 – Rewrite the expression to be in exponential form.	
Step 3 – Isolate the exponential expressions on either side of the =. We then rewrite the expression so that the bases are the same.	
Step 4 - Set the exponents equal to each other (drop the bases) and solve the resulting equation.	

Name ______________________ Date ______________ Class______________

LESSON 15-1

Defining and Evaluating a Logarithmic Function

Reteach

Exponential Form Logarithmic Form

$$b^x = a \qquad \log_b a = x$$

Example

Write the exponential equation in logarithmic form. $3^4 = 81$	Write the logarithmic equation in exponential form. $\log_5 125 = 3$
$3^4 = 81 \to \log_3 81 = 4$	$\log_5 125 = 3 \to 5^3 = 125$

Write the given exponential equation in logarithmic form.

1. $2^6 = 64$ ____________
2. $4^{-2} = \dfrac{1}{16}$ ____________
3. $\left(\dfrac{1}{3}\right)^3 = \dfrac{1}{27}$ ____________

Write the given logarithmic equation in exponential form.

4. $\log_7 49 = 2$ ____________
5. $\log_2 \dfrac{1}{16} = -4$ ____________
6. $\log_8 48 = x$ ____________

You can evaluate logarithms using your knowledge of exponents.

Example

$\log_4 64$	What power of 4 is 64?	$4^? = 64$ $4^3 = 64$ $\log_4 64 = 3$
$\log_2 \dfrac{1}{8}$	What power of 2 is $\dfrac{1}{8}$?	$2^? = \dfrac{1}{8}$ $2^{-3} = \dfrac{1}{8}$ $\log_2 \dfrac{1}{8} = -3$

Evaluate each logarithm.

7. $\log_5 625$
8. $\log_8 2$
9. $\log_3 \dfrac{1}{9}$

Introduction to Logarithms - Application

What is a Logarithm? (Spotlight Task)

As a society, we are accustomed to performing an action and then undoing or reversing that action. Identify the action that undoes each of those named.

1. Putting on a jacket

2. Opening a door

3. Walking forward

4. Depositing money in a bank

In mathematics we also find it useful to be able to undo certain actions.

5. What action undoes adding 5 to a number?

6. What action undoes multiplying a number by 4?

7. What action undoes squaring a number?

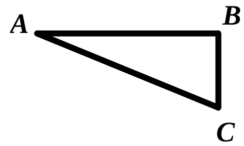
We say that addition and subtraction are inverse operations because one operation undoes the other. Multiplication and division are also inverse operations; squaring and taking the square root are inverse operations.

Inverse operations in mathematics help us solve equations. Consider the equation $2x + 3 = 35$. This equation implies some number (represented by x) has been multiplied by 2; then 3 has been added to the product for a result of 35. To determine the value of x, we subtract 3 from 35 to undo adding 3. This means that 2x must equal to 32. To undo multiplying the number by 2, we divide 32 by 2 and find the number represented by x is 16.

Explain how inverse operations are used in the solution of the following problems.

A2.U5.C1.C.04.task.IntroLogs

8. In right triangle ABC with right angle B, if BC is 8 cm and AC is 17 cm, determine the measure of angle A.



9. If $\sqrt{x+8} = 10$, determine the value of x.

10. Solve $x^3 = 27$ for *x.*

11. Solve $2x = 10$ for *x.*

In problem 8 of Task 3, "Bacteria in the Swimming Pool," we obtained the equation $1500(2)^t = 3000000$ to solve for *t*. This equation is equivalent to equivalent to $2^t = 2000$. Why? While in Task 3 we had no algebraic way to solve this equation because we lacked a strategy to isolate the exponent *t*. Our goal in this current task is to continue our idea of "undoing" to solve an equation; specifically, we need to find an action that will undo raising 2 to a power. This action needs to report the exponent to which 2 has been raised in order to obtain 2000. In order to rewrite $2^t = 2000$ so *t* is isolated, we need to define logarithms. Logarithms allow us to rewrite an exponential equation so that the exponent is isolated. Specifically, if $a = b^c$, then "*c* is the logarithm with base *b* of *a"* and is written as $\log_b a = c$. (We read "$\log_b a = c$" as "log base b of a is c.")

Using logarithms we can write $2^t = 2000$ as $\log_2 2000 = t$. These two expressions are equivalent, and in the expression $\log_2 2000 = t$ we have *t* isolated. Although this is a good thing, we still need a way to evaluate the expression $\log_2 2000$. We know it equals the exponent to which 2 must be raised in order to obtain a value of 2000, but we still don't know how to calculate this value. Hang on…we will get there in the next task! First some preliminary work must be done!

Let's look at a few examples:

***$10^2 = 100$* is equivalent to *$\log_{10} 100 = 2$*.** Notice that 10 is the *base* in both the exponential form and the logarithmic form. Also notice that the logarithm is the exponent to which 10 is raised to obtain 100.

Evaluate *$\log_4 64$*. This question asks for the exponent to which 4 is raised to obtain 64. In other words, 4 to what power equals 64? ________

Consider the following problem: $\log_2 n = 4$. This equation is equivalent to $2^4 = n$; thus $n = 16$.

A2.U5.C1.C.04.task.IntroLogs

The relationship between exponents and logarithms must be understood clearly. The following practice problems will help you gain this understanding.

Rewrite each exponential equation as a logarithmic equation.

12. $6^2 = 36$
13. $10^3 = 1000$
14. $25^{\frac{1}{2}} = 5$

Rewrite each logarithmic equation as an exponential equation.

15. $\log_4 16 = 2$
16. $\log_6 1 = 0$
17. $\log_3 n = t$

18. Evaluate each of the following.
    a. $\log_{10}(0.1)$
    b. $\mathrm{lo}g_3\ 81$
    c. $\log_2 \frac{1}{16}$
    d. $\log_5 5$

19. Between what two whole numbers is the value of $\log_3 18$?

20. Between what two whole numbers is the value of $\log_2 50$?

21. Solve each logarithmic equation for x.
    a) $\log_9 81 = x$

    b) $\log_2 32 = x$

    c) $\log_7 1 = x$

    d) $\log_8 x = 3$

    e) $\log_5(3x + 1) = 2$

f) $\log_6(4x - 7) = 0$

Hopefully you now have an understanding of the relationship between exponents and logarithms. In logarithms, just as with exponential expressions, any positive number can be a base except 1 (we will explore this fact later). Logarithms which use 10 for the base are called common logarithms and are expressed simply as *log x*. It is not necessary to write the base. Calculators are programmed to evaluate common logarithms.

22. Use your calculator to evaluate log 78. First think about what this expression means.

Understanding logarithms can help solve more complex exponential equations. Consider solving the following equation for x:

$$10^x = 350$$

We know that $10^2 = 100$ and $10^3 = 1000$ so *x* should be between 2 and 3. Rewriting

$10^x = 350$ as the logarithmic equation $x = \log 350$, we can use the calculator to determine the value of *x* to the nearest hundredth.

Solve each of the following for *x* using logarithms. Determine the value of x to the nearest hundredth.

23. $10^x = 15$
24. $10^x = 0.3458$
25. $3(10^x) = 2345$
26. $2(10^x) = -6538$

Logarithms that use the irrational number e as a base are of particular importance in many applications. Recall an irrational number is represented by a non-terminating, non-repeating decimal number. The value of e is 2.718281828…. The function $y = \log_e x$ is the natural logarithmic function and has a base of e. The shorthand for $y = \log_e x$ is $y = \ln x$. Calculators are also programmed to evaluate natural logarithms.

Consider $\ln 34$ which means the exponent to which the base e must be raised to obtain 34. The calculator evaluates $\ln 34$ as approximately 3.526. This value makes sense because $e^{3.526}$ is approximately 33.9877, a value very close to 34!

27. Evaluate $\ln 126$. Use an exponential expression to confirm your solution makes sense.

28. Evaluate $\ln e$. Explain why your answer makes sense.

29. If $\ln x = 7$, determine the value of x to the nearest hundredth. HINT: Write the logarithmic equation in exponential form.

30. If $e^x = 85$, determine the value of x to the nearest hundredth. HINT: Write the exponential equation in logarithmic form.

The cards you have been given are to be sorted. There will be six matches of five cards each. You will see a verbal description of the exponential function, a verbal description of the logarithmic function that means the same thing, the logarithmic equation written out, the exponential equation written out, and the solution to the equations. Make the matches, and then be prepared to tell:

a) Of the two equations that you saw, the exponential and the logarithmic, which one helped you find the solution the easiest?

b) How does the solution that you found work for both the logarithmic and the exponential equation?

c) Which ones could you have solved without any work at all except just using your calculator?

STUDENT COPY-CARD SORT-SHOULD BE CUT APART BEFORE BEING PLACED INTO STUDENT HANDS

(A) The logarithm, base two, of 16 is some number.	(£) Two to the power of some number is 16.	(U) $\log_2 16 = x$	(Q) 4	(S) $2^x = 16$
(P) The logarithm, base 16, of 2 is a number.	(B) Sixteen to the power of some number is 2.	(O) $\log_{16} 2 = x$	(W) $\frac{1}{4}$	(G) $16^x = 2$
(Ω) The logarithm with what base of 2 is 16?	(M) Some number to the power of 16 is 2.	(N) $\log_x 2 = 16$	(E) $2^{\frac{1}{16}}$	(V) $x^{16} = 2$
(L) The logarithm with what base of 16 is 2?	(F) Some number to the power of 2 is 16.	(T) $\log_x 16 = 2$	(I) 4	(Z) $x^2 = 16$
(μ) The logarithm, base two, of some number is 16.	(C) Two to the power of 16 is some number.	(R) $\log_2 x = 16$	(K) 65,536	(J) $2^{16} = x$
(H) The logarithm, base 16, of some number is 2.	(X) Sixteen to the power of 2 is some number.	(Y) $\log_{16} x = 2.$	(θ) 256	(Я) $16^2 = x$

Use a calculator to approximate each to the nearest thousandth.

1) $\log 52$

2) $\log 6.2$

3) $\log 7$

4) $\ln 52$

Rewrite each equation in exponential form.

5) $\log_x y = 3$

6) $\log_{19} 361 = 2$

7) $\log_b 164 = -3$

8) $\log_{81} \frac{1}{9} = -\frac{1}{2}$

Rewrite each equation in logarithmic form.

9) $y^{-10} = x$

10) $20^{-8} = x$

11) $u^{-5} = v$

12) $19^2 = 361$

Evaluate each expression.

13) $\log_2 32$

14) $\log_4 \frac{1}{16}$

15) $\log_5 125$

16) $\log_7 343$

Evaluate each expression.

17) $\log_5 1$

18) $\log_{49} 7$

19) $\log_4 \frac{1}{64}$

20) $\log_{49} \frac{1}{7}$

Logarithms as Inverses

Today, we will be learning how to find the inverse of logarithms and exponential functions.

GUIDED NOTES

How to Find the Inverse of Logarithms

Step 1 - Switch the______ and the ______ in the equation.
Step 2 – __________ the log expression.
Step 3 – ___________ the expression to be in ______________ form.
Step 4 - Solve for ______.

Note: If you _____________ with a _______________ , your __________ should be an _____________ function.

SELF CHECK

Find the inverse of the following:

1. $y = 3x + 9$

2. $y = 4x^2 - 16$

Find the inverse of $y = \log_5(x-1) + 2$

Step 1 – Switch the x and the y in the equation.

$$x = \log_5(y-1) + 2$$

Step 2 – Isolate the log expression.

$$x - 2 = log_5(y-1)$$

Step 3 – Rewrite the expression to be in exponential form.

$$5^{x-2} = y - 1$$

Step 4 – Solve for y.

$$5^{x-2} + 1 = y$$

Example!

Find the inverse of $y = \log_3 x$

<u>Step 1</u> – Switch the x and the y in the equation.

$$x = \log_3 y$$

<u>Step 2</u> – Isolate the log expression.

$$x = \log_3 y$$

<u>Step 3</u> – Rewrite the expression to be in exponential form.

$$3^x = y$$

<u>Step 4</u> – Solve for y.

$$\boxed{3^x = y}$$

Example!

Find the inverse of $y = log_5 x + 2$

<u>Step 1</u> – Switch the x and the y in the equation.

$$x = log_5 y + 2$$

<u>Step 2</u> – Isolate the log expression.

$$x - 2 = log_5 y$$

<u>Step 3</u> – Rewrite the expression to be in exponential form.

$$5^{x-2} = y$$

<u>Step 4</u> – Solve for y.

$$\boxed{5^{x-2} = y}$$

Example!

Find the inverse of $y = log(x - 4)$

<u>Step 1</u> – Switch the x and the y in the equation.

$$x = log(y - 4)$$

<u>Step 2</u> – Isolate the log expression.

$$x = log_{10}(y - 4)$$

*When a base is not written for log, it is understood to be 10.

<u>Step 3</u> – Rewrite the expression to be in exponential form.

$$10^x = y - 4$$

<u>Step 4</u> – Solve for y.

$$\boxed{10^x + 4 = y}$$

Find the inverse of $y = \log_4(x + 1) - 3$

<u>Step 1</u> – Switch the x and the y in the equation.

<u>Step 2</u> – Isolate the log expression.

<u>Step 3</u> – Rewrite the expression to be in exponential form.

Example!

Step 4 – Solve for y.

Example!

Find the inverse of $y = \log_7 x$

Step 1 – Switch the x and the y in the equation.

Step 2 – Isolate the log expression.

Step 3 – Rewrite the expression to be in exponential form.

Step 4 – Solve for y.

Example!

Find the inverse of $y = log_3 x + 8$

Step 1 – Switch the x and the y in the equation.

Step 2 – Isolate the log expression.

Step 3 – Rewrite the expression to be in exponential form.

Example!

Step 4 – Solve for y.

Example!

Find the inverse of $y = log(x+6)$

Step 1 – Switch the x and the y in the equation.

Step 2 – Isolate the log expression.

Step 3 – Rewrite the expression to be in exponential form.

Step 4 – Solve for y.

Questions To Ponder

What steps will we need to take to find the inverse of an exponential function? Which steps will be the same as finding the inverse of a logarithm and which will be different?

GUIDED NOTES

How to Find the Inverse of Exponential Functions

Step 1 - ________ the x and the y in the equation.
Step 2 – Isolate the ______________.
Step 3 – Rewrite the expression to be in _________________ form.
Step 4 – Solve for _____.

Note: If you start with an _______________ function, your answer should be a _____________.

Find the inverse of $y = 3^x - 1$

<u>Step 1</u> - Switch the x and the y in the equation.

$$x = 3^y - 1$$

<u>Step 2</u> – Isolate the exponential.

$$x + 1 = 3^y$$

<u>Step 3</u> – Rewrite the expression to be in logarithmic form.

$$log_3(x + 1) = y$$

<u>Step 4</u> – Solve for y.

$$\boxed{log_3(x + 1) = y}$$

Find the inverse of $y = (4)^{\frac{x}{2}}$

<u>Step 1</u> - Switch the x and the y in the equation.

$$x = (4)^{\frac{y}{2}}$$

<u>Step 2</u> – Isolate the exponential.

$$x^2 = (4^{\frac{y}{2}})^2$$

$$x^2 = 4^y$$

<u>Step 3</u> – Rewrite the expression to be in logarithmic form.

$$log_4 x^2 = y$$

Example!

Step 4 – **Solve for y.**

$$\boxed{log_4 x^2 = y}$$

Example!

Find the inverse of $y = \frac{5^{x+1}}{2}$

Step 1 - **Switch the x and the y in the equation.**

$$x = \frac{5^{y+1}}{2}$$

Step 2 – **Isolate the exponential.**

$$2x = 5^{y+1}$$

Step 3 – **Rewrite the expression to be in logarithmic form.**

$$log_5 2x = y + 1$$

Step 4 – **Solve for y.**

$$\boxed{log_5(2x) - 1 = y}$$

Example!

Find the inverse of $y = 4^x + 9$

Step 1 - **Switch the x and the y in the equation.**

Step 2 – **Isolate the exponential.**

Step 3 – **Rewrite the expression to be in logarithmic form.**

<u>Step 4</u> – Solve for y.

Example!

Find the inverse of $y = (7)^{\frac{x}{3}}$

<u>Step 1</u> - Switch the x and the y in the equation.

<u>Step 2</u> – Isolate the exponential.

<u>Step 3</u> – Rewrite the expression to be in logarithmic form.

<u>Step 4</u> – Solve for y.

Example!

Find the inverse of $y = \frac{8^{x-2}}{3}$

<u>Step 1</u> - Switch the x and the y in the equation.

<u>Step 2</u> – Isolate the exponential.

<u>Step 3</u> – Rewrite the expression to be in logarithmic form.

<u>Step 4</u> – Solve for y.

Find the inverse of each function.

1) $y = \log_6 x - 3$

2) $y = \log_5 x^5$

3) $y = 5\log_4 x$

4) $y = \log_6 x + 10$

5) $y = \log_4 (-4x)$

6) $y = \frac{4^x}{3}$

7) $y = 6^x + 9$

8) $y = 2^{\frac{x}{4}}$

9) $y = \frac{3^x}{4}$

10) $y = \log_4 2^x$

Find the inverse of each function.

17) $y = -9\log_6 x$

18) $y = 4\log_4 x$

19) $y = \log x - 3$

20) $y = \log_2 (x+6)$

21) $y = \log_4 (4x)$

22) $y = -4\log_{\frac{1}{4}} x$

23) $y = \log_2 x^3$

24) $y = \ln x - 1$

25) $y = -\dfrac{3^x}{2}$

26) $y = 2^{\frac{x}{3}}$

27) $y = 3^x - 7$

28) $y = 6^x + 10$

29) $y = \dfrac{4^x}{3}$

30) $y = 2^{\frac{x}{4}}$

31) $y = 5^{\frac{x}{5}}$

32) $y = 4^x - 7$

Properties of Logarithms

Today, we will be learning the four basic properties of logarithms.

GUIDED NOTES

Properties of Logarithms

There are four basic properties of logarithms that we will be working with. For every case, the base of the logarithm cannot be equal to 1 and the values must all be positive (no negatives in logs).

Product Rule	
Quotient Rule	
Power Rule	
Change of Base Rule	

Note: The _________ of the logs must be the _________ in order to expand the logarithmic expressions.

Example!

Product Rule

Expand the following logarithms.

1. $\boldsymbol{log_b xy}$ = $log_b x + \ log_b y$
2. $\boldsymbol{log 6a}$ = $log 6 + log\ a$
3. $\boldsymbol{log_3 9b}$ = $log_3 9 + \ log_3 b$

Example!

Quotient Rule

Expand the following logarithms.

1. $\boldsymbol{log_5 \frac{x}{y}} = log_5 x - log_5 y$
2. $\boldsymbol{log_2 \frac{a}{5}} = log_2 a - \ log_2 5$

Example!

Power Rule

Expand the following logarithms.

1. $\boldsymbol{log_5 B^2} = 2log_5 B$
2. $\boldsymbol{log_2 5^x} = xlog_2 5$

Example!

Change of Base Rule

Expand the following logarithms.

1. $\boldsymbol{log_5 B} = \frac{log\ B}{log\ 5}$
2. $\boldsymbol{log_2 5} = \frac{log\ 5}{log\ 2}$

Example!

Expand the following logarithms.

1. $\boldsymbol{log_4 7b} =$

***Hint:** Questions 2 and 3 will require 2 rules to fully expand it.

2. $\boldsymbol{log_2 \frac{MN}{P}}$ =

3. $\boldsymbol{log_7 a^3 b^4}$ =

4. $\boldsymbol{log_7 a}$ =

GUIDED NOTES

Properties of logarithms are not only used to expand logarithms, they are also used to ___________ such expressions.

Note: The __________ of the logs must be the ________ in order to expand the logarithmic expressions.

Example!

Condense the following logarithms.

1. $\boldsymbol{log_4 4 - log_4 17}$ = $log_4 \frac{4}{17}$

2. $\mathbf{log\,5 + log\,2}$ = $log\,(5 \cdot 2)$ = $log\,10$

3. $\boldsymbol{2log_2 m - 4log_2 n}$ = $log_2 m^2 - log_2 n^4$ = $log_2 \frac{m^2}{n^4}$

Questions To Ponder

How many properties were needed in example #3 to condense the expression? Which were they?

Condense the following logarithms.

1. $log_3 6 - \; log_3 12$ =
2. $\log 4 + \log x$ =
3. $5log_2 m + \; 3log_2 n$ =
4. $7log_6 m + \; log_6 n$ =

Which properties were used for each example problem above?

LESSON 16-1

Properties of Logarithms

Reteach

Product Property	Quotient Property	Power Property
$\log_b(mn) = \log_b m + \log_b n$	$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$	$\log_b m^n = n\log_b m$

Use properties to rewrite the expressions as a single logarithm.

Example $\log_4 256 + \log_4 \frac{1}{4}$	**Example** $\log_3 81 - \log_3 3$	**Example** $\log_{\frac{1}{3}}\left(\frac{1}{3}\right)^4$
$\log_4 256 + \log_4 \frac{1}{4} = \log_4\left(256 \bullet \frac{1}{4}\right)$ $= \log_4 64$ $= 3$	$\log_3 81 - \log_3 3 = \log_3\left(\frac{81}{3}\right)$ $= \log_3 27$ $= 3$	$\log_{\frac{1}{3}}\left(\frac{1}{3}\right)^4 = 4\log_{\frac{1}{3}}\left(\frac{1}{3}\right)$ $= 4 \bullet 1$ $= 4$

Rewrite the expression as a single logarithm using the product property. Simplify if possible.

1. $\log_2 \frac{1}{8} + \log_2 128$
2. $\log x + \log y$
3. $\log_{\frac{1}{5}} 20 + \log_{\frac{1}{5}} \frac{1}{100}$

________________________ ________________________ ________________________

Rewrite the expression as a single logarithm using the quotient property. Simplify if possible.

4. $\log_8 3 - \log_8 \frac{1}{192}$
5. $\log_7(ab) - \log_7 b$
6. $\log 2000 - \log 200$

________________________ ________________________ ________________________

Rewrite the expression using the power property. Simplify if possible.

7. $\log_6 6^x$
8. $\log_a a^5$
9. $\log x^2$

________________________ ________________________ ________________________

Independent Practice

Expand each logarithm.

11) $\log_9 \left(\frac{u^2}{v}\right)^3$

12) $\log_4 \left(x^5 y^2\right)$

13) $\log_2 \left(x^2 y^5\right)$

14) $\log_5 \left(x y^5\right)^2$

15) $\log_2 \left(a^6 b^3\right)$

16) $\log_7 \left(\frac{u^4}{v}\right)^6$

Answer Key

1) $\log \frac{2}{3}$

$\log 2 - \log 3$

2) $\log (3 \cdot 11)$

$\log 3 + \log 11$

3) $\log (6 \cdot 7)$

$\log 6 + \log 7$

4) $\log (5 \cdot 11)$

$\log 5 + \log 11$

5) $\log (7 \cdot 8)$

$\log 7 + \log 8$

6) $\log \frac{12}{11}$

$\log 12 - \log 11$

7) $\log \sqrt[3]{x}$

$\frac{\log x}{3}$

8) $\log \sqrt{x}$

$\frac{\log x}{2}$

9) $\log (a \cdot b)$

$\log a + \log b$

10) $\log x^5$

$5 \log x$

11) $\log (u \cdot v)^6$

$6 \log u + 6 \log v$

12) $\log (ab^5)$

$\log a + 5 \log b$

13) $\log 6 - \log 5$

$\log \frac{6}{5}$

14) $\log 12 + \log 5$

$\log 60$

15) $\log 6 + \log 7$

$\log 42$

16) $\log 12 - \log 11$

$\log \frac{12}{11}$

17) $3\log x$

$\log x^3$

18) $6\log a$

$\log a^6$

19) $\log a - \log b$

$\log \frac{a}{b}$

20) $\frac{\log x}{2}$

$\log \sqrt{x}$

21) $\log x + 5\log y$

$\log \left(xy^5\right)$

22) $6\log u - 6\log v$

$\log \frac{u^6}{v^6}$

23) $4\log x + 4\log y$

$\log \left(y^4x^4\right)$

24) $\log u + \log v + \log w$

$\log wvu$

Solving Logarithms

Today, we will be learning how to solve logarithms using their properties.

GUIDED NOTES

How to Solve Logarithms

Step 1 – Using the ____________ of logarithms, _____________ the expression.

Step 2 – If left with a ____________ logarithm, rewrite the expression to be in __________form. Then, move to step 4.

Step 3 – If left with a single logarithm on BOTH sides of the equal sign, apply the equality rule.

Equality Rule	

Step 4 – Solve for the _____________.

Example!

Solve the equation below.

$$\log 3 + \log x = 2$$

Step 1 – Using the properties of logarithms, condense the expression.

$$log\ 3x = 2$$

Step 2 – If left with a single logarithm, rewrite the expression to be in exponential form. Then, move to step 4.

*Remember: If a base is not written for the log, it is understood to be 10.

$$log_{10}3x = 2$$

$$10^2 = 3x$$

Step 3 – If left with a single logarithm on BOTH sides of the equal sign, apply the equality rule.

This step does not apply.

Step 4 – Solve for the variable.

$$10^2 = 3x$$

$$100 = 3x$$

$$\boxed{\frac{100}{3} = x}$$

Solve the equation below.

$$\log_6(x+3) - \log_6(x) = \log_6 10$$

Step 1 – Using the properties of logarithms, condense the expression.

$$log_6\frac{(x+3)}{(x)} = log_6 10$$

Step 2 – If left with a single logarithm, rewrite the expression to be in exponential form. Then, move to step 4.

This step does not apply.

Step 3 – If left with a single logarithm on BOTH sides of the equal sign, apply the equality rule.

$$\cancel{log_6}\frac{(x+3)}{(x)} = \cancel{log_6}10$$

$$\frac{(x+3)}{(x)} = 10$$

Step 4 – Solve for the variable.

$$x + 3 = 10x$$

$$3 = 9x$$

$$\frac{3}{9} = x \quad which\ reduces\ to \quad \boxed{x = \frac{1}{3}}$$

Example!

Solve the equation below.

$$\log 7 + \log x = 1$$

Step 1 – Using the properties of logarithms, condense the expression.

Step 2 – If left with a single logarithm, rewrite the expression to be in exponential form. Then, move to step 4.

Step 3 – If left with a single logarithm on BOTH sides of the equal sign, apply the equality rule

Step 4 – Solve for the variable.

Example!

Solve the equation below.

$$log_7 3 - log_7(-5x) = log_7 37$$

Step 1 – Using the properties of logarithms, condense the expression.

Step 2 – If left with a single logarithm, rewrite the expression to be in exponential form. Then, move to step 4.

Step 3 – If left with a single logarithm on BOTH sides of the equal sign, apply the equality rule

<u>Step 4</u> – Solve for the variable.

If the bases of the logarithms were different, would you still be able to solve? Why or why not?

Solve each logarithmic equation.

1) $\log_5(x+7) = \log_5(2x+3)$

x =

2) $\log_2(x+20) = \log_2 2$

x =

3) $\log_4(3x-2) = \log_4(x+18)$

x =

4) $\log_3(5x+6) = 2\log_3 6$

x =

5) $\log_6(2x-1) = \log_6 27$

x =

6) $\log_5 24 = \log_5(x+2)$

x =

7) $\log_7(x+1) = \log_7(2x+20)$

x =

8) $\log_4(5x-3) = \log_4(2x+36)$

x =

9) $3\log_3 4 = \log_3(x+10)$

x =

10) $\log_6 4x = \log_6 100$

x =

11) $\log_5(2x+2) = \log_5(3x+18)$

x =

12) $\log_2(x+17) = 3\log_2 2$

x =

A2.U5.C1.**F.04.task.SolveLogEquations**

Solving Logarithms - Application

Algebra 2B

Solving Exponential and Logarithmic Equations

20 questions

Questions are at the BOTTOM of the page (ex: First page has question 1)

Answers at the top of the next page (ex: First page has the answer to question 20)

9

$$-7 \cdot 2^{x} = -79$$

$$3.4964$$

$$-4 \cdot 15^{x} = -32$$

$$0.7679$$

$$-5 \cdot 2^{x} = -88$$

4.1375

$$4 \cdot 10^{x+1} = 82$$

$$0.3118$$

$$-3 \cdot 17^{-9x} = -19$$

$$-0.0724$$

$$-4 \cdot 10^{-x} = -24$$

$$-0.7782$$

$$2 \cdot e^{10x-7} + 4 = 70$$

$$1.0497$$

$$-8 \cdot e^{-3x-8} - 2 = -90$$

$$-3.466$$

$$log4 + \log x = 2$$

25

$$\log x - \log 8 = 1$$

80

$$\log x - \log 2 = 1$$

20

$$\log_5(x + 4) - \log_5 x = 4$$

$$\frac{1}{156} \quad or \quad 0.0064$$

$$\log_5(5x) - \log_5 3 = 1$$

3

$$\ln 3 - \ln(-4x - 10) = \ln 9$$

$$-\frac{31}{12} \quad or \quad -2.58\overline{3}$$

$$\ln(x+3) - \ln(x+2) = \ln 16$$

$$-\frac{29}{15}\ \ or -1.9\overline{3}$$

$$\ln 9 + \ln(-4x - 7) = 3$$

$$-2.3079$$

$$\log_4(5x - 4) = \log_4(3x + 8)$$

6

$$\log_{11}(3 - 3x) = \log_{11}(3x + 3)$$

0

$$\log_9(-10x) = \log_9(-9 - x^2)$$

$$No\ Solution$$

$$\log_7(x^2 - 4x) = \log_7(18 - x)$$

$$6\,,-3$$

$$log\,(x+1)+10=11$$

HOMEWORK

Solve each equation.

1) $\log 5x = \log (2x + 9)$

2) $\log (10 - 4x) = \log (10 - 3x)$

3) $\log (4p - 2) = \log (-5p + 5)$

4) $\log (4k - 5) = \log (2k - 1)$

5) $\log (-2a + 9) = \log (7 - 4a)$

6) $2\log_7 -2r = 0$

7) $-10 + \log_3 (n + 3) = -10$

8) $-2\log_5 7x = 2$

9) $\log -m + 2 = 4$

10) $-6\log_3 (x - 3) = -24$

11) $\log_{12} (v^2 + 35) = \log_{12} (-12v - 1)$

12) $\log_9 (-11x + 2) = \log_9 (x^2 + 30)$

13) $\log(16+2b)=\log(b^2-4b)$

14) $\ln(n^2+12)=\ln(-9n-2)$

15) $\log x+\log 8=2$

16) $\log x-\log 2=1$

17) $\log 2+\log x=1$

18) $\log x+\log 7=\log 37$

19) $\log_8 2+\log_8 4x^2=1$

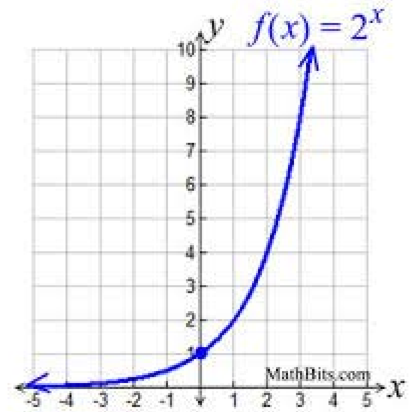
20) $\log_9(x+6)-\log_9 x=\log_9 2$

Term	Definition	Diagram/Visual
Exponential Function		
Exponential Growth		
Exponential Decay		
Transformations		
Domain		
Range		
End Behavior		
X-Intercept		
Y-Intercept		
Asymptotes		

Graphing & Describing Exponential Functions

In this lesson, we will graph and describe exponential functions.

The graph of the parent function $f(x) = 2^x$ is shown. We will use this graph to illustrate how to graph and describe exponential functions.



To graph, make a table, plot the points and connect the points with a smooth curve.

x	-2	-1	0	1	2	3
$f(x) = 2^x$						

Domain:

Range:

***x*-intercept**:
(plug 0 in for y and solve)

***y*-intercept**:
(plug 0 in for x and solve)

Asymptote:

End Behavior:

There are two basic types of exponential functions.

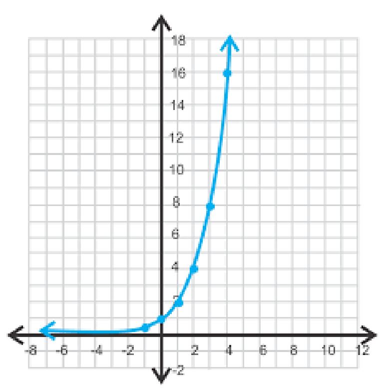
This type is a

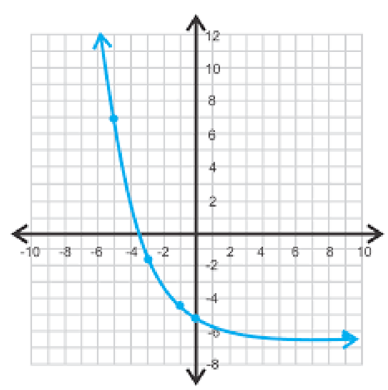
____

function.

$f(x) = b^x$

Where ________





This type is a

____

function.

$f(x) = b^x$

Where ________

Transformations of Exponential Functions

When listing the characteristics of each function, we must also describe the transformations of the function from its parent function.

GUIDED NOTES

Each part of the function helps you describe the transformations.

$a < 0$ ____________
$|a| > 1$ ____________
$|a| < 1$ ____________

$(x - h)$ ________
$(x + h)$ ________

$$f(x) = a \cdot b^{(x-h)} + k$$

b is the ________
$b > 1$ ________
$0 < b < 1$ ____

$+k$ ________
$-k$ ________

List the transformations for each function from the parent rational function.

$y = 2^x - 2$ $\quad$ $y = -3^{x+3}$ $\quad$ $y = 4\left(\frac{1}{2}\right)^x$ $\quad$ $y = \frac{1}{2}(2)^{x-1} + 1$

Example! Graph and describe each function.

$$f(x) = -3(2)^{x-2} + 1$$

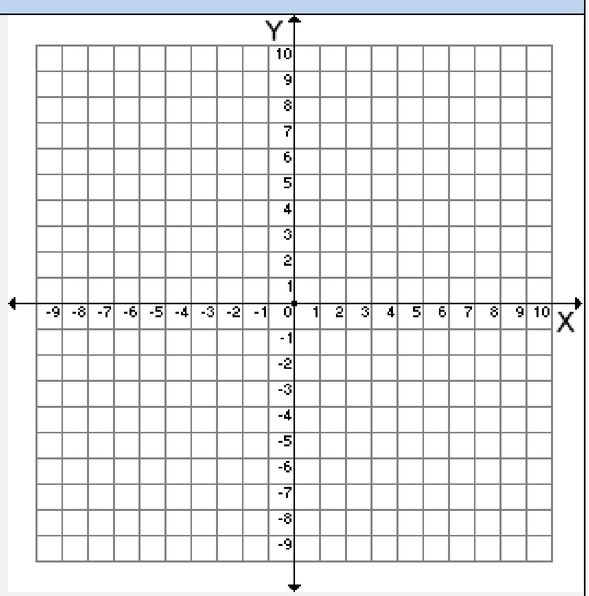
Transformations:

Domain: **Range**:

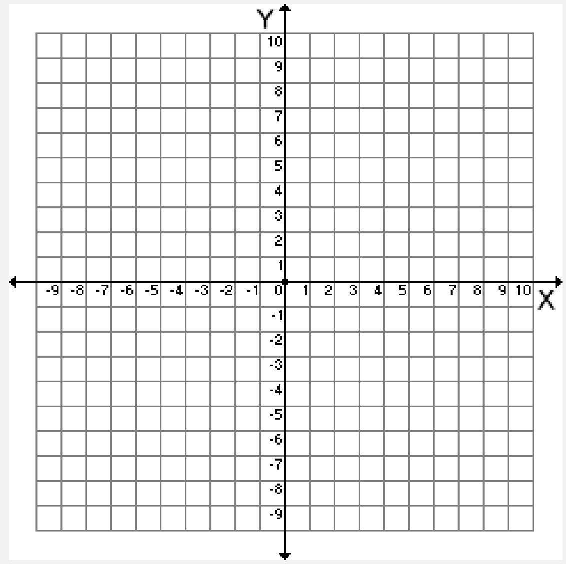
x-intercept: **y-intercept**:

Asymptote:

End Behavior: $As\ x \rightarrow$ ____, $y \rightarrow$ ____
$As\ x \rightarrow$ ____, $y \rightarrow$ ____



$$f(x) = 2\left(\frac{1}{4}\right)^{x-1} - 4$$



Transformations:

Domain: **Range**:

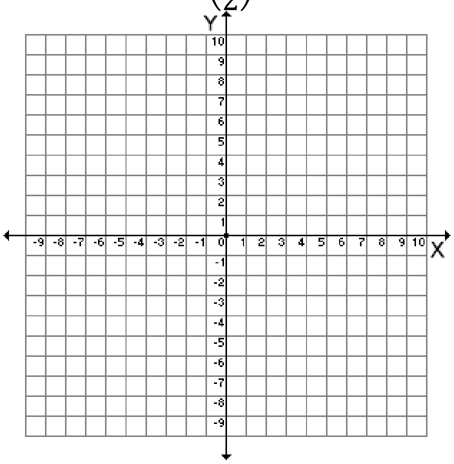
x-intercept: **y-intercept**:

Asymptote:

End Behavior: $As\ x \to$ ____, $y \to$ _____
$As\ x \to$ _____, $y \to$ _____

SELF CHECK Graph and describe each function.

$$y = -2\left(\frac{1}{2}\right)^{x-1} + 2$$



Transformations:

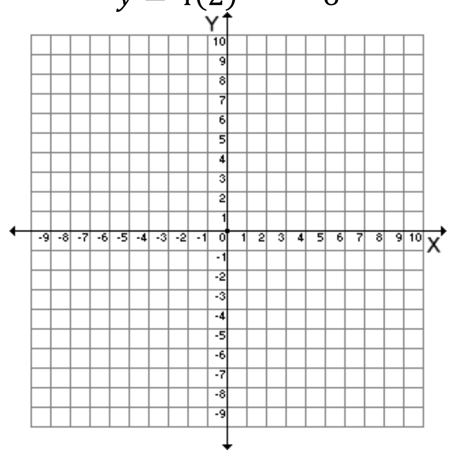
Domain: **Range**:

x-intercept: **y-intercept**:

Asymptote:

End Behavior: $As\ x \to$ ____, $y \to$ _____
$As\ x \to$ _____, $y \to$ _____

$$y = 4(2)^{x+2} - 6$$



Transformations:

Domain: **Range**:

x-intercept: **y-intercept**:

Asymptote:

End Behavior: $As\ x \to$ ____, $y \to$ _____
$As\ x \to$ _____, $y \to$ _____

Questions To Ponder

1. Will the domain be the same for every exponential function?
2. Can you transform the graph of $f(x) = b^x$ so that it has a turning point?

Graphing Exponential Functions

1. $f(x) = \frac{1}{3}(3)^x - 2$

Transformation(s):

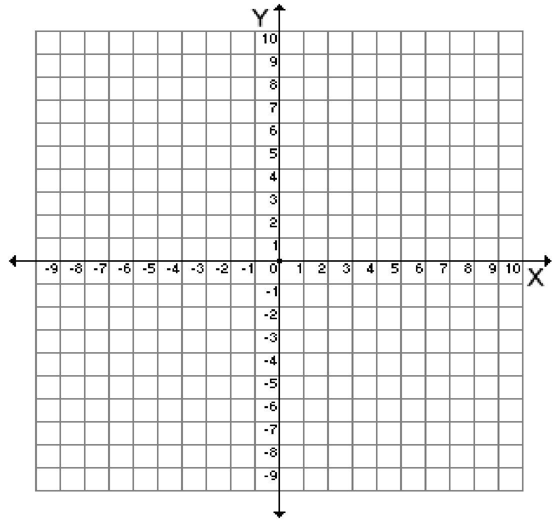
Domain:

Range:

Asymptote:

x-intercept: y-intercept:

End Behavior:



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2. $f(x) = 3\left(\frac{1}{3}\right)^{x+3}$

Transformation(s):

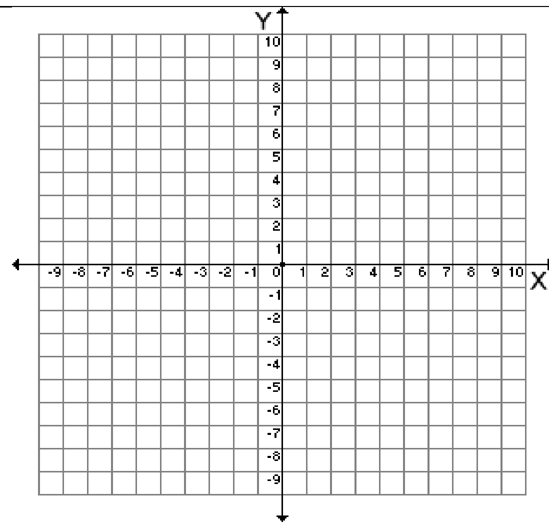
Domain:

Range:

Asymptote:

x-intercept: y-intercept:

End Behavior:



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3. $f(x) = -2(2)^{x+1} - 2$

Transformation(s):

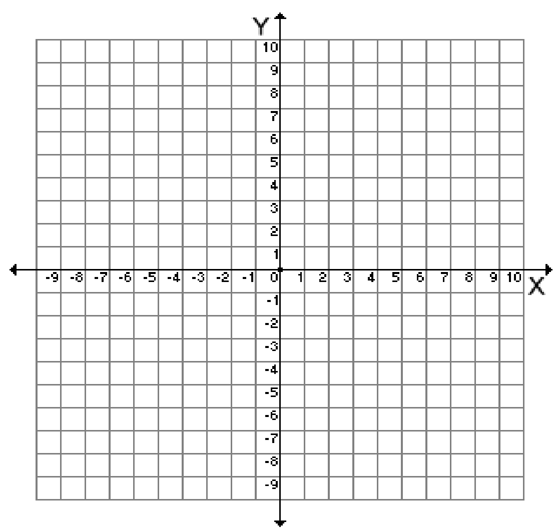
Domain:

Range:

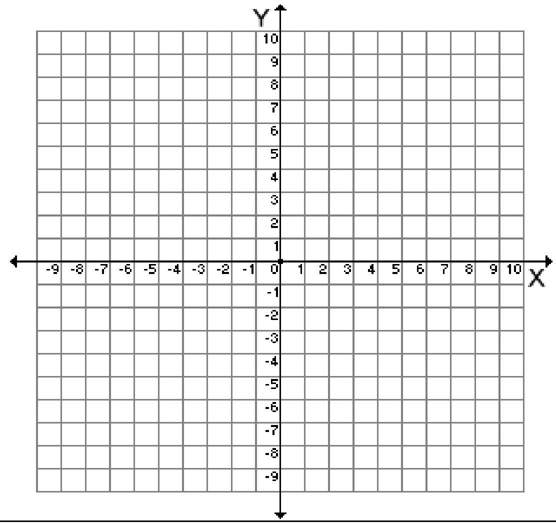
Asymptote:

x-intercept: y-intercept:

End Behavior:



4. $f(x) = \left(\frac{2}{3}\right)^{x-2} + 4$



Transformation(s):

Domain:

Range:

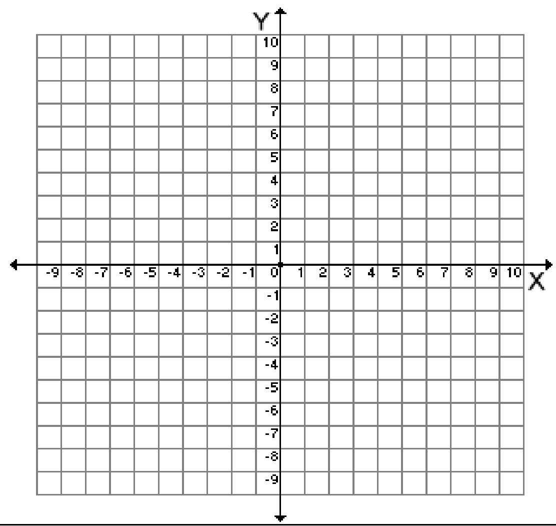
Asymptote:

x-intercept: y-intercept:

End Behavior:

---

5. $f(x) = -3\left(\frac{1}{2}\right)^{x+1} + 1$



Transformation(s):

Domain:

Range:

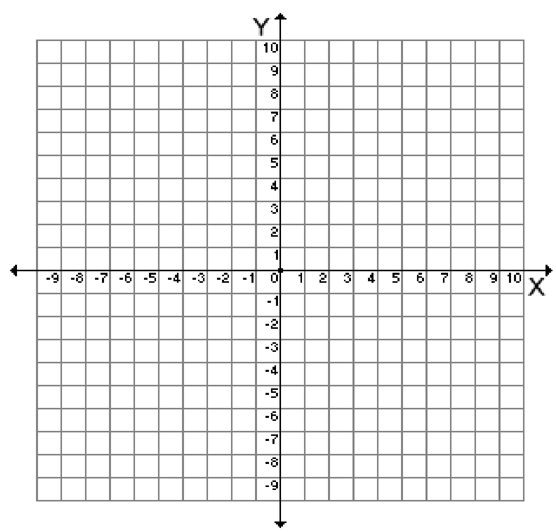
Asymptote:

x-intercept: y-intercept:

End Behavior:

---

6. $f(x) = 4^x - 1$



Transformation(s):

Domain:

Range:

Asymptote:

x-intercept: y-intercept:

End Behavior:

Graphing Exponential Functions - Application

Graphs of Exponential Functions

In Coordinate Algebra one of the functions studied was the exponential function. By definition an exponential function is $y = a^x$ where $a > 0$ and $a \neq 1$. An exponential function returns powers of a base number a. The input of the exponential function is the exponent and the output is the number obtained when the base number is raised to that exponent.

1. For each function given, represent the function as a table and then use these points to graph the function on the coordinate plane paper below. *Be sure to label you graphs*.

a.

$f(x) = 2^x$

x	$f(x)$
6	
4	
3	
2	
1	
0	
-1	
-2	
-3	
-4	

b.

$f(x) = 3^x$

x	$f(x)$
6	
4	
3	
2	
1	
0	
-1	
-2	
-3	
-4	

c.

$f(x) = 4^x$

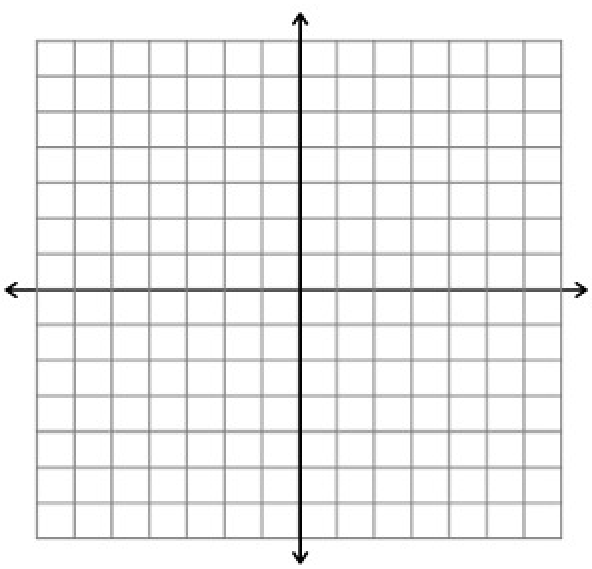
x	$f(x)$

d.

$f(x) = 10^x$

x	$f(x)$

2. Describe characteristics of their graphs:



Domain: Range:

Increasing: Decreasing:

Asymptotes:

x-intercept: y-intercept:

What happens as x approaches positive infinity?

What happens as x approaches negative infinity?

What do the functions have in common?

How does the graph of the exponential function change as the base a changes?

3. The symbol e represents the irrational number 2.718281828…. Recall an irrational number is represented by a non-terminating, non-repeating decimal number. e is one of those important numbers in mathematics like π that keeps showing up in all kinds of places. $y = e^x$ is the **natural exponential function**.

Use graphing technology to graph $y = 2^x$, $y = 3^x$, and $y = e^x$. How do their graphs compare? What do you notice about the graph of $y = e^x$ in relationship to the graphs of $y = 2^x$ and $y = 3^x$?

4. Use graphing technology to graph each function.

a. $y = 2^{-x}$
b. $y = 3^{-x}$
c. $y = 4^{-x}$
d. $y = 10^{-x}$

How do these graphs compare to those in part (1) above? Use what you know about transformations of functions to explain the relationship between the graphs of $y = 2^x$ and $y = 2^{-x}$.

Does the same relationship hold for $y = 3^x$ and $y = 3^{-x}$? For $y = 4^x$ and $y = 4^{-x}$? In general, what is the relationship between the graphs of $y = a^x$ and $y = a^{-x}$?

5. Graph $y = \left(\frac{1}{2}\right)^x$. Compare its graph to $y = 2^{-x}$. What do you observe?

Use properties of exponents to explain the relationship between $\left(\frac{1}{2}\right)^x$ and 2^{-x}.

Do your observations about the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^{-x}$ now make sense?

6. Graph $y = 2^x + 3$. How does this graph compare to that of $y = 2^x$?

Based on what you know about transformations of functions, **describe in words** how $y = 2^x + 3$ transforms the graph of the parent function $y = 2^x$.

Discuss what you notice about the domain, range, intercepts, and asymptote of $y = 2^x + 3$.

7. Graph $y = 2^{x-5}$. How does this graph compare to that of $y = 2^x$?

Based on what you know about transformations of functions, **describe in words** how $y = 2^{x-5}$ transforms the graph of the parent function $y = 2^x$.

Discuss what you notice about the domain, range, intercepts, and asymptote of $y = 2^{x-5}$.

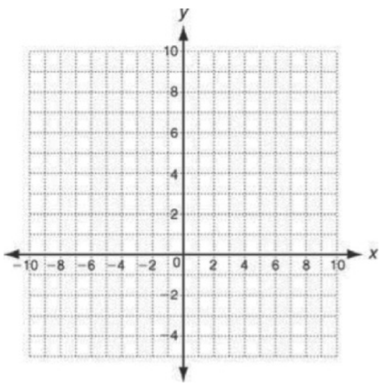
8. The exponential function $y = a^x$ is defined for all real numbers $a > 0$ and $a \neq 1$.

a. Why do you think the function is not defined for bases that are negative real numbers? Often to determine why something cannot be true, it helps to see what would happen if it were true!! So…explore what would happen for negative values of a; for example, see what would happen if $a = -2$. Set up a table of values to see if you can determine a reasonable explanation for why the base is not allowed to be negative in an exponential function.

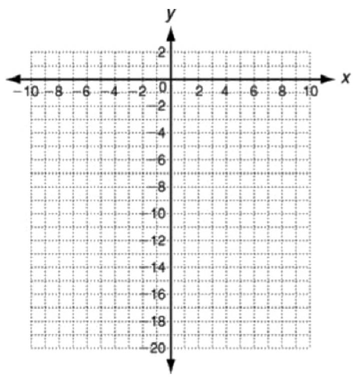
b. Why do you think the function is not defined for a base of 0 or a base of 1? Explore the functions $y = a^x$ for $a = 0$ and $a = 1$. Can you offer a reasonable explanation for excluding values of 0 and 1 for the base of an exponential function?

Graph and describe each function.

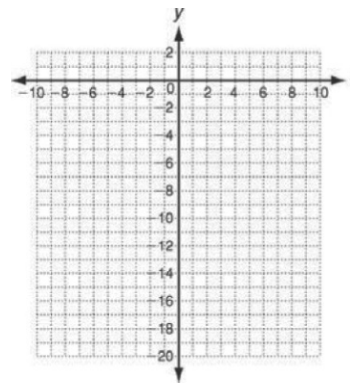
1. $y = \left(\frac{1}{2}\right)^x - 3$



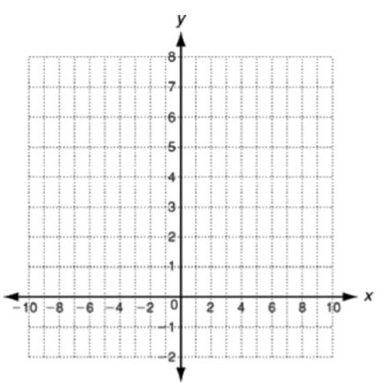
2. $y = -2(3)^x$



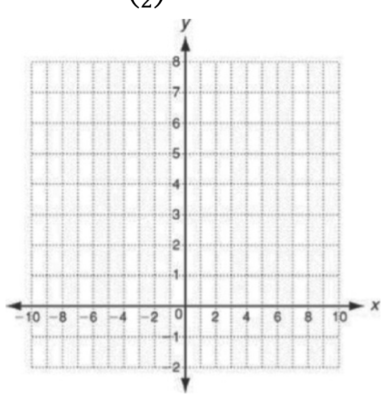
3. $y = -(0.5)^{x+3} - 3$



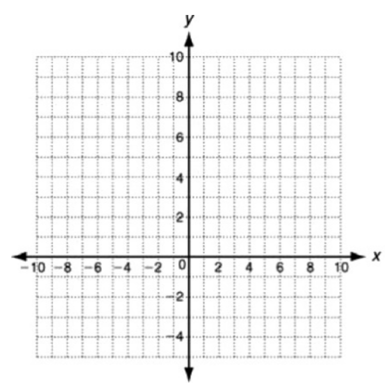
4. $y = \frac{1}{2}(2)^{x-1} + 3$



5. $y = 3\left(\frac{1}{2}\right)^x$



6. $y = 5(2)^x$



A2.U5.C2.**H.01.Vocab.ApplicationExp**

Term	Definition	Diagram/Visual
Exponential Function		
Exponential Growth		
Exponential Decay		
Growth Factor		
Growth Rate		
Decay Factor		
Decay Rate		

Application of Exponential Functions

In this lesson, we will find a function that models a given situation. Then use the graph of the function to make a prediction.

GUIDED NOTES

An ________________________ has the form $f(t) = a(1+r)^t$ where $a > 0$ and r is a constant percent increase (expressed as a decimal) for each unit increase in time, t. The base $1 + r$ is called the _______________, and the constan percent r, is called the _______________.

An ________________________ has the form $f(t) = a(1-r)^t$ where $a > 0$ and r is a constant percent decrease (expressed as a decimal) for each unit increase in time, t. The base $1 - r$ is called the _______________, and the constan percent r, is called the _______________.

Example!

A) Tony purchased a rare guitar in 2000 for $12,000. Experts estimate that its value will increase by 14% per year. Use a graph to find the number of years it will take for the value of the guitar to be $60,000. Write a function to model the growth in value for the guitar.

$$f(t) = a(1+r)^t$$

B) At the same time that Tony bought the $12,000 guitar, he also considered buying another rare guitar for $15,000. Experts estimated that this guitar would increase in value by 9% per year. Determine after how many years the two guitars will be worth the same amount. Write a function to model the growth in value for the second guitar.

$$g(t) = a(1 + r)^t$$

Use a graphing calculator to graph the two functions.

Use the graph to predict when the two guitars will be worth the same amount.

Use the intersection feature to find the *t*-value where $g(t) =$ __________

So, the two guitars will be worth the same amount __________ years after ____________.

Example!

C) The value of a truck purchased new for $28,000 decreases by 9.5% each year. Write an exponential function for this situation and graph it using a calculator. Use the graph to predict after how many years the value of the truck will be $5000.

"Purchased new for $28,000..." ________________

"...decreases by 9.5% each year." ________________

Substitute parameter values. ________________________

Simplify. ________________________________

Graph the function with a graphing calculator. Use WINDOW to adjust the graph settings so that you can see the function and the function values that are important.

Find when the value reaches $5000 by finding the intersection between $V_T(t) = 28{,}000(0.905)^t$ and $V_T(t) = 5000$ on the calculator.

The intersection is at the point $(17.26, 5000)$, which means after 17.26 years, the truck will have a value of $5000.

D) The value of a sports car purchased new for $45,000 decreases by 15% each year. Write an exponential function for the depreciation of the sports car, and plot it along with the previous example. After how many years will the two vehicles have the same value if they are purchased at the same time?

"Purchased new for $45,000________________

"...decreases by 15% each year." ________________

Substitute parameter values. ________________

Simplify. ____________________________

Add this plot to the graph for the truck value from Example C and find the intersection of the two functions to determine when the values are the same.

The intersection point is ________________.

After ______________years, the values of both vehicles will be ______________.

SELF CHECK	Find the function that models the given situation. Then use the graph of the function to make a prediction.

On federal income tax returns, self-employed people can depreciate the value of business equipment. Suppose a computer valued at $2765 depreciates at a rate of 30% per year. Use a graphing calculator to determine the number of years it will take for the computer's value to be $350.
John researches a baseball card and finds that it is currently worth $3.25. However, it is supposed to increase in value 11% per year. In how many years will the card be worth $26?

1. What is the difference between the graph of an exponential growth function and an exponential decay function?

1. Odette has two investments that she purchased at the same time.
Investment 1 cost \$10,000 and earns 4% interest each year.
Investment 2 cost \$8000 and earns 6% interest each year.

a. Write exponential growth functions that could be used to find $v_1(t)$ and $v_2(t)$, the values of the investments after t years.

b. Find the value of each investment after 5 years. Explain why the difference between their values, which was initially \$2000, is now less.

2. If A is deposited in a bank account at r% annual interest, compounded annually, its value at the end of n years $V(n)$ can be found using the formula $V(n) = A\left(1+\frac{r}{100}\right)^n$. Suppose that \$5000 is invested in an account paying 4% interest. Find its value after 10 years.

3. A quantity of insulin used to regulate sugar in the bloodstream breaks down by about 5% each minute after the injection. A bodyweight-adjusted dose is generally 10 units. How long does it take for the remaining insulin to be half of the original injection?

4. Colleen's office equipment is depreciating at a rate of 9% per year. She paid \$24,500 for it in 2009. Write the function $f(n)$ that expresses the value of the equipment after n years. What will the equipment be worth in 2015 to the nearest hundred dollars?

5. Stringed instruments like guitars and pianos create a note when a string vibrates back and forth. The distance that the middle of the string moves from the center is called the amplitude (a), and for a guitar, it starts at 0.75 mm when a note is first struck. Amplitude decays at a rate that depends on the individual instrument and the note, but a decay rate of about 25% per second is typical. Calculate the time it takes for an amplitude of 0.75 mm to reach 0.1 mm.

How Long Does It Take?

1. **A Population Problem:** A new solar system was discovered far from the Milky Way in 1999. After much preparation, NASA decided to send a group of astronauts to explore Exponentia, one of the planets in the system. Upon landing on the planet, the astronauts discovered life on the planet. Scientists named the creatures Viètians (vee-et-ee-ans), after the French mathematician François Viète who led the way in developing our present system of notating exponents. After observing the species for a number of years, NASA biologists determined that the population was growing by 10% each year.
   a. The estimated number of Viètians was 1 million in 1999 and their population increases 10% a year. Complete the table to show the population for the next 4 years after 1999.

Years since 1999	0	1	2	3	4
Population in millions	1				

   b. Write an equation for the population of Exponentia, P, as a function of the number of years, t, since 1999. How can you express the population as an expression in the table rather than as a computed value to help you see patterns to create the function?

   c. What was the population in 2005? What will the population be in 2015 if the population growth rate remains the same?

Use technology to graph the function in part (b).

i. In the context of this problem about the population of Exponentia, what are the domain and range?

Domain: ________________ Range: _________________

ii. What are some characteristics of the graph you can identify?

Suppose you want to know when the population reached 2 million. Write an equation that could be solved to answer this question. Determine the answer graphically and algebraically.

2. Suppose there are 25 bacteria in a Petri dish, and the number of bacteria doubles every 4 hours.
   a. How many bacteria will there be in 4 hours? In 8 hours? 1 hour? 2 hours? Record your answers in the table. Explain how you came up with your answers. (You can return to your answers later to make any corrections if you find your strategy was incorrect.)

Time (hours)	0	1	2	4	8
Number of bacteria	25				

   b. Write a function for the number of bacteria present after *t* hours. What does your exponent need to represent? How can you determine these exponents if you know the number of hours that have passed?

   c. Use the function to check your answers that you wrote in the table of part (a). Do you need to make any corrections? If so, make these corrections.

   d. Use your function to determine the number of bacteria after 24 hours.

   e. Determine how long it will take to have 5000 bacteria. Determine the answer graphically and algebraically.

   f. The bacteria double every 4 hours. Suppose we want to know the **growth rate per hour**. Use properties of exponents to rewrite the function you obtained in part (b) so the exponent is *t*, not $\frac{t}{4}$. How can you now determine the growth rate per hour?

3. Suppose for a particular patient and dosing regimen a drug reaches its peak level of 300 mg in the bloodstream. The drug is then eliminated from the bloodstream at a rate of 20% per hour.
   a. How much of the drug remains in the bloodstream 2 hours after it reaches its peak level of 300 mg? How much is there 5 hours after the peak level? Make a table of values to record your answers. So that a pattern is more apparent, write the expressions used to obtain your answers.

Time (hours) since reaching peak level	0	1	2	3	4	5
Amount of drug (mg) in bloodstream	300					

   b. Using your work from part (a), write expressions for each computed value using the initial amount, 300 mg.

c. Write a function f that gives the amount of the drug in the patient's bloodstream t hours after reaching its peak level.

d. Use the function you wrote in part (c) to compute the amount of the drug after 1 hour, 2 hours, 3 hours, 4 hours, and 5 hours. Are these amounts the same as those you wrote in the table in part (a)?

e. Use technology to graph the function. Explain how to use the graph to determine how long it will take to have less than 10 mg of the drug in the bloodstream.

f. Write an equation that you could solve to determine when exactly 10 mg of the drug remains in the bloodstream. Solve the equation algebraically. Can you use the solution to this equation to answer the question in part (e)?

4. Which of the problems in this section represent exponential growth? Which represent exponential decay?

A2.U5.C2.**H.05.HW.ApplicationExp**

1. The annual sales for a fast food restaurant are $650,000 and are increasing at a rate of 4% per year. Write the function $f(n)$ that expresses the annual sales after n years. Then find the annual sales after 5 years.

2. If a basketball is bounced from a height of 15 feet, the function $f(x) = 15(0.75)^x$ gives the height of the ball in feet of each bounce, where x is the bounce number. What will be the height of the 5th bounce? Round to the nearest tenth of a foot.

3. Starting with 25 members, a club doubled its membership every year. Write the function $f(n)$ that expresses the number of members in the club after n years. Then find the number of members after 6 years.

4. During a certain period of time, about 70 northern sea otters had an annual growth rate of 18%. Write the function $f(n)$ that expresses the population of sea otters after n years. Then find the population of sea otters after 4 years.

5. In 1995, the population of a town was 33,500. It is decreasing at a rate of 2.5% per decade. Write the function, $f(n)$, that expresses the population of the town after n decades. What is the expected population of the town in the year 2025 to the nearest hundred?

6. The value of a company's equipment is $25,000 and decreases at a rate of 15% per year. Write the function, $f(n)$, that expresses the value of the equipment after n years. Then find the value of the equipment in year eight.

A2.U5.C2.**I.O1.Vocab.GraphLog**

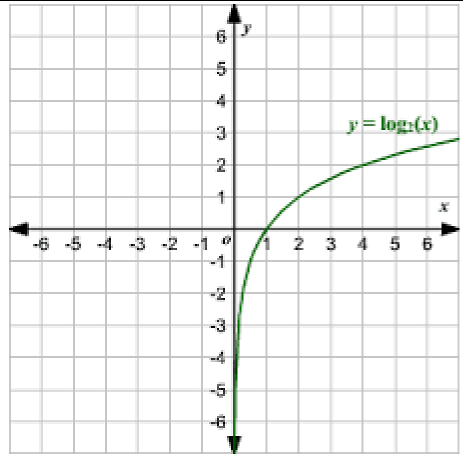
Term	Definition	Diagram/Visual
Logarithmic Function		
Transformations		
Domain		
Range		
End Behavior		
X-Intercept		
Y-Intercept		
Asymptotes		

Graphing & Describing Logarithmic Functions

In this lesson, we will graph and describe logarithmic functions.

GUIDED NOTES

The graph of the logarithmic function $f(x) = \log_2 x$ is shown. We will use this function to illustrate how to graph and describe exponential functions.



To graph, make a table, plot the points and connect the points with a smoth curve.

x	-2	-1	0	1	2	3
$f(x) = \log_2 x$						

Domain:

Range:

***x*-intercept**:
(plug 0 in for y and solve)

***y*-intercept**:
(plug 0 in for x and solve)

Asymptote:
(always $x = h$)

End Behavior:

Transformations of Logarithmic Functions

GUIDED NOTES

Each part of the function helps you describe the transformations.

a<0 ______________
a>1 ______________
a<1 ______________

$(x - h)$ ___________
$(x + h)$ ___________

$$f(x) = a\log_b(x - h) + k$$

$+k$ __________
$-k$ __________

List the transformations for each function from the parent rational function.

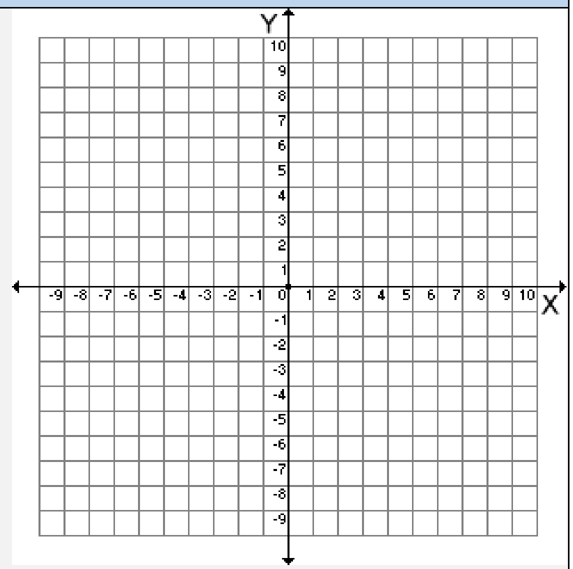
$g(x) = -2\log_2(x - 1) - 2$

$g(x) = 2\log(x + 2) + 4$

$g(x) = \frac{1}{2}\log_3(x + 1) + 2$

Example! Graph and describe each function.

$$f(x) = \log(x-1)+5$$



Transformations:

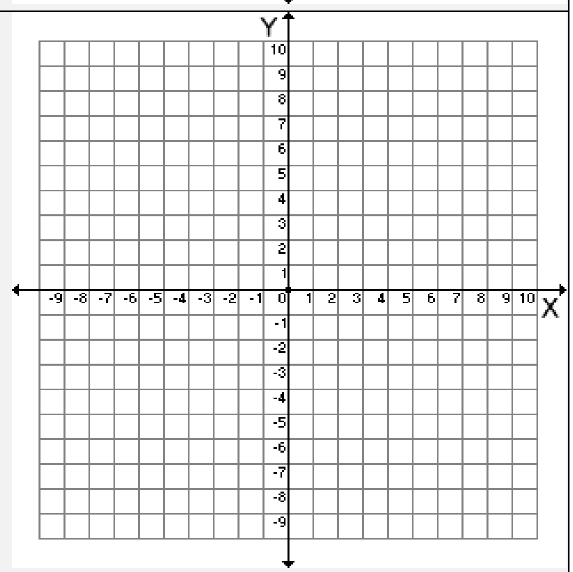
Domain: **Range**:

x-intercept: **y-intercept**:

Asymptote:

End Behavior: $As\ x \to$ _______, $y \to$ _____
$As\ x \to$ ______, $y \to$ _______

$$f(x) = \frac{1}{2}\log_2(x+1)+2$$



Transformations:

Domain: **Range**:

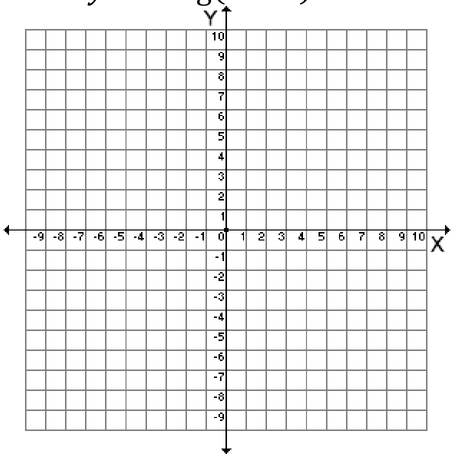
x-intercept: **y-intercept**:

Asymptote:

End Behavior: $As\ x \to$ _____, $y \to$ _____
$As\ x \to$ _____, $y \to$ _____

SELF CHECK Graph and describe each function.

$y = 2\log(x+2) + 4$



Transformations:

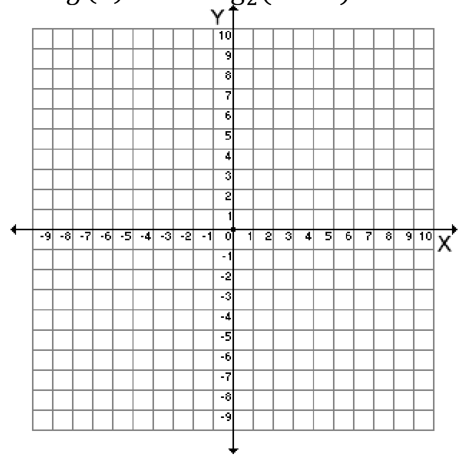
Domain: **Range:**

x-intercept: **y-intercept:**

Asymptote:

End Behavior: $As\ x \to$ ____, $y \to$ ____
$As\ x \to$ ____, $y \to$ ____

$g(x) = -2\log_2(x-1) - 2$



Transformations:

Domain: **Range:**

x-intercept: **y-intercept:**

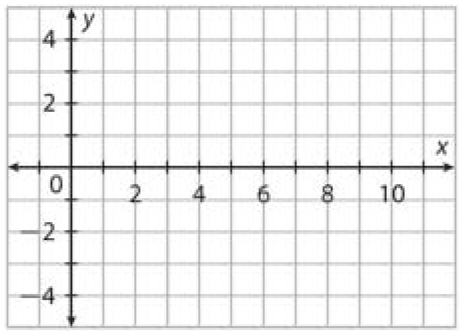
Asymptote:

End Behavior: $As\ x \to$ ____, $y \to$ ____
$As\ x \to$ ____, $y \to$ ____

Questions To Ponder

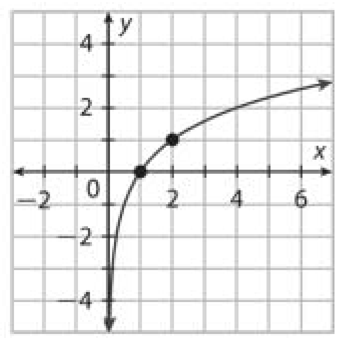
1. Why is the equation of the asymptote $x = h$?
2. Which transformations of $f(x) = \log_b x$ change the function's range?

1. Graph the logarithmic functions $f(x) = \log_2 x$, $f(x) = \log x$, and $f(x) = \ln x$ on the same coordinate plane. To distinguish the curves, label the point on each curve where the y-coordinate is 1.

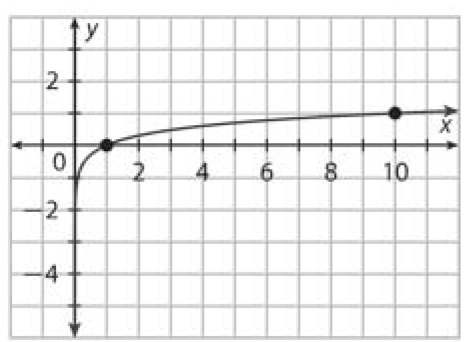


Graph each function on the same graph with its parent function. Then list the transformations, domain, range, asymptote, intercepts, and end behavior.

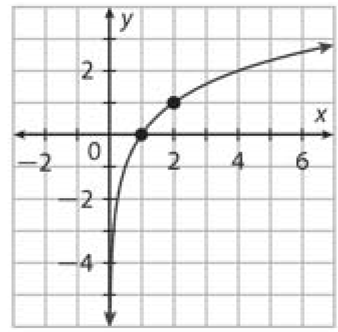
2. $g(x) = -4\log_2(x+2) + 1$



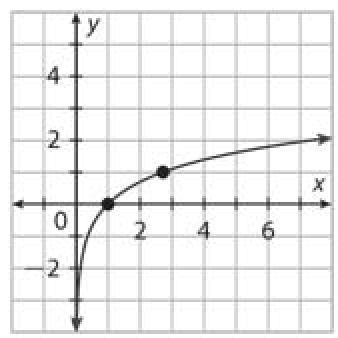
3. $g(x) = 3\log(x-1) - 1$



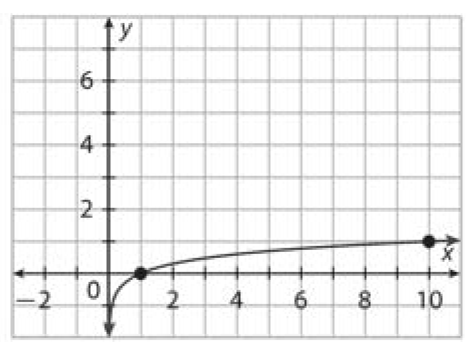
4. $g(x) = \frac{1}{2}\log_2(x-1) - 2$



5. $g(x) = -4\ln(x-4) + 3$



6. $g(x) = -2\ \log(x+2) + 5$

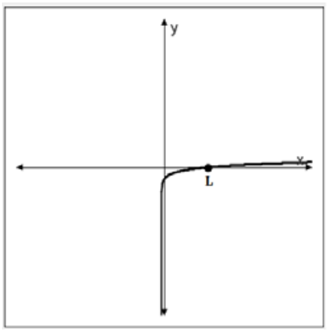


Part 1 amd 3: PRE/POST-ASSESSMENT

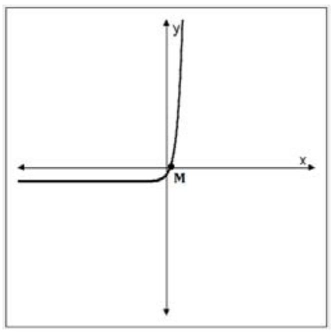
Graphing Exponential & Logarithmic Functions

Match each graph below to its function.

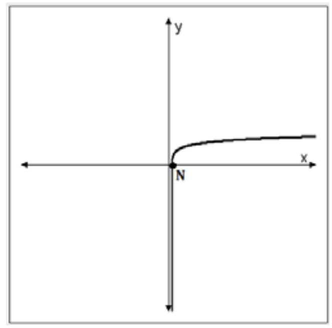
A)



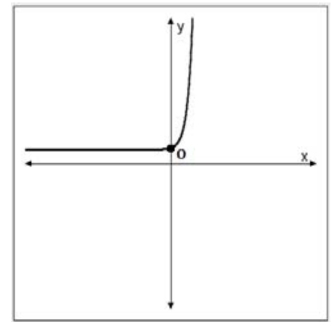
B)



C)



D)



1) $f(x)=2^{x+1}-4$ matches graph _________. I can tell because

____

2) $f(x)=2^{x-1}+4$ matches graph _________. I can tell because

____

3) $f(x)=\log_2(x+1)-4$ matches graph _________. I can tell because

____

4) $f(x)=\log_2(x-1)+4$ matches graph _________. I can tell because

____

5) What is the ordered pair for point M on the graph above? ____________________Explain how you found your answer.____.

6) a)The asymptote of $g(x) = 4^{x-1} - 8$ is $y = -8$. Explain *algebraically* why this is the case.

____

____

b) How can the asymptote of $g^{-1}(x)$ be found?

____

____

7) Ms. Math asked Becky to find the Domain and Range of g(x), and she asked Sherry to find the Domain and Range of $g^{-1}(x)$. Together they figured out a way to cut their work in half. What did they do?

____

____

Part 2: COLLABORATIVE ACTIVITY

Card Set Option 1

1-1 $f(x) = 3^{(x+1)} - 6$

Domain: ______All Real Numbers________

Range:______________________________

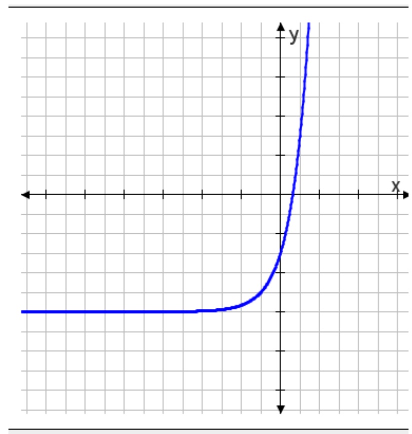
Asymptote:___________________________

x-intercept: _______(-0.63, 0)_____________

y-intercept: ____________________________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

1-b



1-2 $f(x) = 3^{(x-6)} + 1$

Domain: _____________________________

Range:___________[1,∞)____________________

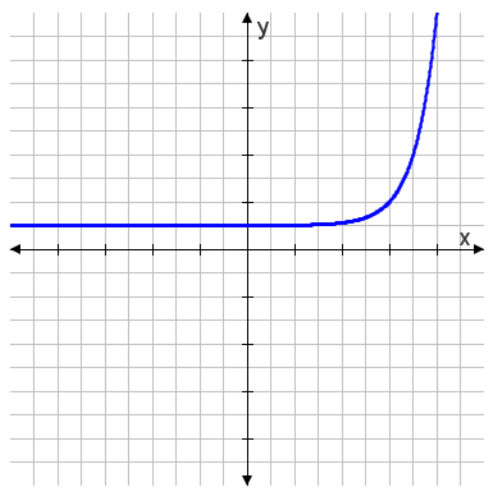
Asymptote:__________________________

x-intercept: ___________________________

y-intercept: ________≈(0, 1.001)____________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

1-c



1-3 $f(x) = 3^{(x-1)} + 6$

Domain: _____________________________

Range:______________________________

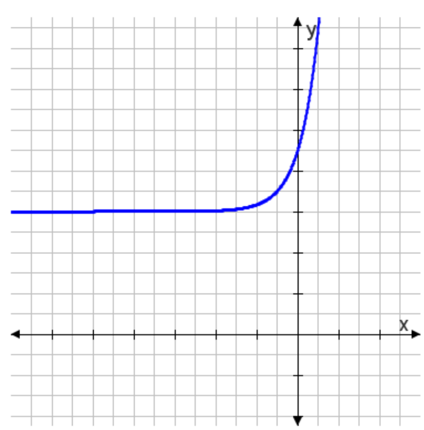
Asymptote:______y=6___________________

x-intercept: _____NONE__________________

y-intercept: ___________________________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

1-a



Card Set Option 1, page 2

1-4 $f(x) = \log_3(x - 6) + 1$

Domain: ______(6,∞)__________________________

Range:_________________________________

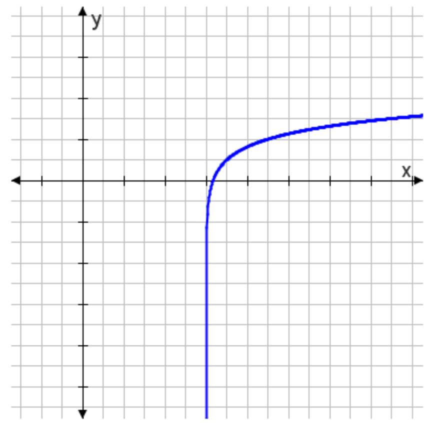
Asymptote:____________________________

x-intercept: ______(9,0)_______________

y-intercept: _____________________________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$ ___

1-e



1-5 $f(x) = \log_3(x + 6) - 1$

Domain: _________(-6, 0)_________________

Range:_________________________________

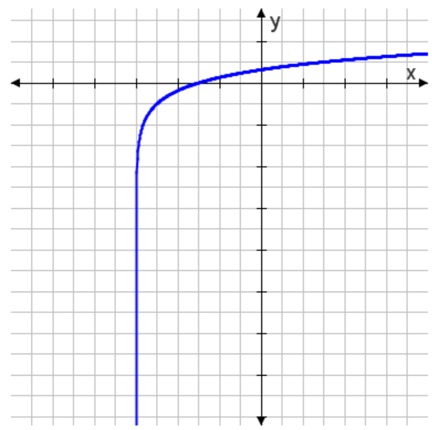
Asymptote:____________________________

x-intercept: _____________________________

y-intercept: ___________≈(0, 0.63)__________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

1-d



1-6 $f(x) = \log_3(x + 1) + 6$

Domain: _______________________________

Range:_______All Real Numbers______________

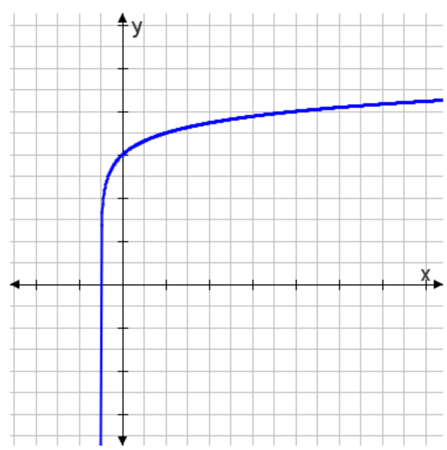
Asymptote:____________________________

x-intercept: ______≈(1.001, 0)___________

y-intercept:_____________________________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

1-f



Card Set Option 2

2-1 $f(x) = 3^{(x+1)} - 6$

Domain: ______________________________

Range: ______________________________

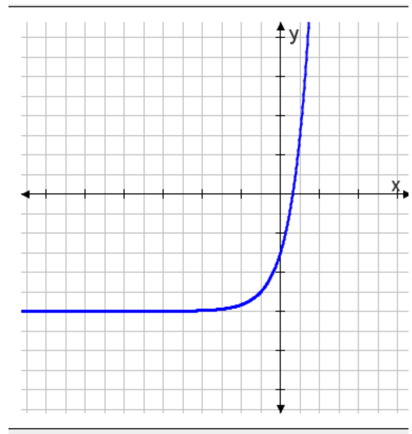
Asymptote: ___________________________

x-intercept: ___________________________

y-intercept: ___________________________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

2-b



2-2 $f(x) = 3^{(x-6)} + 1$

Domain: ______________________________

Range: ______________________________

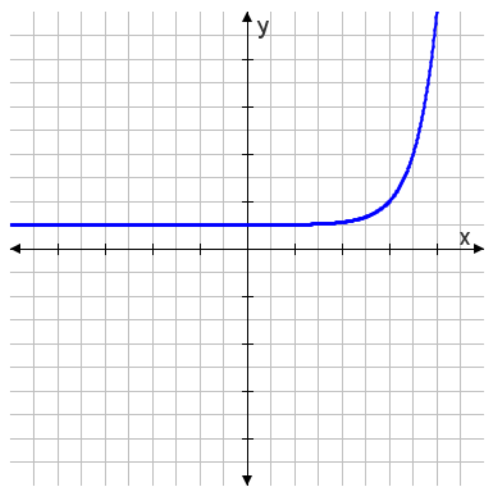
Asymptote: ___________________________

x-intercept: ___________________________

y-intercept: ___________________________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

2-c



2-3 $f(x) = 3^{(x-1)} + 6$

Domain: ______________________________

Range: ______________________________

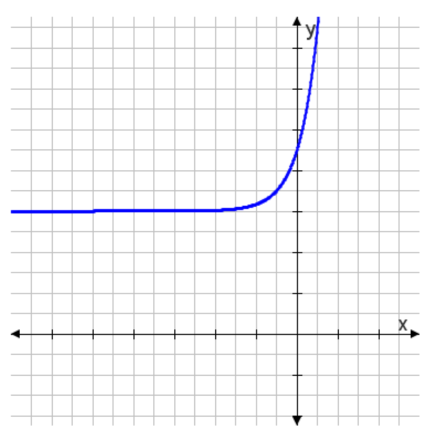
Asymptote: ___________________________

x-intercept: ___________________________

y-intercept: ___________________________

End Behaviors: $As\ x \to \infty, y \to$
$As\ x \to \infty, y \to$

2-a



Card Set Option 2, page 2

2-4 $f(x) = \log_3(x - 6) + 1$

Domain: ______________________________

Range: ______________________________

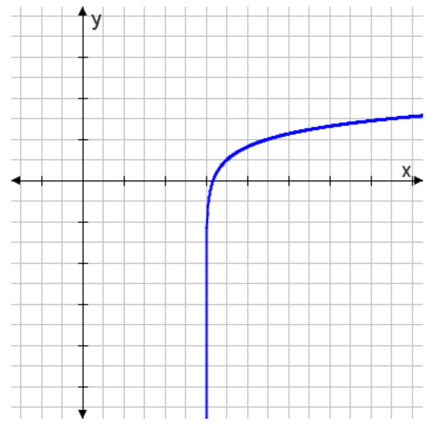
Asymptote: ___________________________

x-intercept: ____________________________

y-intercept: ____________________________

End Behaviors: $As\ x \rightarrow \infty, y \rightarrow$
$As\ x \rightarrow \infty, y \rightarrow$

2-e



2-5 $f(x) = \log_3(x + 6) - 1$

Domain: ______________________________

Range: ______________________________

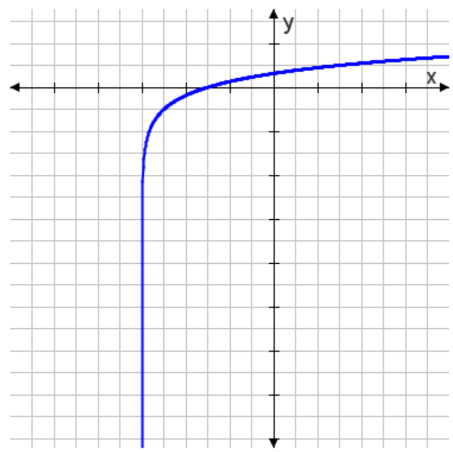
Asymptote: ___________________________

x-intercept: ____________________________

y-intercept: ____________________________

End Behaviors: $As\ x \rightarrow \infty, y \rightarrow$
$As\ x \rightarrow \infty, y \rightarrow$ ___________

2-d



2-6 $f(x) = \log_3(x + 1) + 6$

Domain: ______________________________

Range: ______________________________

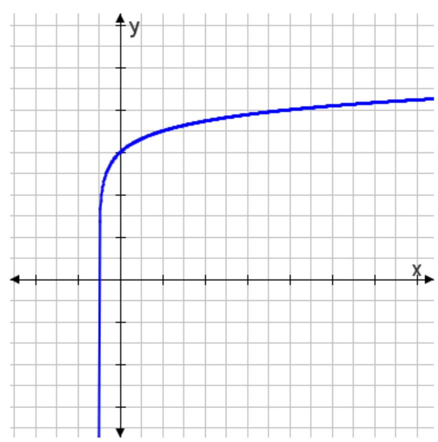
Asymptote: ___________________________

x-intercept: ____________________________

y-intercept: ____________________________

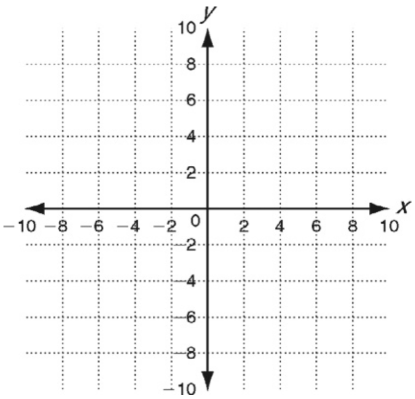
End Behaviors: $As\ x \rightarrow \infty, y \rightarrow$
$As\ x \rightarrow \infty, y \rightarrow$ _______________

2-f

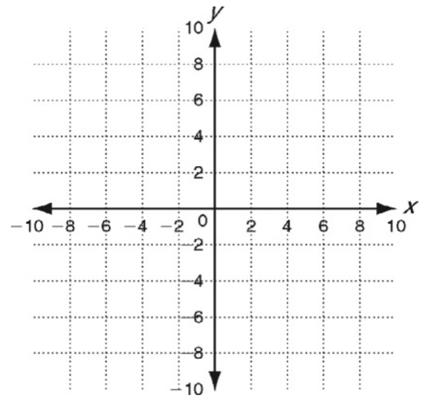


Graph and describe each function.

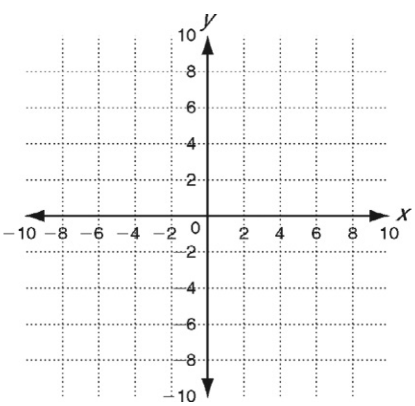
1. $g(x) = 5\log_2(x+2) - 1$



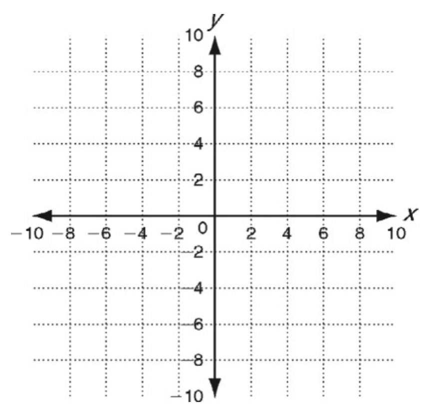
2. $f(x) = 3\log_4(x+6)$

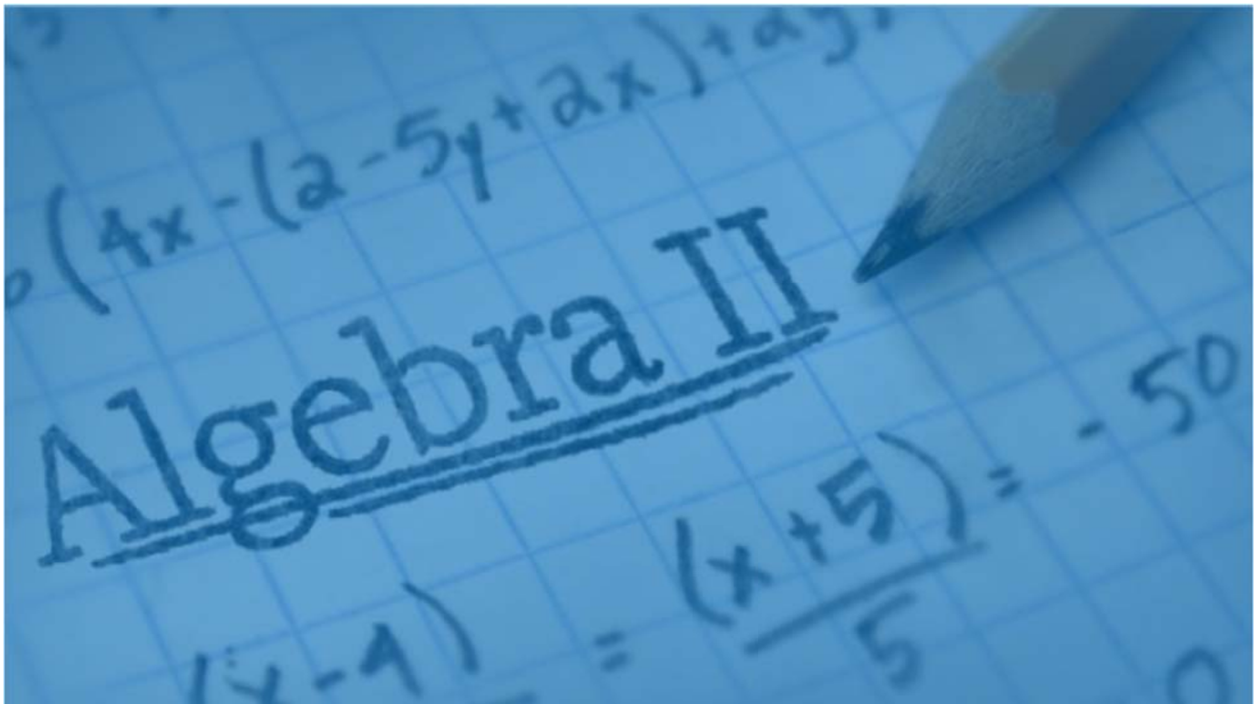


3. $g(x) = -2\log_8(x+9) + 3$



4. $g(x) = -\log(x+5) + 2$





Unit 6

Mathematical Modeling

Algebra 2

Unit 6: Mathematical Modeling

Concept 1: Composition and Inverse Functions

Lesson A: Function Composition	(A2.U6.C1.A.____.FunctComposition)
Lesson B: Verify Inverses Using Compositions	(A2.U6.C1.B.____.InversesByComposition)
Lesson C: Find Inverses	(A2.U6.C1.C.____.FindInverses)
Lesson D: Graph Inverses, Is Inverse a Function	(A2.U6.C1.D.____.GraphInverses)

Concept 2: Absolute Value and Piecewise Functions

Lesson E: Graph Absolute Value, ID Characteristics	(A2.U6.C2.D.___.GraphAbsValue)
Lesson F: Graph Piecewise Functions	(A2.U6.C2.E.____.Piecewise)
Lesson G: Graph Step Functions	(A2.U6.C2.F.____.StepFunctions)

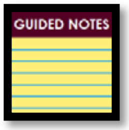
Concept 3: Solving Systems of Linear Equations and Linear Programming with Applications

Lesson H: Solve Systems and of Linear Inequalities	(A2.U6.C3.F.____.InequalitySystems)
Lesson I: Linear Programming	(A2.U6.C3.G. ____.LinearProgramming)

A2.U6.C1.**A.01.Vocab.CompositionOfFunctions**

Term	Definition	Notation	Diagram/Visual
Function Composition			
Composite Function			

Compostion of Functions



Composition of Functions

A way of combining two functions is to form the **composition** of one with the other.

For instance, if $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$f(g(x)) = f(x + 1)$

$= (x + 1)^2$.

This composition is denoted as $f \circ g$ and reads as "f composed with g."

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 2.36.)

Here are the steps we can use to find the composition of two functions:

Step 1: Rewrite the composition in a different form. For example, the composition (f o g)(x) needs to rewritten as f(g(x)).

Step 2: Replace each occurrence of x found in the outside function with the inside function. For example, in the composition of (f o g)(x) = f(g(x)), we need to replace each x found in f(x), the outside function, with g(x), the inside function.

Step 3: Simplify the answer.

(f o g)(x) = f(g(x)), the g function is inside of the f function

(g o f)(x) = g(f(x)), the f function is inside of the g function

If f(x) = –4x + 9 and g(x) = 2x – 7, find (f o g)(x).

(f o g)(x) = f(g(x)) Rewrite the composition in a different form.

= -4(2x - 7) + 9 Replace each occurrence of x in f(x) with g(x) = 2x – 7.

= -8x +37 Simplify the answer by distributing and combining like terms.

Thus, (f o g)(x) = -8x + 37

Example! If $h(x) = 3x – 5$ and $g(x) = 2x^2 – 7x$ find $(g \circ h)(x)$.

$(g \circ h)(x) = g(h(x))$ Rewrite the composition in a different form.

$= 2(3x - 5)^2 - 7(3x - 5)$ Replace each occurrence of x in g(x) with $h(x) = 3x – 5$.

$= 2(9x^2 – 30x + 25) – 7(3x – 5)$ Simplify the answer by distributing and combining like terms.

$= 18x^2 - 60x + 50 - 21x + 35$

$= 18x^2 – 81x + 85$

Thus, $(g \circ h)(x) = 18x^2 – 81x + 85$

SELF CHECK If $f(x) = x^2 – 4x + 2$ and $g(x) = 3x – 7$, find $(f \circ g)(x)$.

Questions To Ponder Does it matter the order in which I place one function into the other function?

Now it is your turn to try a few practice problems on your own.

Problem 1: If g(x) = –6x + 5 and h(x) = –9x – 11, find (g o h)(x).

Problem 2: If f(x) = $\sqrt{2x-5}$ and g(x) = $5x^2$– 3, find (g o f)(x).

Problem 3: If f(x) = –2x + 9 and g(x) = $-4x^2$ + 5x – 3, find (f o g)(x).

Problem 4: If f(x) = x – 3 and g(x) = $4x^2$ – 3x – 9, find (g o f)(x).

Problem 5: If g(x) = $\sqrt[3]{x-4}$ and h(x) = x^3 + 4, find (h o g)(x).

Composition of Functions – Application

Part 1: Applications of Function Composition

In the mail, you receive a coupon for $5 off of a pair of jeans. When you arrive at the store, you find that all jeans are 25% off.

Let x represent the original cost of the jeans.

1. Write a function, f(x), that represents the effect of your original coupon.

2. Write a function, g(x), that represents the effect of the 25% discount at the store.

3. Write a function, h(x), that represents how much you would pay if you use the mail coupon first followed by applying the discount from the store.

4. Write a function, j(x), that represents how much you would pay if you use the store discount first, followed by the mail coupon.

5. You find a pair of jeans for $36. How much would you pay for it using both functions h(x) and J(x).

6. If you only have $40 with you, what's the most expensive pair of jeans you can purchase? (do not consider tax).

7. Determine when you would want to use h(x) applying the $5 coupon first, and determine when you would want to use j(x) applying the 25% off first.

Part 2: Composite Functions and their domain and range

Carrie, marine biologist is performing experiments along the continental slope off of the coast of baja California where the biodiversity is very dynamic. Scientists have proven that biodiversity is closely linked to the temperature of the water. Carrie wants to monitor the temperature of the ocean at different depths along the continental slope, to help record the changes in biodiversity due to changes in the temperature of the water. At the same time she does not want the robot to crash into the continental slope, so she needs to take into effects the speed of the currents.

Earlier scientists have found that the speed of ocean current as a function of depth. The speed, S, depends on depth, d, according to the following formula.

$$S(d) = 3d + 1$$

Where S is measured in meters per second and d is measured in meters.

Suppose that the depth of a research robot depends on time, t, according to the formula:

$$d(t)=(1/27)t^2$$

1. Use function composition to write the speed of the current at the depth of the robot as a function of time. Give an exact expression.

2. What is the speed of the current at the depth of the robot after 9 seconds? Round your answer, if necessary, to the nearest integer.

3. What is the realistic domain of the robot and what does that represent?

4. What is the realistic range of the robot and what does that represent? (It might help to find the vertex of d(t))

Notes Composite Functions and Application

Function Review:

f(x) = 3x + 2

x = input = 4

f(4) =

Composite Functions -> the embedding of functions

f (x) = 3x + 2 ***g***(x) = 4x – 1

f(***g***(x))

x = input = 3

f(***g***(3))

g(3) = 4(3) – 1 ***f***(x) = 3x + 2

= 12 – 1 = 11 ***f***(11)=

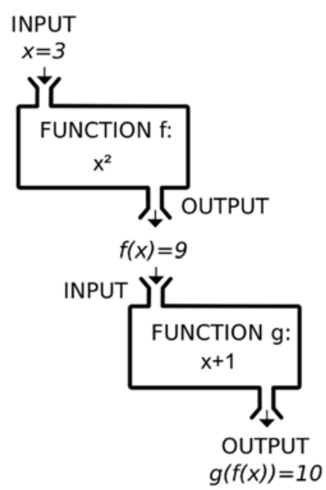
f(***g***(x)) = ***f***(4x - 1) b/c ***g***(x) = 4x – 1

= 3(4x – 1) + 2

= 12x – 3 + 2

= 12x – 1

A way of thinking about composite functions is taking the output of one function and making it the input of another. (Like a Double Function Machine)



Example 2: $f(x) = x^2 - 3$ $g(x) = 2x - 1$

$f(g(2)) = f(3)$ ***b/c*** $g(2) = 2(2) - 1 = 3$

$= 3^2 - 3$

$= 6$

$f(g(x)) = f(2x - 1)$

$= (2x - 1)^2 - 3$

Part 1: Basic Practice

I. Let $f(x) = 2x - 1$, $g(x) = 3x$, and $h(x) = x^2 + 1$. Compute the following:

1. $f(g(x))$
2. $f(h(x))$
3. $g(h(x))$
4. $f(g(-3))$
5. $f(h(7))$
6. $g(h(0))$

II. Let f(x) = 9 – x , g(x) = x^2 + x, and h(x) = x – 2. Compute the following:

7. g(f(x))

8. f(g(x))

9. h(f(x))

10. g(f(-3))

11. f(g(11))

12. h(f(-6))

Composition of Functions

Let $f(x) = 2x - 1$, $g(x) = 3x$, and $h(x) = x^2 + 1$.

1. $f(g(x))$
2. $f(h(x))$
3. $g(h(x))$
4. $h(f(x))$
5. $h(g(f(x)))$

Let $f(x) = 9 - x$, $g(x) = x^2 + x$, and $h(x) = x - 2$.

6. $f(h(x))$
7. $h(g(x))$
8. $g(h(x))$
9. $g(h(f(x)))$
10. $h(g(f(x)))$

A2.U6.C1.**B.01.Vocab.InverseByComposition**

Term	Definition	Notation	Diagram/Visual
Inverse Functions			

Composition of a function and its inverse function

If both the domain and the range are all real numbers for a function f(x), and if f(x) has an inverse $f^{-1}(x)$, then:

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

If they both are not equal to x, then it is not an inverse.

Determine if f(x) = 3 + 2x and g(x) = $\frac{x-3}{2}$ are inverses by finding f(g(x)) and g(f(x))

$f(g(x)) = 3 + 2\left(\frac{x-3}{2}\right) = 3 + x - 3 = x$

$g(f(x)) = \frac{(3+2x)-3}{2} = \frac{2x}{2} = x$

SELF CHECK

Determine if f(x) = x + 2 and g(x) = x -2 are inverses by finding f(g(x)) and g(f(x))

Determine whether the two functions given are inverses by finding f(g(x)) and g(f(x)).

Problem 1. $f(x) = \frac{2}{3}x - \frac{1}{4}$ $g(x) = \frac{12x+3}{8}$

Problem 2. $f(x) = \frac{2}{3}x + 5$ $g(x) = \frac{3x - 15}{2}$

Problem 3. $f(x) = (x + 10)^2$ $g(x) = \sqrt{x} + 10$

Problem 4. $f(x) = x^2 - 8$ $g(x) = \sqrt{x - 8}$

Problem 5. $f(x) = \frac{x-11}{3}$ $g(x) = 3x + 11$

EXPONENTS AND LOGARITHMS AS INVERSES (ILLUSTRATIVE MATHEMATICS)

Let f be the function defined by $f(x) = 10^x$ and g be the function defined by $g(x) = log_{10}(x)$.

a. Sketch the graph of $y = f(g(x))$. Explain your reasoning.

b. Sketch the graph of $y = g(f(x))$. Explain your reasoning.

c. Let f and g be any two inverse functions. For which values of x does $f(g(x)) = x$? For which values of x does $g(f(x)) = x$?

Inverses by Composition - Find, Verify

For the following pairs of functions f and g find the composition functions f(g(x)) and g(f(x)). Also, ascertain whether they are inverses or not.

Problem 1. $f(x) = 2x - 5$ $g(x) = x^2 - 3x$

Problem 2. $f(x) = \sqrt{3x - 1}$ $g(x) = x^2 + 1$

Problem 3. $f(x) = \sqrt{3x - 1}$ $g(x) = \frac{x^2+1}{3}$

Problem 4. $f(x) = x^2 - 5$ $g(x) = \sqrt{x + 5}$

Problem 5 $f(x) = (3x-4)^2$ $g(x) = \frac{\sqrt{x}+4}{3}$

A2.U6.C1.**C.01.Vocab.FindInverses**

Term	Definition	Notation	Diagram/Visual
Inverse Functions			

GUIDED NOTES

Inverse Functions

We know that a set of ordered pairs can represent a function.

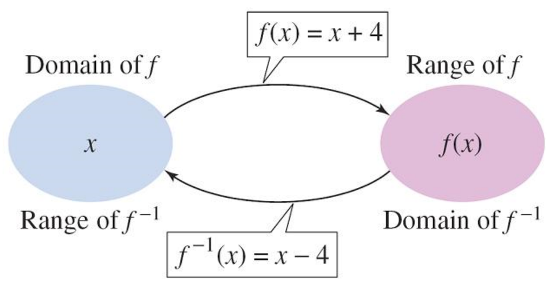
For instance, the function $f(x) = x + 4$ from the set
$A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$f(x) = x + 4$: $\{(1, 5), (2, 6), (3, 7), (4, 8)\}$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}.

It is a function from the set B to the set A, and can be written as follows.

$f^{-1}(x) = x - 4$: $\{(5, 1), (6, 2), (7, 3), (8, 4)\}$



FINDING INVERSE FUNCTIONS ALGEBRAICALLY

To find the inverse function algebraically

1. *Change f(x) to y.*
2. *Switch the x and y in the function*
3. *Solve for y*
4. *Change y to* $f^{-1}(x)$

Given the table of f(x), complete the table for $f^{-1}(x)$.

f(x) $f^{-1}(x)$

X	-3	*-2*	*-1*	*0*		*2*
f(x)	*10*	*6*	*4*	*1*		*-3*

1) $f(x) = x^2$

2) $f(x) = 3 - 4x$

3) $f(x) = \frac{x^3}{2} - 7$

4) $f(x) = \frac{x+3}{x-2}$

5) $f(x) = \sqrt[3]{2x-1}$

INVERTIBLE OR NOT (ILLUSTRATIVE MATHEMATICS)

The table below shows some input-output pairs of two functions f and g that agree for the values that are given but some of their output values are missing.

t	0	15	30	45	60	75	90	105	120
f(t)	0	0.5		1.3	2	2.7		4	
g(t)	0	0.5		1.3	2	2.7		4	

a. Complete the table in a way so that f could be invertible and so that g is definitely not invertible.

b. Graph both functions and explain from the graph why f is invertible and g is not.

c. Come up with two real life situations that f and g could be representing.

d. Find and interpret the value $f^{-1}(4)$ in terms of these contexts.

Find the inverse of each function.

1) $f(x)=(x-2)^3 + 3$

7) $f(x) = \sqrt[3]{x+1} + 2$

2) $f(x) = \frac{4}{x+2}$

8) $f(x) = \frac{-2x+1}{3}$

3) $f(x)= -(x-1)^3$

4) $f(x)= \sqrt{5x-1}$

5) $f(x)= \frac{10-x}{5}$

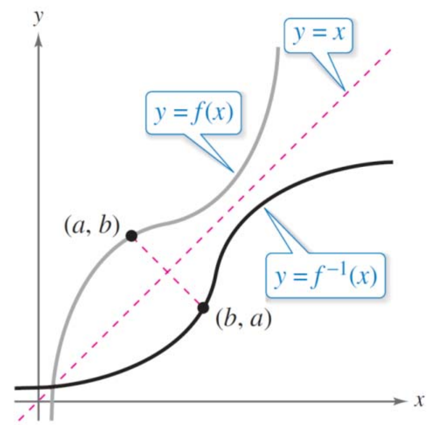
6) $f(x)=-(x-1)3$

Graphing Inverse Functions

GUIDED NOTES

The Graph of an Inverse Function

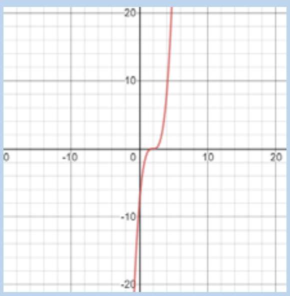
The graphs of a function f and its inverse function f^{-1} are related to each other in the following way.
If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa.
This means that the graph of f^{-1} is a reflection of the graph of f in the line y = x.



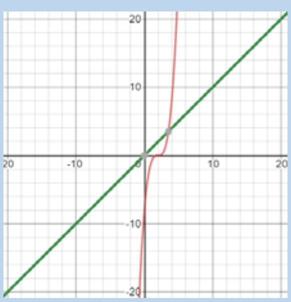
Graph $f(x) = (x-2)^3$ and its inverse.

First we graph $f(x) = (x-2)^3$

x	-1	0	1	2	3	4
f(x)	-27	-8	-1	0	1	8

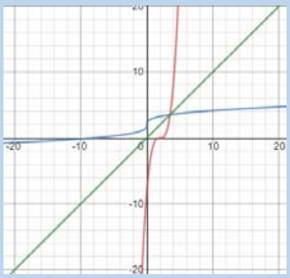


Then we can graph $f(x) = x$, because reflecting over $f(x) = x$ gives us the graph of the inverse.



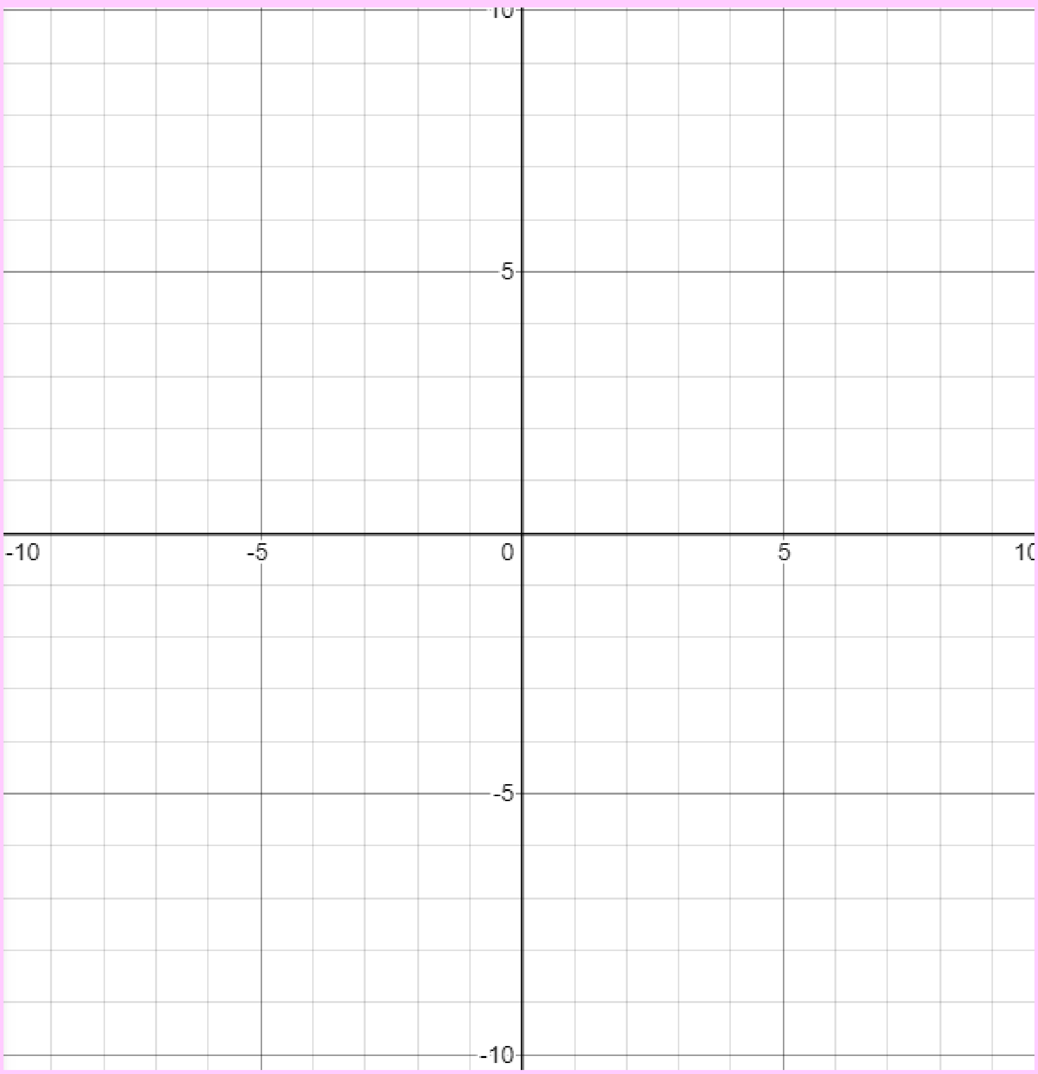
Now we can graph the inverse by simply switching the x and y values, or graphically by plotting points that are mirror images across f(x) = x

x	-27	-8	-1	0	1	8
f(x)	-1	0	1	2	3	4



SELF CHECK

Graph $f(x) = \frac{1}{2}x^2$ and its inverse.

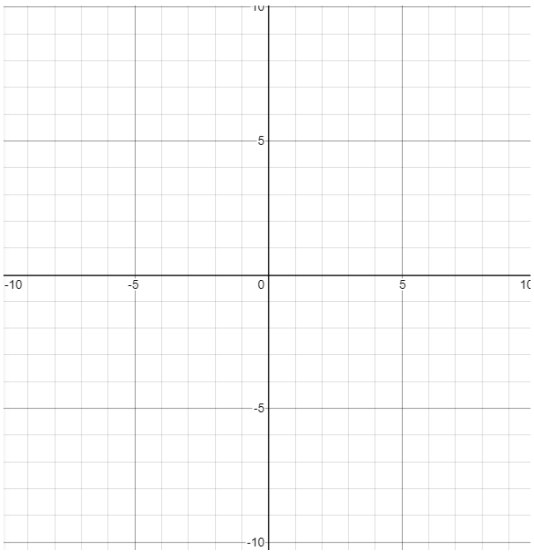


Questions To Ponder

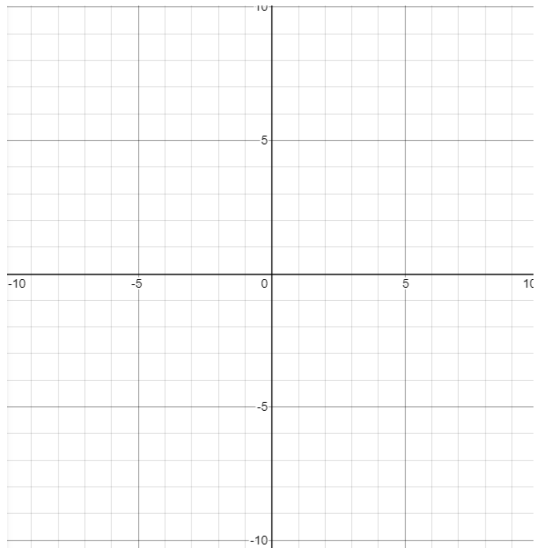
Will the inverse of a function always be a function as well?

Graph the function and its inverse.

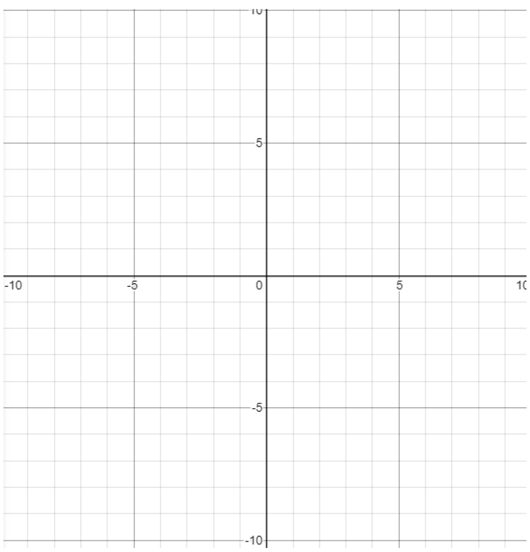
1) $f(x) = 5 - 3x$



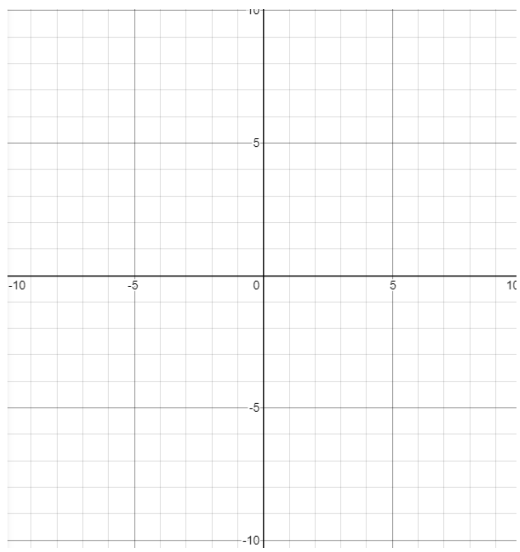
2) $f(x) = (x-2)^3 + 1$



3) $f(x) = \sqrt{x+3}$



4) $f(x) = \frac{2}{x}$



Inverse Graphing

1. Graph $f(x) = x^2 + 1$ and its inverse. Restrict the domain of f(x) so that $f^{-1}(x)$ is a function.

2. Graph $f(x) = x^3 + 1$ and its inverse. Restrict the domain of f(x) so that $f^{-1}(x)$ is a function.

3. Graph $f(x) = x^3 - 1$ and its inverse. Restrict the domain of f(x) so that $f^{-1}(x)$ is a function.

4. Graph $f(x) = |x^3 - 1|$ and its inverse. Restrict the domain of f(x) so that $f^{-1}(x)$ is a function.

5. Which of the following functions are one-to-one? For each of the functions find the inverse and, if necessary, restrict the domain of the original function so that the inverse is a function.

a) $f(x) = x + 4$	h) $f(x) = \sqrt{x}$
b) $f(x) = 2x$	i) $f(x) = \sqrt[3]{x}$
c) $f(x) = \dfrac{4}{x + 7}$	j) $f(x) = \sqrt{2x + 3}$
d) $f(x) = \dfrac{x + 4}{x - 3}$	k) $f(x) = \sqrt[3]{2x + 3}$
e) $f(x) = x^3 - 1$	l) $f(x) = 5$
f) $f(x) = x^4 - 1$	m) $f(x) = x^2 - 2x + 2$
g) $f(x) = (x - 2)^2 + 1$	n) $f(x) = 3x^2 - 6x + 1$

6. Show that each of the following functions are inverses by showing that $f(g(x)) = x$.

a) $f(x) = x^2 - 4;\ g(x) = \sqrt{x + 4}$

b) $f(x) = \dfrac{1}{x - 1};\ g(x) = \dfrac{1}{x} + 1$

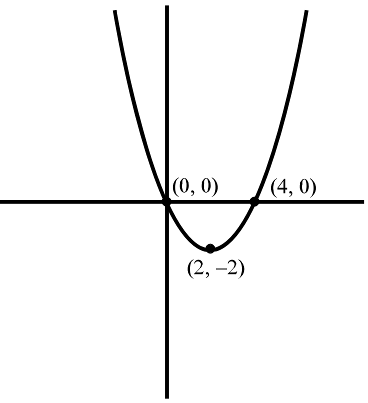
c) $f(x) = 2x + 3;\ g(x) = \dfrac{x - 3}{2}$

d) $f(x) = \dfrac{2x + 1}{2x - 1};\ g(x) = \dfrac{x + 1}{2(x - 1)}$

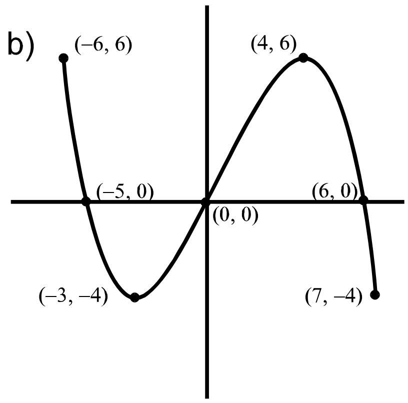
7. What conditions must be placed on a, b, c, and d in $f(x) = \dfrac{ax + b}{cx + d}$ so that $f^{-1}(x) = f(x)$?

8. Graph the inverse of each of the following functions. Where the function is not one-to-one, restrict the domain of the function so that the inverse will be a function.

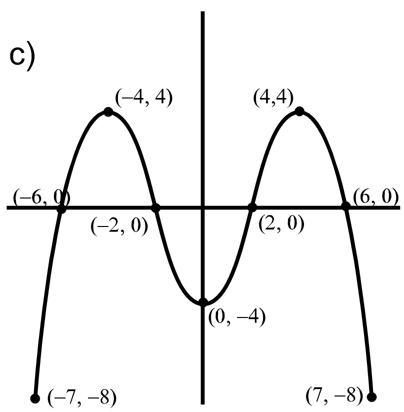
a)



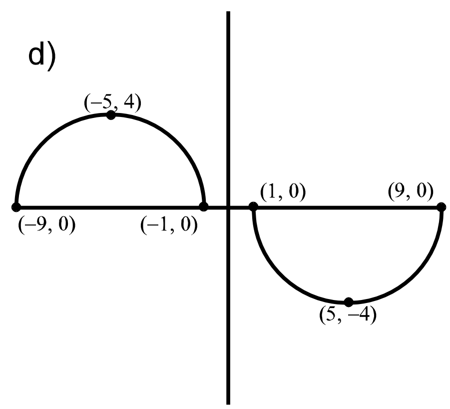
b)



c)

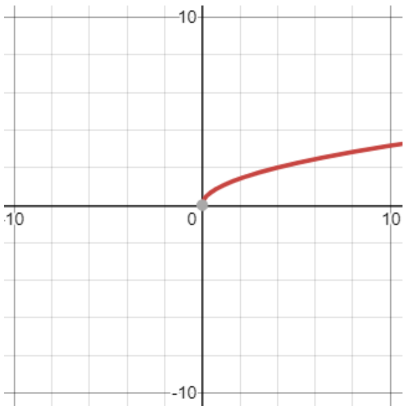


d)

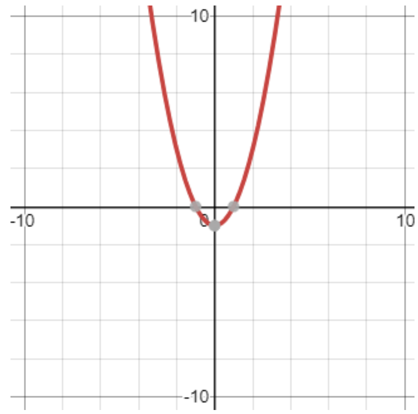


Graph the inverse for each relation below (put your answer on the same graph).

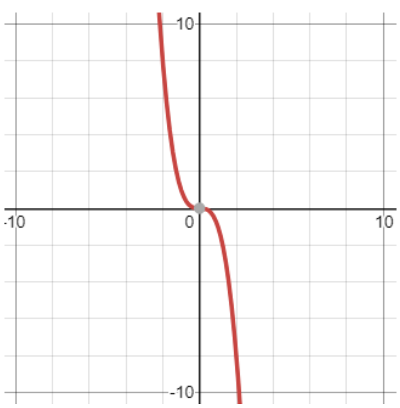
1.



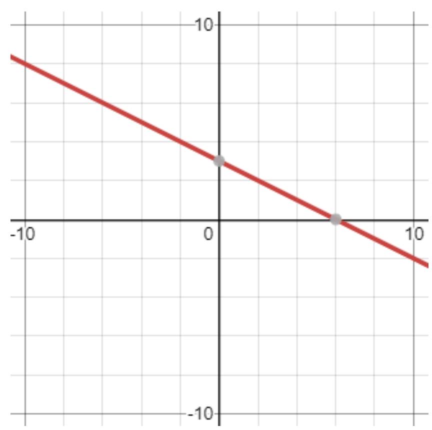
2.



3.



4.



Absolute Value Functions (The graphs always look like the letter V)
Absolute value is the distance from zero, hence it is always positive after you after you apply absolute value.

What effect does each one have on the parent graph?

$$y = a\,|\,x - h\,|\, + k$$

- a: stretch or shrink; up/down
- h: left/right
- k: up/down

This means the NEW vertex of the transformed function is (h,k)

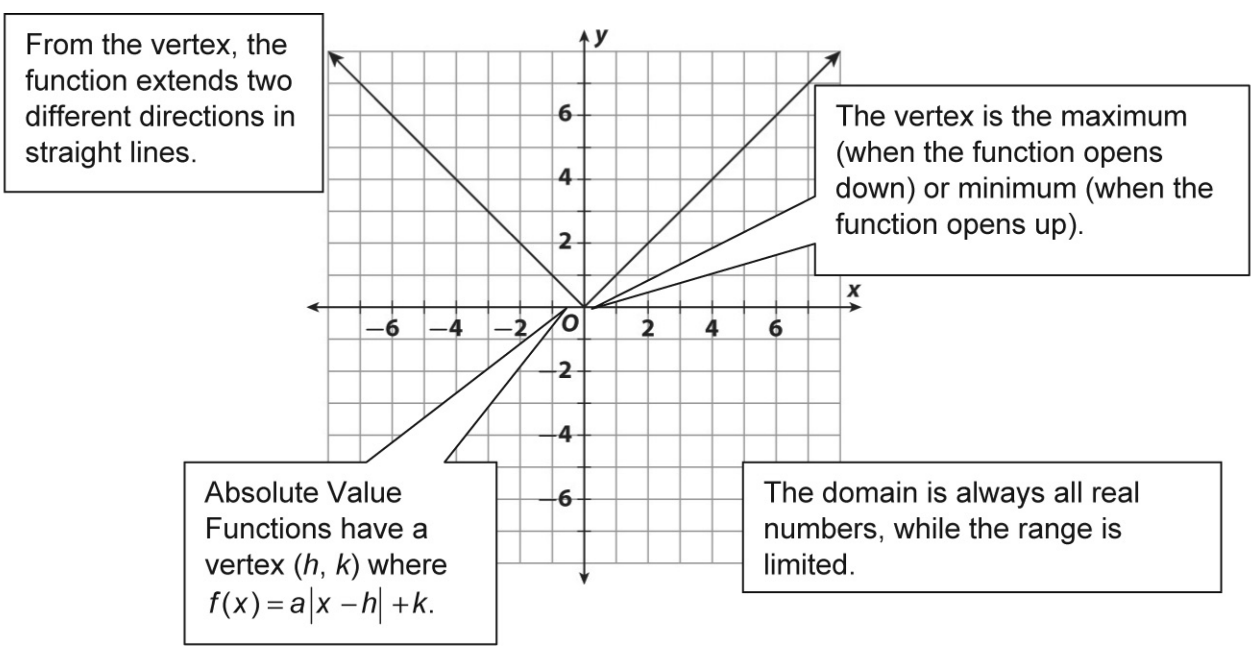
Steps for graphing absolute value functions:

1) Find the vertex
2) Create a table to represent function with vertex in the middle row.
3) Graph the points
4) Tell characteristics

Knowing the characteristics of a function helps when graphing.

Interval of increase is $(0, \infty)$ and the interval of decrease is $(-\infty, 0)$

End behavior as $x \to -\infty, f(x) \to \infty$ $\quad x \to \infty, f(x) \to \infty$



In this example the range is $[0, \infty)$

Example!

Determine the vertex (opposite of h, k) of the following functions.
State whether the graph will open up or down. ("a" positive or negative)
State whether the vertex will be a maximum or minimum.

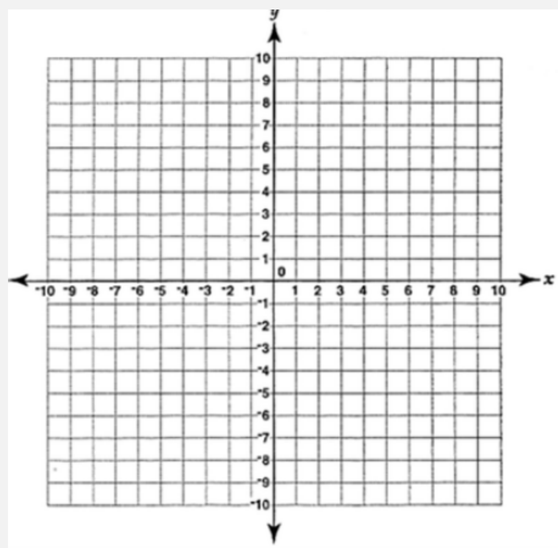
1) y = 2 |x - 2| + 3
2) y = -|x + 5| - 6
3) y = -2|x + 2|
4) y = 1/3 |x| + 5
5) y = |x|

For each absolute value function: Graph the function then identify the domain, the range, the vertex (and indicate if it is a max or min), end behavior, intervals of increase/decrease. Then graph the function.

6. $f(x) = -2|x-4| + 3$

Vertex: 4, 3 Max or Min

Table

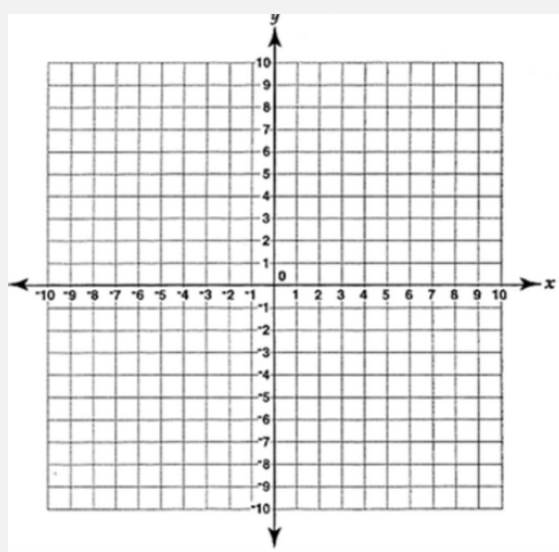


Domain: Range:
Interval of Increase:
Interval of Decrease:
End Behavior:
X int: Y int:

7. $f(x) = \frac{1}{2}|x+4| - 1$

Vertex: ________ Max or Min

Table

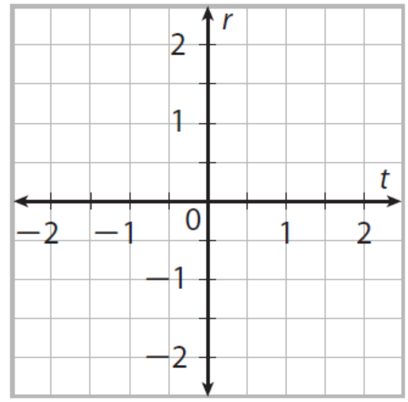


Domain: Range:
Interval of Increase:
Interval of Decrease:
End Behavior:

X int: Y int:

Example 8

A rainstorm begins as a drizzle, builds up to a heavy rain, and then drops back to a drizzle. The rate r (in inches per hour) at which it rains is given by the function $r = -0.5|t - 1| + 0.5$, where t is the time (in hours). Graph the function. Determine for how long it rains and when it rains the hardest.



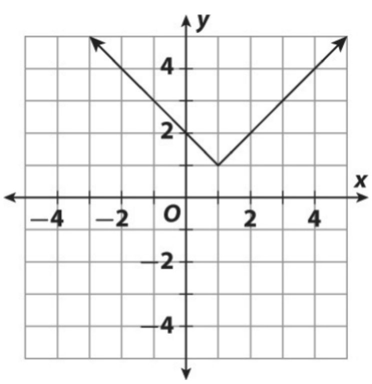
SELF CHECK

Evaluate each expression for $x = -1$ and $x = 2$.

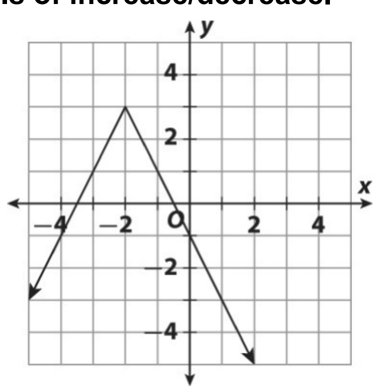
1. $|x| + 1$ ____________________
2. $|x - 4|$ ____________________
3. $3|x + 1|$ ____________________
4. $|2x|$ ____________________
5. $-|x| + 5$ ____________________
6. $-2|x - 3|$ ____________________

For each absolute value graph, identify the domain, the range, the vertex (and indicate if it is a max or min), end behavior, intervals of increase/decrease.

7.

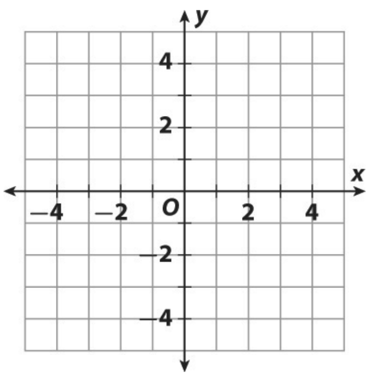


8.

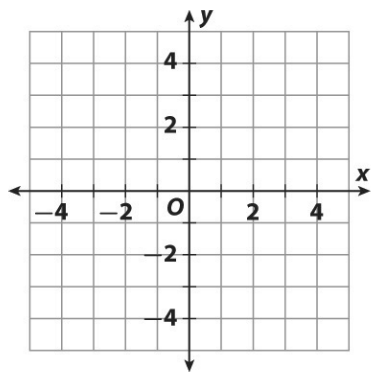


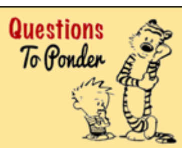
For each absolute value function: Graph the function then identify the domain, the range, the vertex (and indicate if it is a max or min), end behavior, intervals of increase/decrease.

9. $f(x) = |x| - 4$



10. $f(x) = |x + 3|$





What do you notice about the slope of each branch?

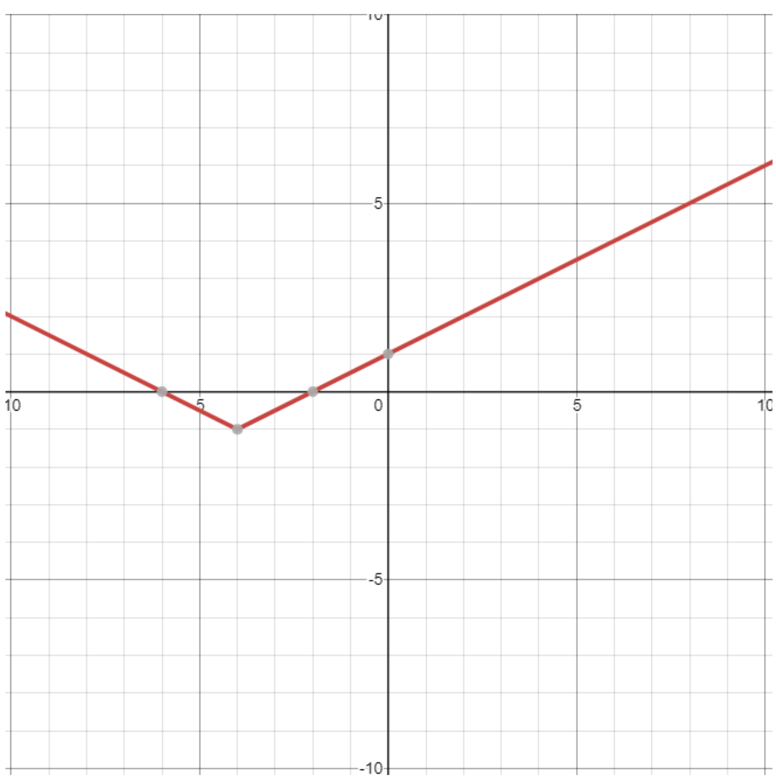
How does that relate to ***a*** (the coefficient in front of the absolute value symbol)?

Can the absolute value function open to the side?

7. Domain: all reals; Range: $[-1, \infty)$; Vertex: (-4, -1) min, End Behavior $x \to -\infty, f(x) \to \infty$; $x \to \infty, f(x) \to \infty$
Interval of Increase $(-4, \infty)$, Interval of Decrease $(-\infty, -4)$
X int (-6, 0) (2, 0)
Y int (0, 1)

Sample Table

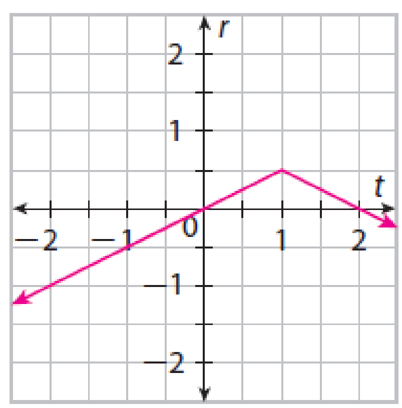
X	-8	-7	-6	-5	-4	-3	-2	-1	0
Y	1	0.5	0	-0.5	-1	-0.5	0	0.5	1



8.

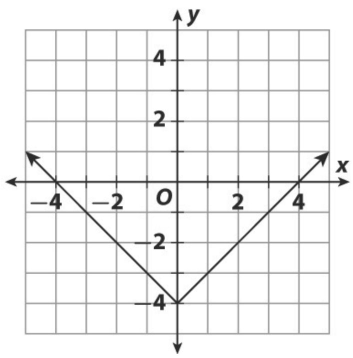
A rainstorm begins as a drizzle, builds up to a heavy rain, and then drops back to a drizzle. The rate r (in inches per hour) at which it rains is given by the function $r = -0.5|t - 1| + 0.5$, where t is the time (in hours). Graph the function. Determine for how long it rains and when it rains the hardest.

Since there can't be negative rainfall, the negative values can be discarded. Therefore, it rains for a total of 2 hours. The vertex of the graph is at (1, 0.5), so it rains the hardest at 1 hour.

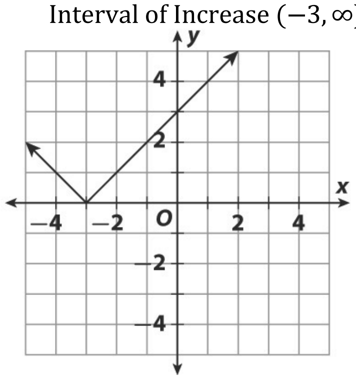


Self Check

1. 2; 3
2. 5; 2
3. 0; 9
4. 2; 4
5. 4; 3
6. –8; –2
7. Domain: all reals; Range: $[1, \infty)$, Vertex: $(1, 1)$ min, End Behavior $x \to -\infty, f(x) \to \infty \qquad x \to \infty, f(x) \to \infty$
   Interval of Increase $(1, \infty)$ Interval of Decrease $(-\infty, 1)$
8. Domain: all reals; Range: $(-\infty, 3]$; Vertex: (□2, 3) max, End Behavior $x \to -\infty, f(x) \to -\infty \qquad x \to \infty, f(x) \to -\infty$
   Interval of Increase $(-\infty, -2)$, Interval of Decrease $(-2, \infty)$
9. Domain: all reals; Range: $[-4, \infty)$; Vertex: (0, -4) min, End Behavior $x \to -\infty, f(x) \to \infty \qquad x \to \infty, f(x) \to \infty$
   Interval of Increase $(0, \infty)$, Interval of Decrease $(-\infty, 0\ )$



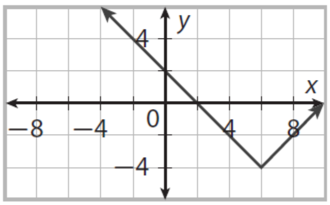
10. Domain: all reals; Range: $[0, \infty)$; Vertex: (-3, 0) min, End Behavior $x \to -\infty, f(x) \to \infty \qquad x \to \infty, f(x) \to \infty$
    Interval of Increase $(-3, \infty)$, Interval of Decrease $(-\infty, -3\ )$

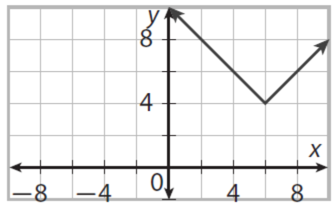


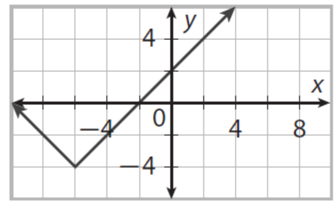
Name ____ Period _______ Date____________

1. Match each graph with its function.

____________ $y = |x + 6| - 4$ ____________ $y = |x - 6| - 4$ ____________ $y = |x - 6| + 4$

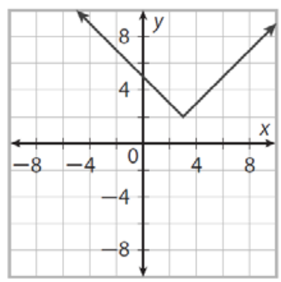
A 

B 

C 

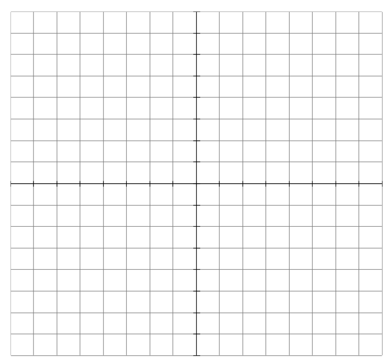
2. Explain the Error. Explain why the graph shown is not the graph of $f(x) = |x + 3| + 2$.

What is the correct equation shown in the graph?



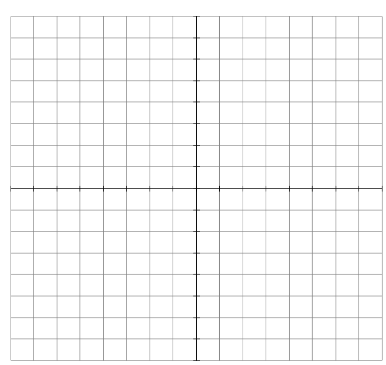
Graph and tell the characteristics of the following functions

3. $f(x) = |x - 4| + 3$
Table:



Vertex: Max or Min
Domain: Range:
Interval of Increase: Interval of Decrease:
End Behavior:
X int: Y int:

4. $f(x) = -\frac{1}{2}|x + 4| - 2$
Table:

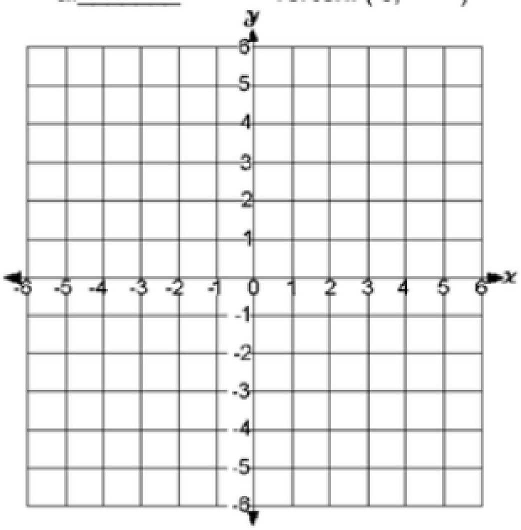


Vertex: Max or Min
Domain: Range:
Interval of Increase: Interval of Decrease:
End Behavior:
X int: Y int:

Name:______________________________ Date:______________ Class/ Day:_______ **Exit Card: Graphing Abs Day 1**

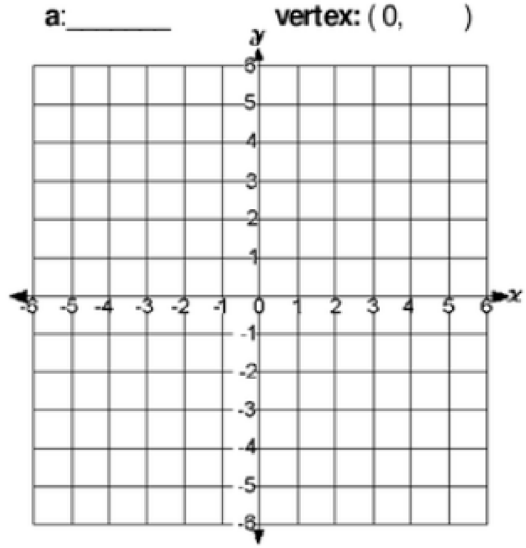
f(x) = | x| - 1

a:_______ **vertex:** (0,)



f(x) = | x+ 4| - 1

a:_______ **vertex:** (0,)



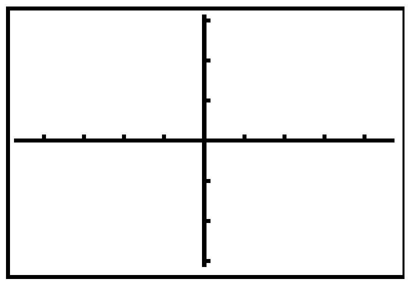
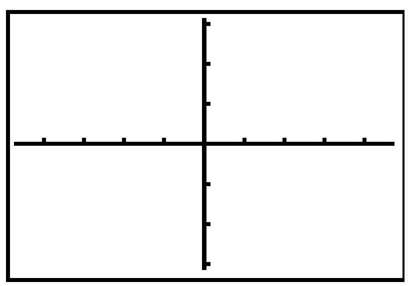
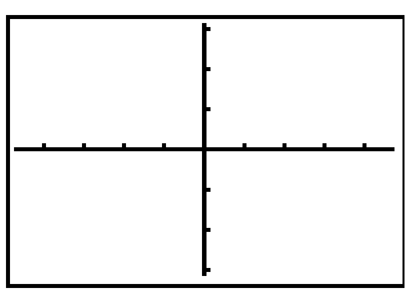
Tonight's Homework is:_________________

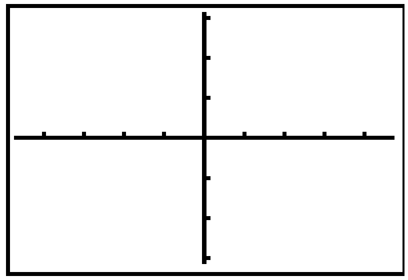
Circle One --> I understand I still need more help I do not understand.

Circle One --> I (will/will not) come to tutoring for this assignment.

INVESTIGATING TRANSFORMATIONS OF ABSOLUTE VALUE GROUP TASK

In your group, investigate each set of equations. Each person will choose an equation #1-4, then sketch the graph after using graphing calculator. Compare with others in your group. Record your findings in the appropriate column.

ABSOLUTE VALUE	SKETCH OF GRAPH	OBSERVATIONS: Compare your graph to "Parent." How did the graphs change? What in the equation caused these changes?
1) "Parent": $y = \|x\|$ 1. $y = \|x\| + 1.5$ 2. $y = \|x\| - 1$ 3. $y = \|x\| - 2$ 4. $y = \|x\| + 1$		
2) "Parent": $y = \|x\|$ 1. $y = \|x + 2\|$ 2. $y = \|x - 3\|$ 3. $y = \|x + 1\|$ 4. $y = \|x - 3\|$		
3) "Parent": $y = \|x\|$ 1. $y = -3\|x\|$ 2. $y = 2\|x\|$ 3. $y = -2\|x\|$ 4. $y = 3\|x\|$		

4) "Parent": $y = \|x\|$ 1. $y = 0.2\|x\|$ 2. $y = -.2\|x\|$ 3. $y = ½\|x\|$ 4. $y = -½\|x\|$		

In your group, discuss how changes in the parent function "Parent", $y = |\, x \,|$ will affect the graph of the function.

5. $y = |\, x \,| + c$ (outside changes)

6. $y = |\, x - h \,|$ (inside changes)

7. $y = a|\, x \,|$ (coefficient changes, when "a" is + ? When "a" is $-a$?)

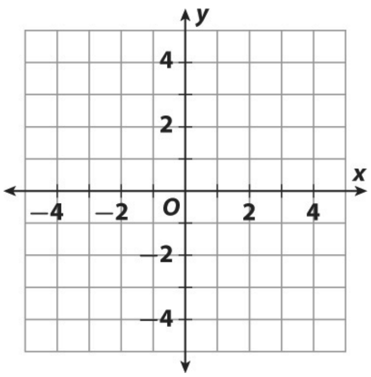
8. $y = a|\, x \,|$ (coefficient changes, when "a" is closer to zero? When "a" is farther from zero?)

A2.U6.C2.**E.05.HW.GraphAbsValue**

Name ______________________________________ Date__________________ Class _________________

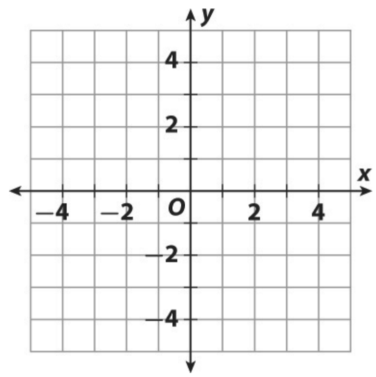
Graph each function. Then identify the key characteristics.

1. $f(x) = |x| + 2$



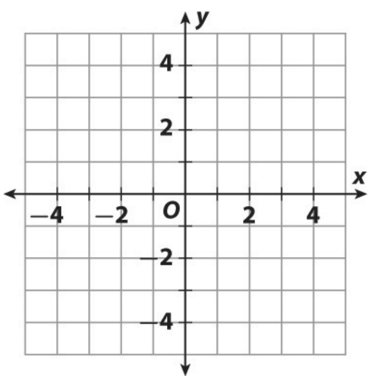
Vertex: Max/Min
Domain: Range:
Interval of Increase:
Interval of Decrease:
End Behavior:
X int: Y int:

2. $f(x) = -|x - 4|$



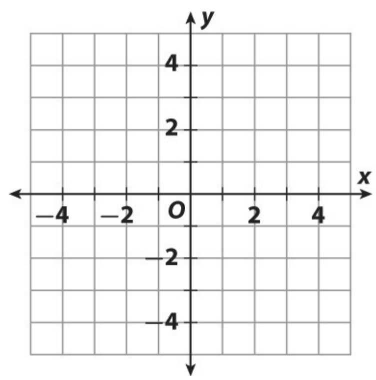
Vertex: Max/Min
Domain: Range:
Interval of Increase:
Interval of Decrease:
End Behavior:

3. $f(x) = -3|x| + 5$



Vertex: Max/Min
Domain: Range:
Interval of Increase:
Interval of Decrease:
End Behavior:
X int: Y int:

4. $f(x) = |x + 1| - 1$



Vertex: Max/Min
Domain: Range:
Interval of Increase:
Interval of Decrease:
End Behavior:
X int: Y int:

Charles meets with customers on a daily basis. He uses the function $\boldsymbol{f(x) = 5|x - 8| + 20}$ to calculate how many dollars he charges, ***x***, per hour, for his time.

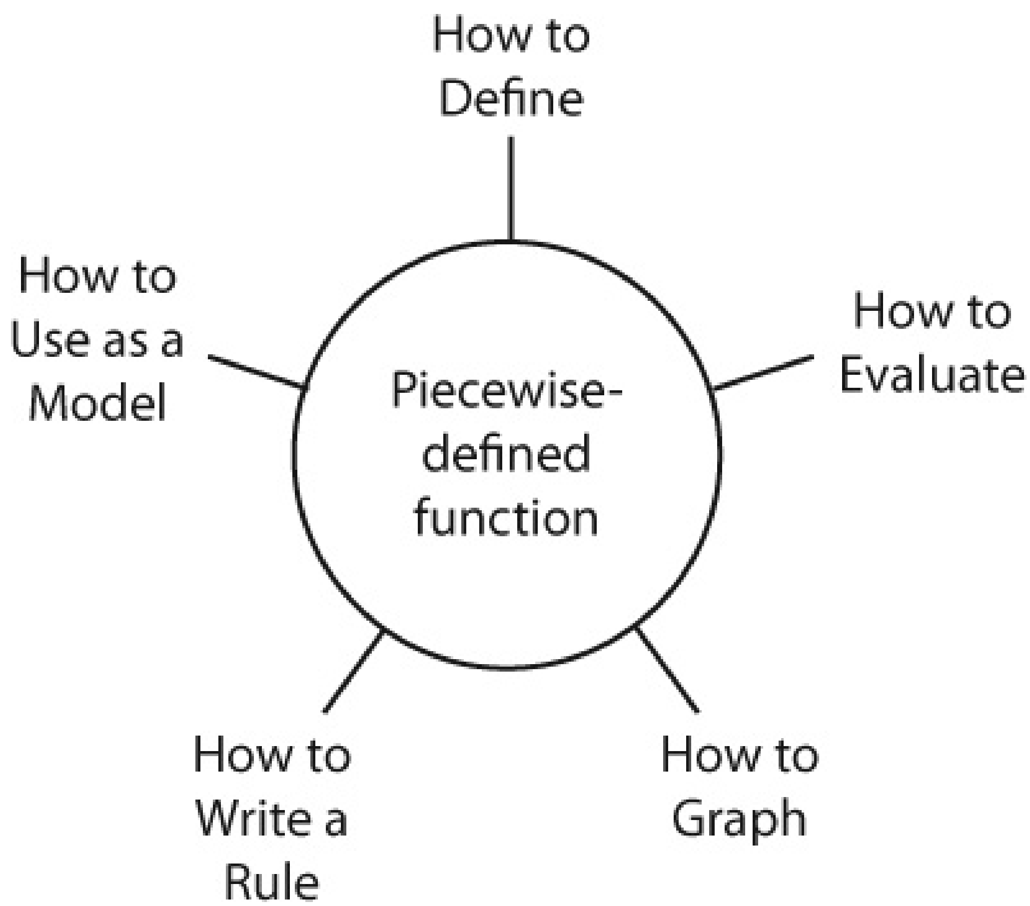
5. How much does Charles charge per hour if a customer hires him for 3 hours? _________________

6. Find the lowest hourly rate that Charles charges. (Hint this is the minimum value of the function) ___________

Show your work.

A2.U6.C2.**F.01.Vocab.Piecewise**

Piecewise-defined functions differ from other functions in several ways. To understand the differences, you might find it helpful to identify *clusters* of information about piecewise-defined functions and display them in a diagram like the one shown here. Then, as you study the lesson, jot down some notes for each cluster.



A2.U6.C2.**F.01.Vocab.Piecewise**

SAMPLE ANSWER KEY

Define: as a function that has different rules for different parts of its domain

Evaluate: Find the part of the domain that contains *x*. Use the rule associated with that part of the domain.

To Graph: You use dots when the graph makes a transition from one rule to another. You use a closed dot at the point (*x*, *y*) if *x* is included in the domain for the rule. You use an open dot at the point (*x*, *y*) if *x* is not included in the domain for the rule.

Write a rule: You usually must write one equation for each distinct piece of the graph. If the greatest integer function is involved, you might be able to use the greatest integer notation [ ] to write just one equation.

Model: word problem example of some sort. Sample answer: The situation involves different intervals, and there is a different way to calculate a result over each interval.

GUIDED NOTES

PIECEWISE FUNCTIONS

A piecewise function is a function that is a combination of two or more functions.

A *function rule* tells how to find the output values of a function when you know the input values. Many functions have just one rule. However, a **piecewise-defined function** has different rules for different parts of its domain.

> **Domain** is the set of all possible inputs for a function.

Problem 1

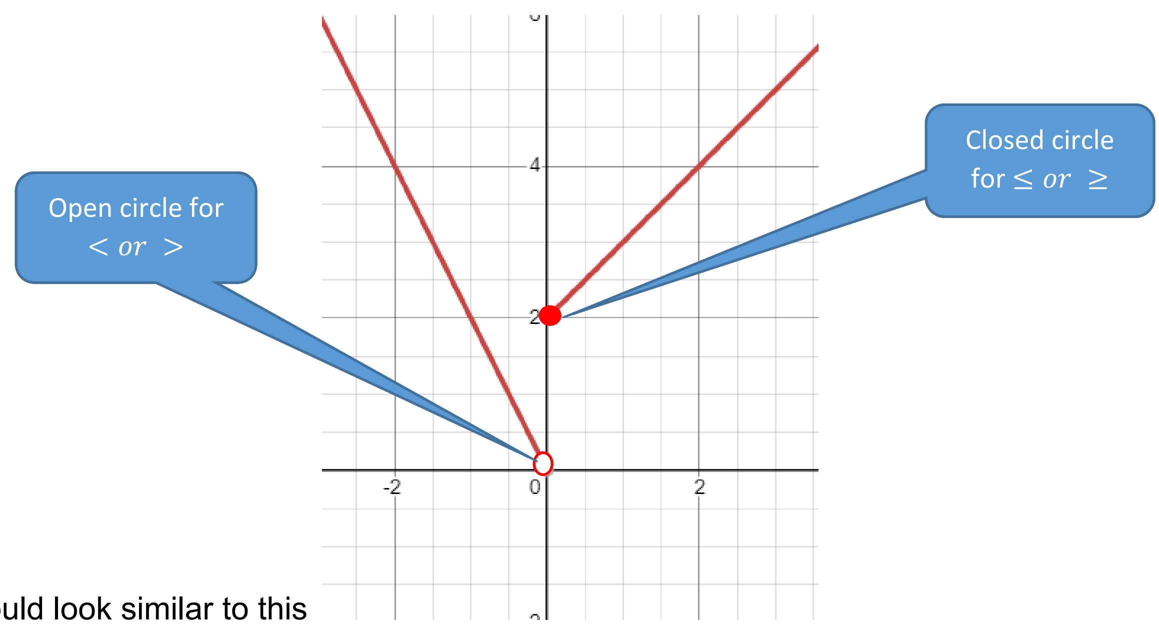
Find $f(x)$ for each x-value in the table.

$$f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x + 2 & \text{if } x \geq 0 \end{cases}$$

$x < 0$ | $x \geq 0$

input→	**x**	–3	–2	–1	–0.9	–0.1	0	0.1	0.9	1	2	3
output→	**f(x)**	6	4	2	0.18	0.2	2	2.1	2.9	3	4	5

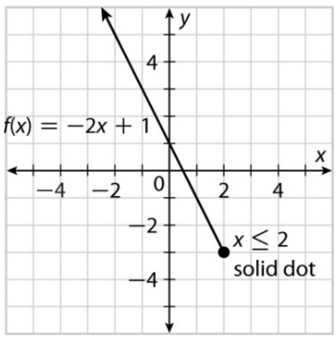
The rule is $f(x) = -2x$. The rule is $f(x) = x + 2$.



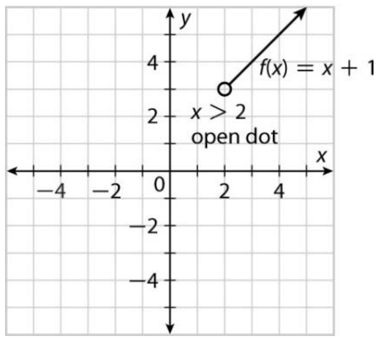
Graph would look similar to this

You graph a piecewise function such as $f(x) = \begin{cases} -2x+1 & x \leq 2 \\ x+1 & x > 2 \end{cases}$ by graphing one piece at a time. Here are the steps. Remember that $\leq$ and $\geq$ require solid dots and $<$ or $>$ require open dots as you can see here.

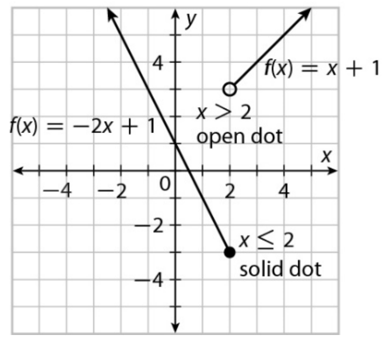
First Interval



Second Interval



Both Intervals



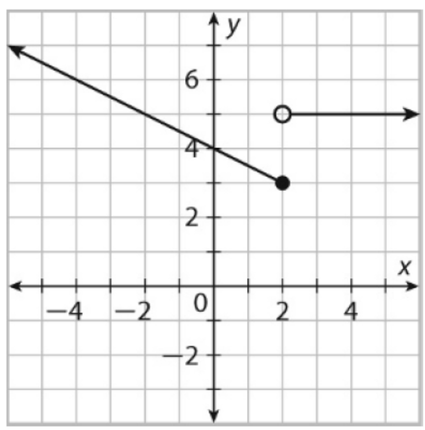
Problem 2

Write an equation for the function that is graphed.

This piece of the graph has slope $-\frac{1}{2}$. The y-intercept is 4. Write an equation in slope-intercept form.

$$y = -\frac{1}{2}x + 4$$

The rule changes when $x > 2$.



On this piece of the graph, the value of y is always 5. Write an equation for a constant function.

$$y = 5$$

An equation for the function is

$$f(x) = \begin{cases} -\frac{1}{2}x + 4 & \text{if } x \leq 2 \\ 5 & \text{if } x > 2 \end{cases}$$

You can often represent a graph such as this one by writing a piecewise-defined function.

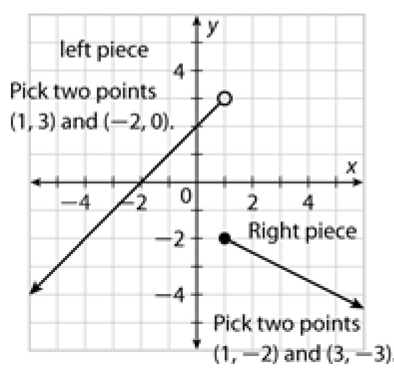
For the left piece
Use **(1, 3) and (–2, 0).**

For the right piece
Use **(1, –2) and (3, –3).**

Find equations for the lines containing these pairs.

$$y - 3 = \frac{3-0}{1-(-2)}(x-1)$$
$$y - 3 = x - 1$$
$$y = x + 2$$

$$y - (-2) = \frac{-2-(-3)}{1-3}(x-1)$$
$$y + 2 = -0.5(x-1)$$
$$y + 2 = -0.5x + 0.5$$
$$y = -0.5x - 1.5$$

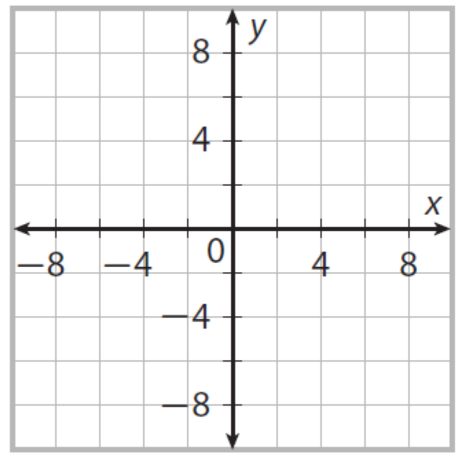


The absolute value function $|x|$, can be defined piecewise as $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$. When x is nonnegative, the function simply returns the number. When x is negative, the function returns the opposite of x.

Complete the input-output table for $f(x)$.

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

x	**f(x)**
–8	
–4	
0	
4	
8	



Piecewise Functions can be used to model certain situations.

Travel On her way to a concert, Maisee walks at a speed of 0.03 mile per minute from her car for 5 minutes, waits in line for a ticket for 3 minutes, and then walks to her seat for 4 minutes at a speed of 0.01 mile per minute.

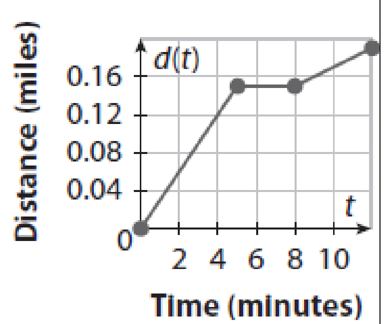
Express the Maisee's distance traveled d (in miles) as a function of time t (in minutes).

For $0 \leq t \leq 5$, $m = 0.03$ and $b = 0$, so $d(t) = 0.03t$.

For $5 < t \leq 8$, $m = 0$ and $b = 0.15$, so $d(t) = 0.15$.

For $8 < t \leq 12$, $m = 0.01$ beginning at (8, 0.15), so $d(t) = 0.01t + 0.07$.

$$d(t) = \begin{cases} 0.03t & \text{if } 0 \leq t \leq 5 \\ 0.15 & \text{if } 5 < t \leq 8 \\ 0.01t + 0.07 & \text{if } 8 < t \leq 12 \end{cases}$$



Example!

1.

Find $f(-3)$, $f(-0.2)$, $f(0)$, and $f(2)$ for $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$

$-3 < 0$, so $f(-3) = -(-3) = \square$

$0 \geq 0$, so $f(0) = \square + 1 = \square$

$-0.2 < 0$, so $f(-0.2) = \square = \square$

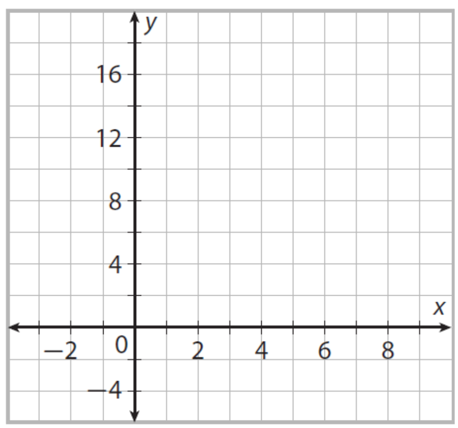
$2 \geq 0$, so $f(2) = \square + \square = \square$

2.

Find $f(-2)$, $f(-0.4)$, $f(3.7)$, and $f(5)$ for $f(x) = \begin{cases} -x & \text{if } x < 2 \\ 2x + 3 & \text{if } 2 \leq x < 4 \\ x^2 & \text{if } x \geq 4 \end{cases}$

3. Graph

$$f(x) = \begin{cases} x & \text{if } x < 2 \\ 2x + 3 & \text{if } 2 \leq x < 4 \\ x^2 & \text{if } x \geq 4 \end{cases}$$



4.

Travel On his way to class from his dorm room, a college student walks at a speed of 0.05 mile per minute for 3 minutes, stops and talks to a friend for 1 minute, and then to avoid being late for class, runs at a speed of 0.10 mile per minute for 2 minutes.

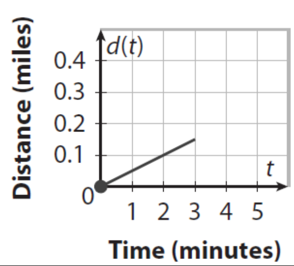
Express the student's distance traveled d (in miles) as a function of time t (in minutes).

For $0 \leq t \leq 3$, $m = \square$ and $b = 0$, so $d(t) = \square t$.

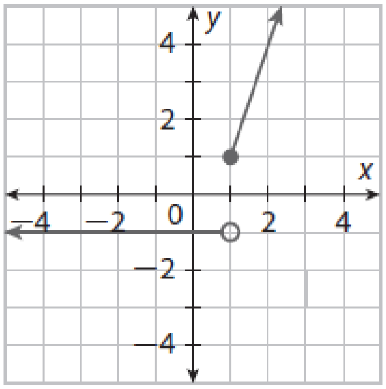
For $3 < t \leq \square$, $m = 0$ and $b = 0.15$, so $d(t) = 0.15$.

For $4 < t \leq 6$, $m = \square$ beginning at $\square$, so $y = \square x - \square$.

$$d(t) = \begin{cases} \square t & \text{if } 0 \leq t \leq 3 \\ 0.15 & \text{if } 3 < t \leq \square \\ \square t - \square & \text{if } 4 < t \leq 6 \end{cases}$$



5. Write an equation for each piecewise function



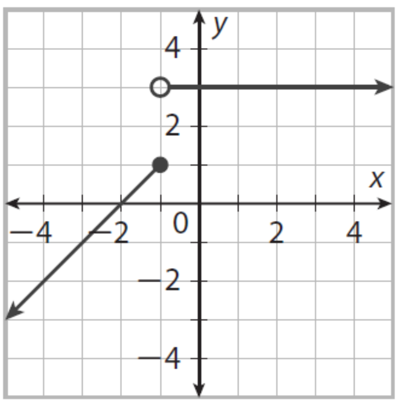
Find the equation of the ray on the right.

$m = \frac{1-4}{1-2} = \frac{-3}{-1} = 3$

Because the point (1, 1) is on the ray, $y - 1 = 3(x - 1)$, so $y = 3x - 2$

The equation of the line that contains the horizontal ray is $y = -1$.

The equation for the function is $y = \begin{cases} -1 & \text{if } x < 1. \\ 3x - 2 & \text{if } x \geq 1 \end{cases}$



Find the equation for the ray on the left.

$m = \frac{1 - \square}{-1 - \square} = \frac{\square}{\square} = \square$

Because the point (−1, 1) is on the ray, $y \,\square = \square\,(x\,\square)$,

so $y = \square$

The equation of the horizontal ray is $y = \square$.

The equation for the function is $y = \begin{cases} \square\, x + \square & \text{if } x \leq -1 \\ \square & \text{if } x > -1 \end{cases}$

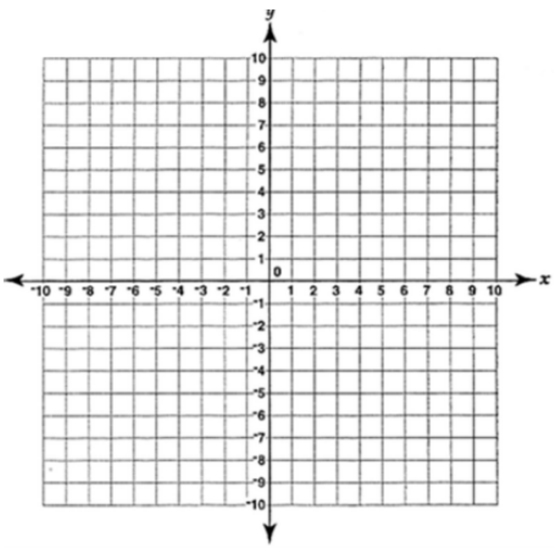
SELF CHECK

1.

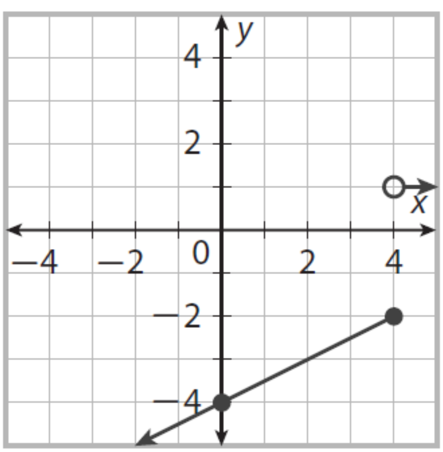
$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$$

Make a table

x	−3	−2	−1	0	1	2
***f*(*x*)**						



2. Write the equation for the piecewise function



3. Find $f(0)$, $f(2)$, and $f(4)$ for $f(x) = \begin{cases} 8 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$.

Can the pieces ever overlap?

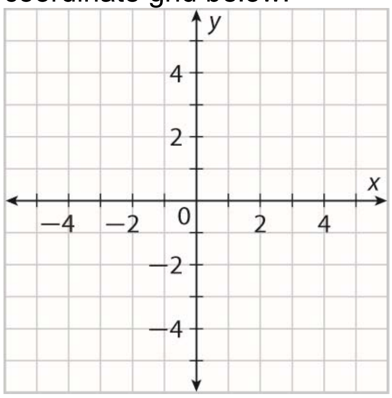
Algebra 2 Unit #6 Concept# 2F

A2.U6.C2.**F.03.ClassWk.Piecewise**

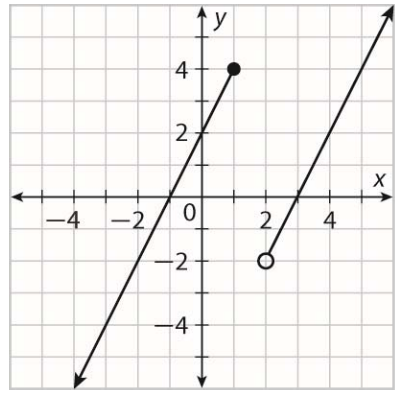
Independent Practice

Name___________________________________ Period _______ Date________

1. Graph $f(x)=\begin{cases}-0.5x+2 & x<2\\ 2x-6 & x\geq 2\end{cases}$ on the coordinate grid below.



2. For the graph below, write a piecewise defined function to represent it.



Evaluate each piecewise function for the given values.

3. Find $f(-3)$, $f(-2.1)$, $f(0.6)$, and $f(3.3)$ for $f(x)=\begin{cases}-1 & \text{if } x\leq 0\\ 3x & \text{if } 0<x<1.\\ x+3 & \text{if } x\geq 1\end{cases}$

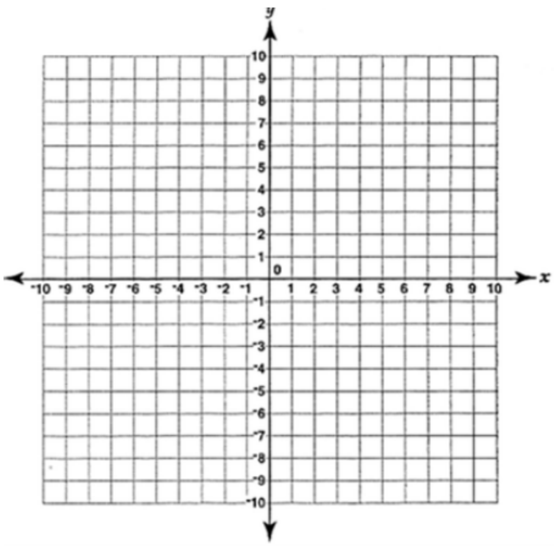
4. Find $f(-4)$, $f(-2.9)$, and $f(1.9)$ for $f(x)=\begin{cases}-5 & \text{if } x\leq -3\\ x+2 & \text{if } -3<x\leq 0.\\ x^2+7 & \text{if } x\geq 0\end{cases}$

5. Find $f(-3)$, $f(-1)$, and $f(1)$ for $f(x)=\begin{cases}\frac{2}{x} & \text{if } x\leq -2\\ x & \text{if } -2<x\leq 0\\ 1 & \text{if } x\geq 0\end{cases}$

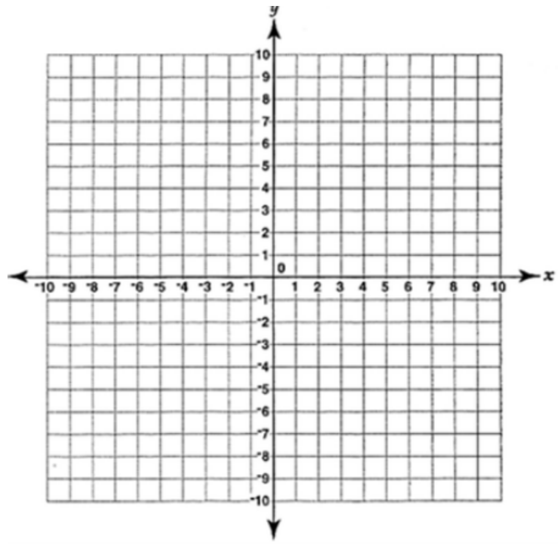
6. Find $f(-2)$, $f(-1)$, $f(0)$, $f(4)$, and $f(9)$ for $f(x)=\begin{cases}-x^2 & \text{if } x\leq -2\\ 2x & \text{if } -2<x<2\\ x+6 & \text{if } 2\leq x\leq 4\\ \sqrt{x}+8 & \text{if } x>4.\end{cases}$

Graph each piecewise function

7. $f(x)=\begin{cases}-x+1 & \text{if } x<0\\ x & \text{if } x\geq 0\end{cases}$



8. $f(x)=\begin{cases}-1 & \text{if } x<1\\ 2x-2 & \text{if } x\geq 1\end{cases}$



9. Write a piecewise function for the given situation and fill in the table and graph.

Finance A garage charges the following rates for parking (with an 8 hour limit):

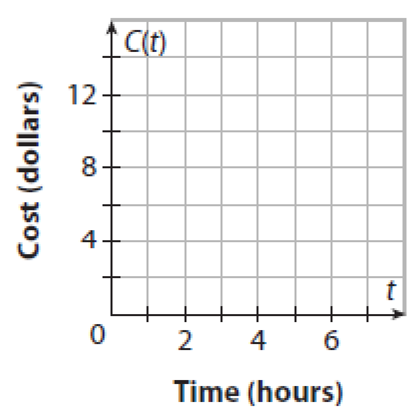
\$4 per hour for the first 2 hours

\$2 per hour for the next 4 hours

No additional charge for the next 2 hours

Express the cost C (in dollars) as a function of the time t (in hours) that a car is parked in the garage.

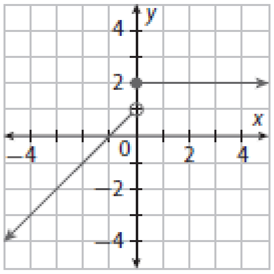
$C(t) =$

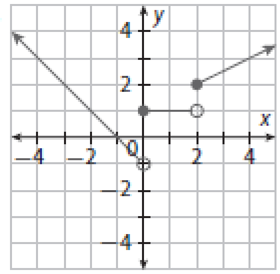


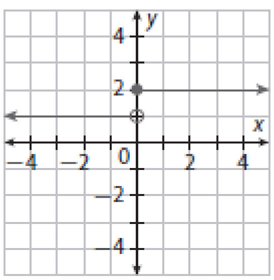
t	0	1	2	3	4
C(t)					

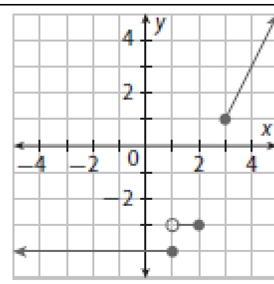
t	5	6	7	8
C(t)				

Write a piecewise equation (be sure to include the domains) for each graph.

10. 

11. 

12. 

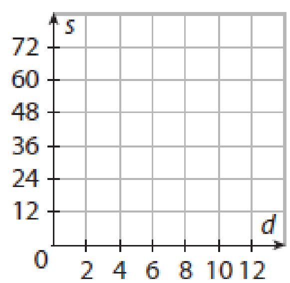
13. 

Traveling - Application

Suppose someone is traveling from New York City to Miami, Florida. The following table describes the average speeds at various intervals on this 1200-mile trip.

Distance Traveled (hundreds of miles)	Average Speed (mi/h)
$0 < d \leq 2$	37.7
$2 < d \leq 4$	46.6
$4 < d \leq 6$	63.3
$6 < d \leq 8$	45.5
$8 < d \leq 10$	64.4
$10 < d \leq 12$	49.9

A. Graph the distance function. Make sure to use appropriate labels.

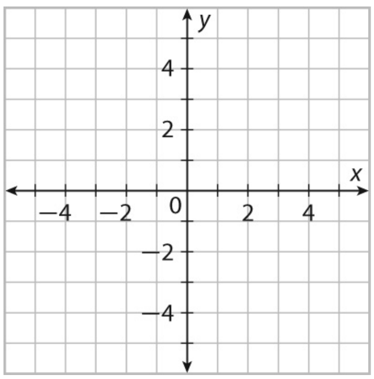


B. Write the piecewise function that is given by the table.

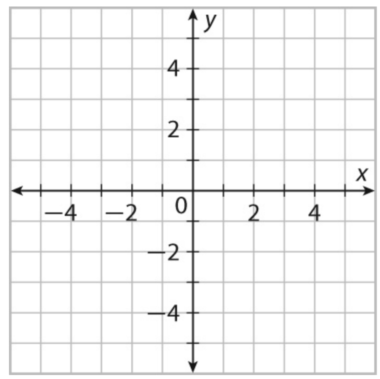
C. Suppose the destination was changed from Miami, Florida to Minneapolis, Minnesota instead. Explain why it is not okay to use the piecewise function created for the trip from New York to Miami when traveling to Minneapolis, even though the distance is comparable.

Graph each piecewise-defined function.

1. $f(x)=\begin{cases}0.5x-1.5 & x<-1\\ x+1 & -1\le x\le 3\\ 4 & x>3\end{cases}$

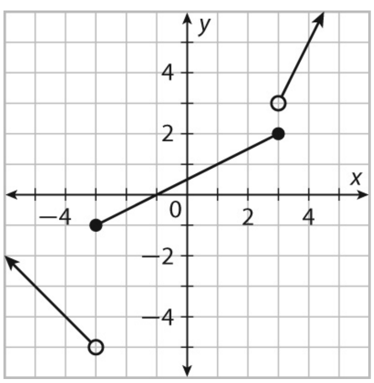


2. $f(x)=\begin{cases}-4x-16 & x<-3\\ 0.5x-4.5 & -3\le x<3\\ -2 & x\ge 3\end{cases}$



Write equations to complete the definition of each function.

3.



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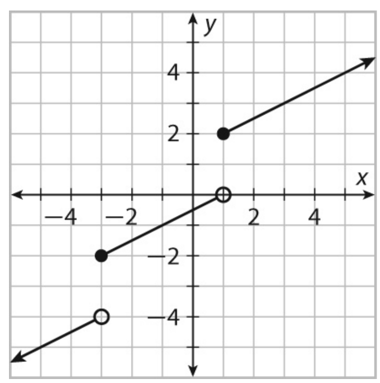
4.



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5. The graph at the right shows shipping cost as a function of purchase amount.

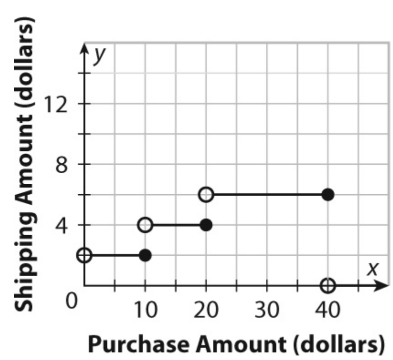
   Find the shipping cost for each purchase amount.

   purchase amount: \$8.49 ______________

   purchase amount: \$20.00 _____________

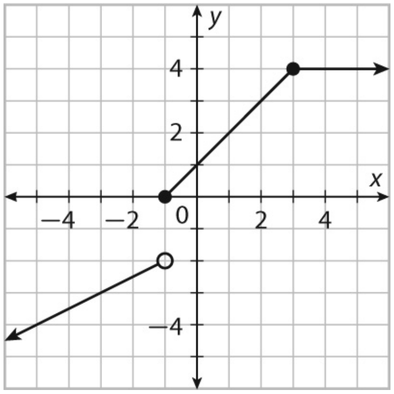
   purchase amount: \$89.50 _____________

   purchase amount: \$40.01 _____________

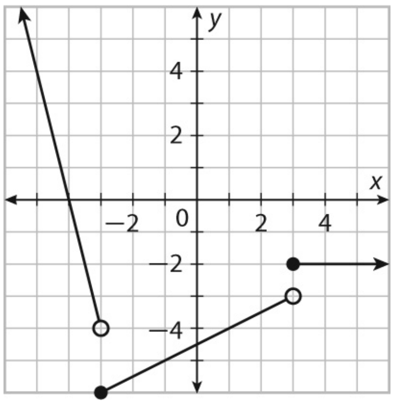


ANSWER KEY

1.



2.



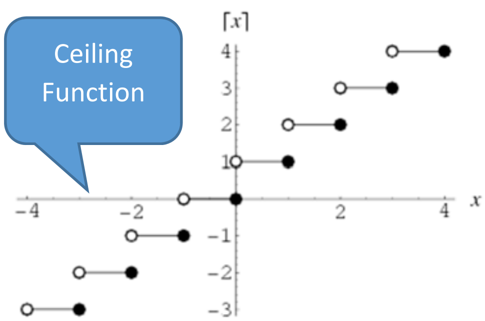
3. $f(x)=\begin{cases}-x-8 & x<-3\\ 0.5x+0.5 & -3\le x\le 3\\ 2x-3 & x>3\end{cases}$

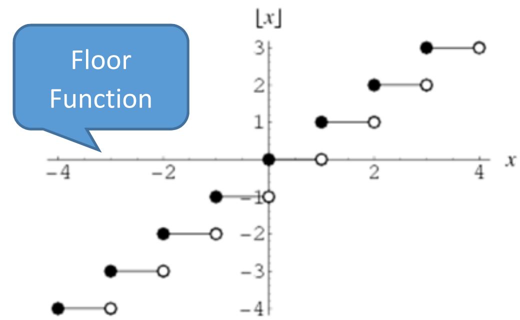
4. $f(x)=\begin{cases}0.5x-2.5 & x<-3\\ 0.5x-0.5 & -3\le x<1\\ 0.5x+1.5 & x\ge 1\end{cases}$

5. $2.00; $4.00; $0.00; $0.00

Step function, a piecewise function in which each function rule is a constant function. To evaluate a piecewise function for a given value of x, substitute the value of x into the rule for the part of the domain that includes x. The graph of a step function resembles a set of steps.

The **greatest integer function** is a piecewise function whose rule is denoted by $\lfloor x \rfloor$, which represents the *greatest integer less than or equal to* x. Basically, the greatest integer function rounds down a real number to the nearest integer. For example: [2] = 2; [1.5] = 1; [-3.1] = -4; [-6.9] = -7.





The Floor and Ceiling Functions

The **floor symbol** $\lfloor\ \rfloor$ and the **ceiling symbol** $\lceil\ \rceil$ are defined as follows.

Definition of Greatest Integer/Least Integer

$\lfloor x \rfloor$ = the greatest integer less than or equal to x, and

$\lceil x \rceil$ = the least integer greater than or equal to x.

The **floor function** is the function f with $f(x) = \lfloor x \rfloor$, for all real numbers x. It is also called the **greatest-integer function**, or the **rounding-down function**. On some calculators and in some computer languages it is called the **int function**. Another notation you may see for the floor function is $f(x) = [\![x]\!]$.

The **ceiling function** is the function f with $f(x) = \lceil x \rceil$, for all real numbers x. It is also called the **rounding-up function**.

GUIDED

Example 1

Evaluate each of the following.

a. $\lfloor 5\frac{7}{8} \rfloor$ b. $\lfloor -4.2 \rfloor$ c. $\lceil \pi \rceil$ d. $\lceil 13 \rceil$

Solution

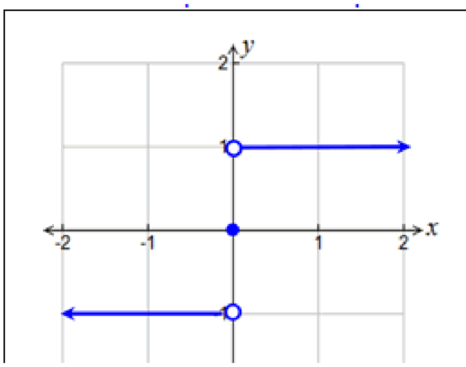
a. $\lfloor 5\frac{7}{8} \rfloor$ is the greatest integer less than or equal to $5\frac{7}{8}$. So, $\lfloor 5\frac{7}{8} \rfloor$ = __?__.

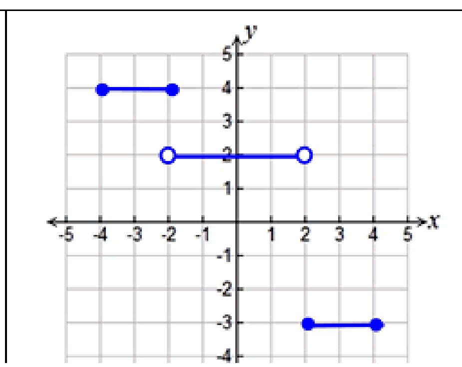
b. $\lfloor -4.2 \rfloor$ is the __?__ less than or equal to −4.2. So, $\lfloor -4.2 \rfloor$ = __?__.

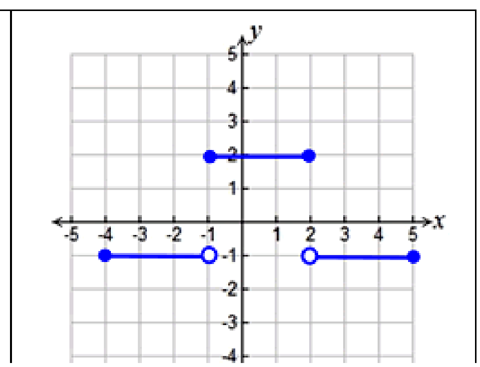
c. $\lceil \pi \rceil$ is the least integer greater than or equal to $\pi \approx$ __?__.
So, ⌈ __?__ ⌉ = __?__

d. $\lceil 13 \rceil$ is the __?__ greater than or equal to 13. So, ⌈ __?__ ⌉ = __?__.

Other step functions







Example!

Example 2

Find $f(-3)$, $f(-2.9)$, $f(0.7)$, and $f(1.06)$ for $f(x) = \lfloor x \rfloor$.

The greatest integer function $f(x) = \lfloor x \rfloor$ can also be written in the form below.

$$f(x) = \begin{cases} \vdots \\ -3 & \text{if } -3 \le x < -2 \\ -2 & \text{if } -2 \le x < -1 \\ -1 & \text{if } -1 \le x < 0 \\ 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } 2 \le x < 3 \\ \vdots \end{cases}$$

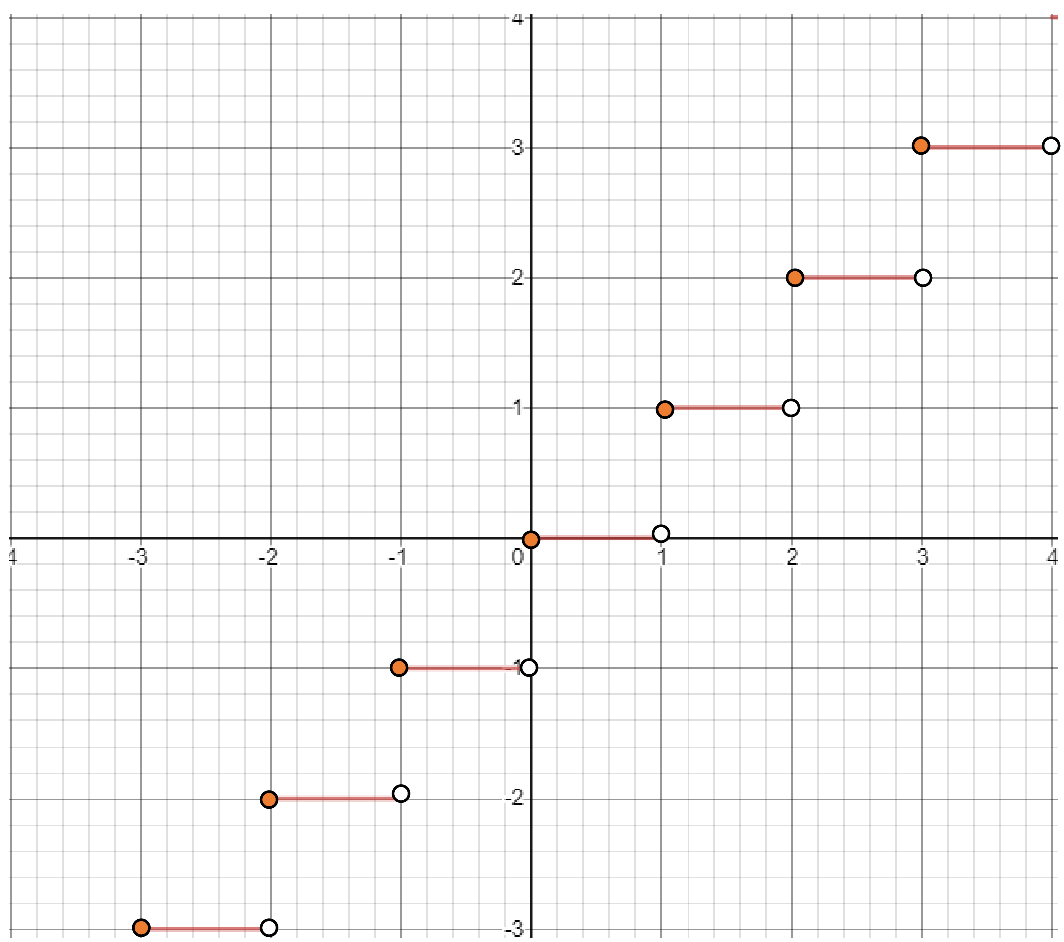
-3 is in the interval $-3 \le x < -2$, so $f(-3) = -3$.

-2.9 is in the interval $-3 \le x < -2$, so $f(-2.9) = -3$.

0.7 is in the interval $0 \le x < 1$, so $f(0.7) = 0$.

1.06 is in the interval $1 \le x < 2$, so $f(1.06) = 1$.

The graph would look like this



Example 3

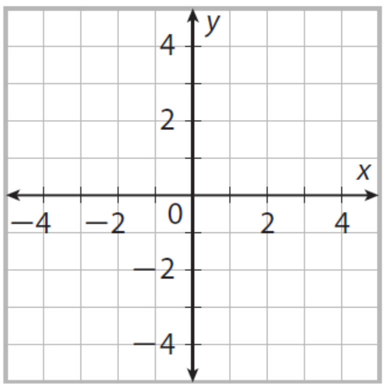
Find $f(-6)$, $f(-2.2)$, $f(1.4)$ and $f(3.6)$ for $f(x) = -2\lfloor x \rfloor$.

Example 4

Find $f(-2.8)$, $f(-1.2)$, $f(0.4)$, and $f(1.6)$ for $f(x) = \lfloor x \rfloor^2$

Example 5 Graph

$f(x) = 2\lfloor x \rfloor - 2$



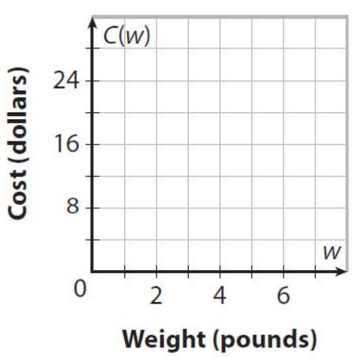
Example 6 Application Problem

Cost Analysis The cost to send a package between two cities is \$8.00 for any weight less than 1 pound. The cost increases by \$4.00 when the weight reaches 1 pound and again each time the weight reaches a whole number of pounds after that.

Express the shipping cost C (in dollars) as a function of the weight (in pounds). Express your answer in terms of the greatest integer function $\lfloor w \rfloor$.

$C(w) =$

w	0.5	1	1.5	2	2.5
C(w)					

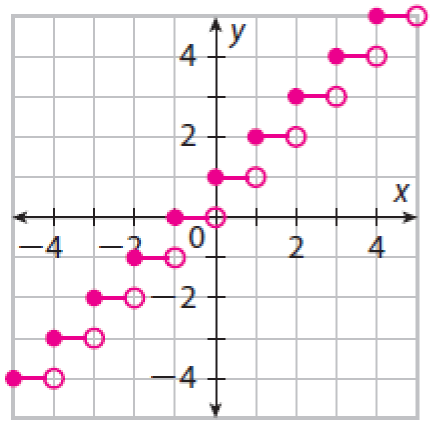


SELF CHECK

1. *Find f(-4), f(-3.1), f(1.2) and f(2.8) for* $f(x) = [x]$

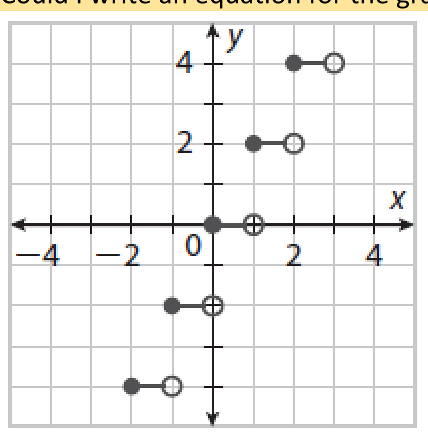
2. *Graph*

$f(x) = \lfloor x \rfloor + 1$



Questions To Ponder

Could I write an equation for the graph below. Express the answer in terms of $\lfloor x \rfloor$.



Name ____ Period__________ Date ________

1. The graph at the right shows shipping cost as a function of purchase amount.

Find the shipping cost for each purchase amount.

purchase amount: \$5.49 ____________

purchase amount: \$40.00 ____________

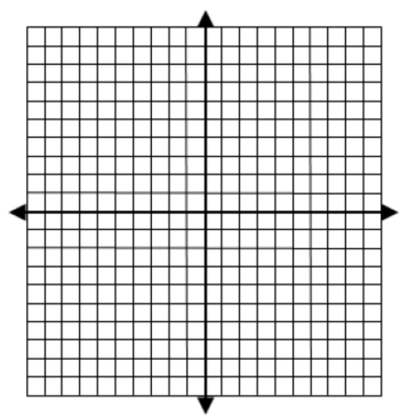
purchase amount: \$199.50 ____________

purchase amount: \$40.01 ____________

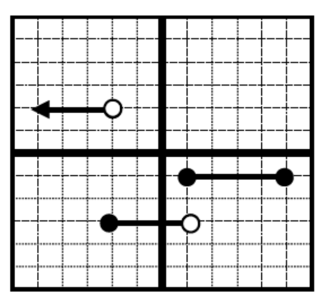
purchase amount of \$12.00 ____________

2. Graph the following function on the graph to the right.

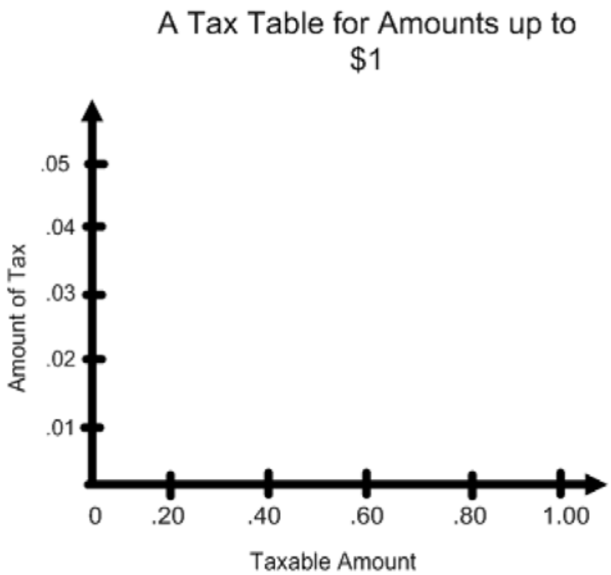
$$f(x)=\begin{cases} 3 & \text{if} \quad x<-2 \\ -2 & \text{if} \quad -2\le x<1 \\ 1 & \text{if} \quad x\ge 1 \end{cases}$$



3. Write the equation of the function to the right.

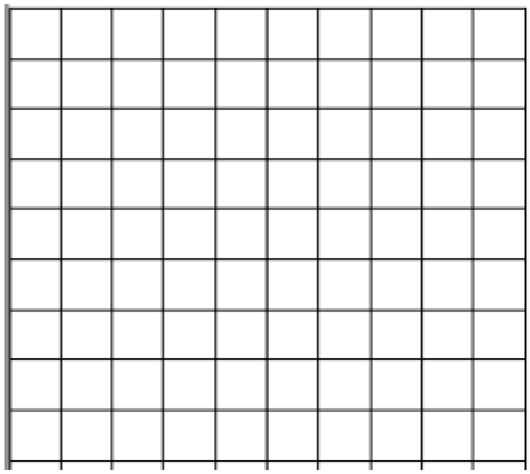


4. In many states a "sales tax" is added to most goods that you buy. The tax rate varies from state to state. Let's suppose that your particular state issues a sales tax on any goods purchased.
You are selling candy bars. The taxable amounts and tax imposed up to $1 are shown below.
   - For amounts between $0.01 and $0.20, the tax is $.01.
   - For amounts greater than $0.20 and less than or equal to $0.40, the tax is $0.02.
   - For amounts greater than $0.40 and less than or equal to $0.60, the tax is $0.03.
   - For amounts greater than $0.60 and less than or equal to $0.80, the tax is $0.04.
   - For amounts greater than $0.80 and less than or equal to $1.00, the tax is $0.05.



   a. A candy bar costs $0.55. What is the total cost with tax?
   b. Your aunt purchased three candy bars at $0.55 each. What is the total cost with tax?
   c. Someone purchased 4 candy bars at $0.55 each. They gave you $2 and a quarter. Is this enough money to cover the candy bars and the tax? Explain your answer.

5. **You are selling candy bars. The profit you make and money you spend is as follows.**
   - For amounts earned between $0 and $2, you spend $1.
   - For amounts greater than $2 and less than or equal to $4, you spend $2.
   - For amounts greater than $4 and less than or equal to $6, you spend $3.
   - For amounts greater than $6 and less than or equal to $8, you spend $4.
   - For amounts greater than $8 and less than or equal to $10, you spend $5.



Floor and Ceiling (STEP) Functions

1. Victor started working at a car dealership washing cars. He gets $50 per day, plus $10 per car for car washing. He does not get paid for a partial car wash – he must finish the car to get paid for washing it at all.

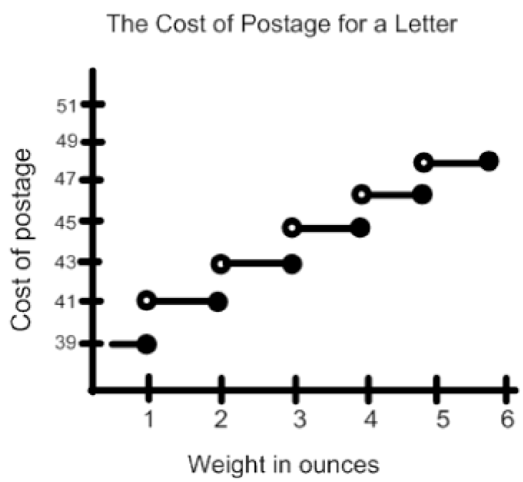
   a. Write an equation that would show how much he gets paid, p(x), where x is the number of cars he washes in a day.
   b. Create a graph that shows how much he would make for up to 20 cars.
   c. Would this be a "floor" or a "ceiling" function?
   d. What is the domain of this function?
   e. What is the range of this function?

2. Victor uses part of his earnings at the car wash to pay for doggie day care for his dog, Frank the Tank. The Endless Love Pet Palace charges $3 per hour, but they charge for the whole hour, even if you dropped off several minutes after the hour began, and even if you picked up before the end of the hour. For instance, if you dropped off your dog at 7:45 am and picked your dog up at 9:05 am that same day, you would be charged for the 7:00 hour ($3), for the 8:00 hour ($3), and the 9:00 hour ($3), for a grand total of $9 that day. It's worth it, because Frank the Tank really enjoys going to "school," Victor just knows he needs to never be late picking up.

   a. Write an equation that would show how much he pays, p(x), for doggie daycare when x is the number of hours the Endless Love Pet Palace Charges you for?
   b. Create a graph that shows how much he will pay for up to 24 hours.
   c. Would this be a "floor" or a "ceiling" function?
   d. What is the domain of this function?
   e. What is the range of this function?

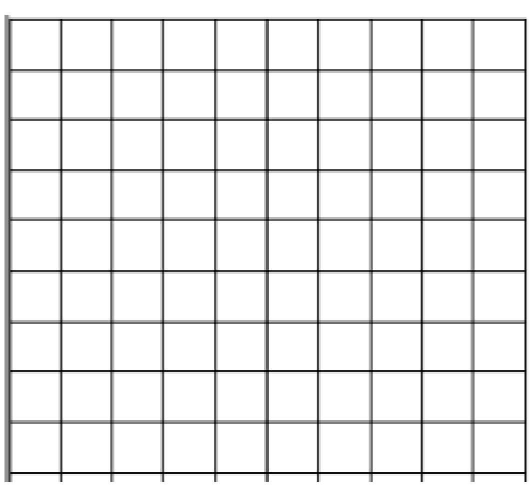
Name ___________________________________ Period ______ Date ________

1. **Find the amount of postage (in cents) needed for a letter weighing the following number of ounces**
   a) 1.5 ounces _____
   b) 4 ounces _____
   c) 1.1 ounces _____
   d) 2.7 ounces _____
   e) What do you think will happen if the letter weighs more than 6 ounces? _____________________



2. **A wholesale t-shirt manufacturer charges the following prices for t-shirt orders:**
   - $20 per shirt for shirt orders up to 20 shirts.
   - $15 per shirt for shirt between 21 and 40 shirts.
   - $10 per shirt for shirt orders between 41 and 80 shirts.
   - $5 per shirt for shirt orders over 80 shirts.

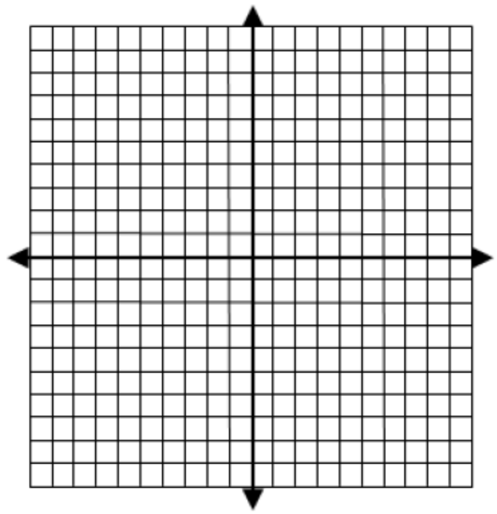
   Sketch a graph of this discontinuous function.



You've ordered 40 shirts and must pay shipping fees of $10. How much is your total order?

3. Graph

$$f(x)=\begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \\ 3 & 3 \le x < 4 \end{cases}$$



GUIDED NOTES

A **system of linear inequalities** consists of two or more linear inequalities that have the same variables. The **solutions** of a system of linear inequalities are all the ordered pairs that make all the inequalities in the system true.

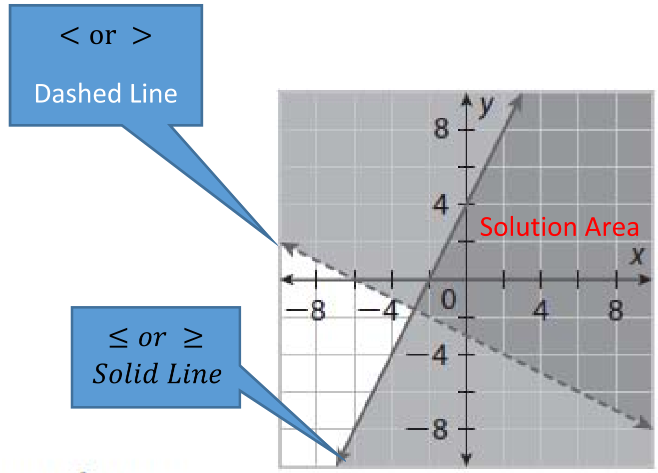
$$\begin{cases} -6x + 3y \leq 12 \\ y > \frac{1}{2}x - 3 \end{cases}$$

Solve the first inequality for y.

$$-6x + 3y \leq 12$$
$$3y \leq 6x + 12$$
$$y \leq 2x + 4$$

Graph the system.

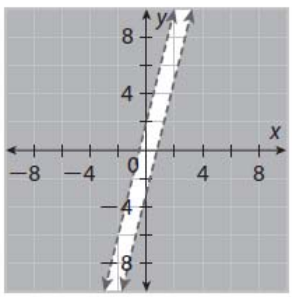
$$\begin{cases} y \leq 2x + 4 \\ y > \frac{1}{2}x - 3 \end{cases}$$



(0, 0) and (2, 8) are solutions. (−6, −4) and (−4, 4) are not solutions.

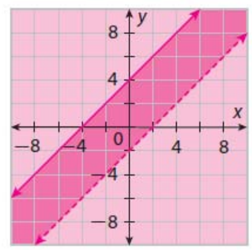
Other examples:

$$\begin{cases} y < 4x - 3 \\ y > 4x + 2 \end{cases}$$



This system has no solution.

$$\begin{cases} y > x - 2 \\ y \leq x + 4 \end{cases}$$

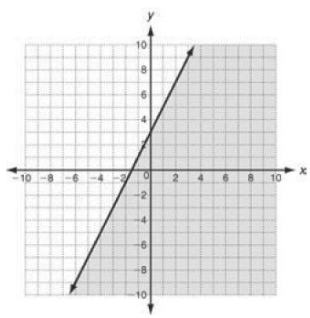


The solutions are all points __between__ the parallel lines and on the __solid__ line.

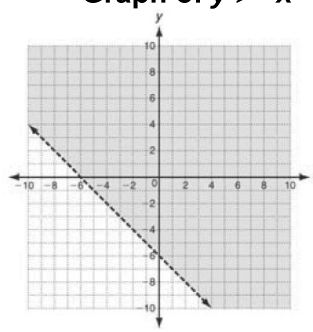
Problem 1

You can graph a system of linear inequalities by combining the graphs of the inequalities.

Graph of $y \leq 2x + 3$

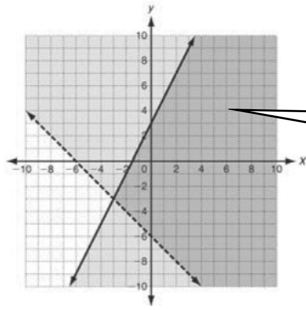


Graph of $y > -x - 6$



Graph of the system

$$\begin{cases} y \leq 2x + 3 \\ y > -x - 6 \end{cases}$$

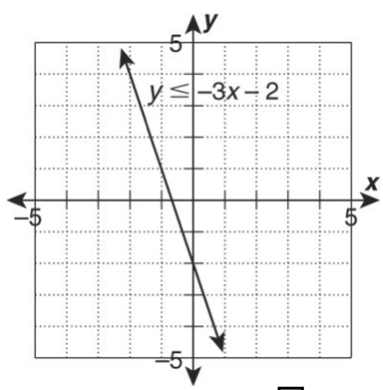


All solutions are in this double shaded area.

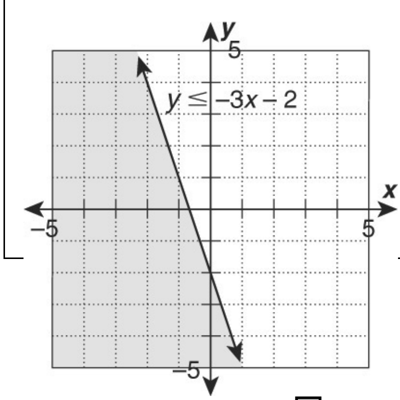
Two ordered pairs that are solutions: (3, 4) and (5, –2)

Problem 2

Graph 1:



Graph 2:



$$\begin{cases} y + 3x \leq -2 \\ y > \frac{1}{2}x - 2 \end{cases}$$

Equation 1

$y + 3x \leq -2$

Step 1: Rewrite in terms of *y*.
$y + 3x \leq -2$

$y \leq -3x - 2$

Step 2: Identify the slope and *y*-intercept.

$m = -3;\ b = -2$

Step 3: Boundary line: Solid or Dashed?
$\leq \longrightarrow$ solid

Step 4: Shade over or under?

$y \leq -3x - 2 \longrightarrow$ under

Equation 2

$y > \frac{1}{2}x - 2$

Step 1: Already in slope-intercept form

$y > \frac{1}{2}x - 2$

Step 2: Identify the slope and *y*-intercept.

$m = \frac{1}{2};\ b = -2$

Step 3: Boundary line: Solid or Dashed?
$> \longrightarrow$ dashed

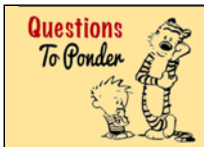
Step 4: Shade over or under?

$y > \frac{1}{2}x - 2 \longrightarrow$ over

Graph 3:

Graph 4:

Graph 5: Where do they overlap?



1. Explain why (−3, 1) is a solution to the above system.
2. Explain why (4, 2) is NOT a solution to the above system

Example 1

Solve the system of equations by graphing.

$$\begin{cases} x + 3y > 3 \\ -x + y \leq 6 \end{cases}$$

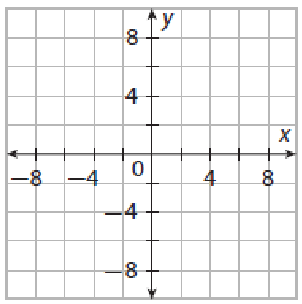
(A) First look at $x + 3y > 3$. The equation of the boundary line is ____________.

(B) What are the x-and y-intercepts?

(C) The inequality symbol is $>$ so use a ________ line.

(D) Shade ________ the boundary line for solutions that are greater than the inequality.

(E) Graph $x + 3y > 3$.



(F) Look at $-x + y \leq 6$. The equation of the boundary line is ____________.

(G) What are the x-and y-intercepts?

(H) The inequality symbol is $\leq$ so use a ______ line.

(I) Shade ________ the boundary line for solutions that are less than the inequality.

(J) Graph $-x + y \leq 6$ on the same graph as $x + 3y > 3$.

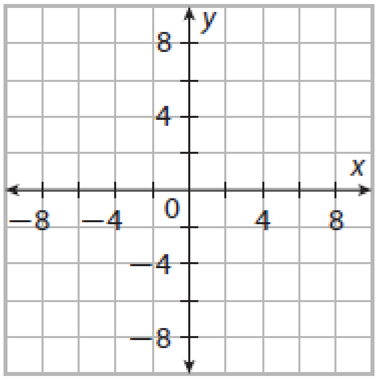
(K) Identify the solutions. They are represented by the ______________ shaded regions.

(L) Check your answer by using a point in each region. Complete the table.

Ordered Pair	Satisfies $x + 3y > 3$?	Satisfies $-x + y \leq 6$?	In the overlapping shaded regions?
(0, 0)			
(2, 3)			
(−8, 2)			
(−4, 6)			

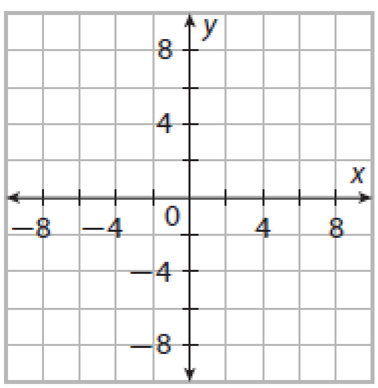
Example 2

$$\begin{cases} y \leq x + 3 \\ y < -3 \end{cases}$$



Example 3

$$\begin{cases} x > 2 \\ y \leq -\frac{1}{2}x - 2 \end{cases}$$



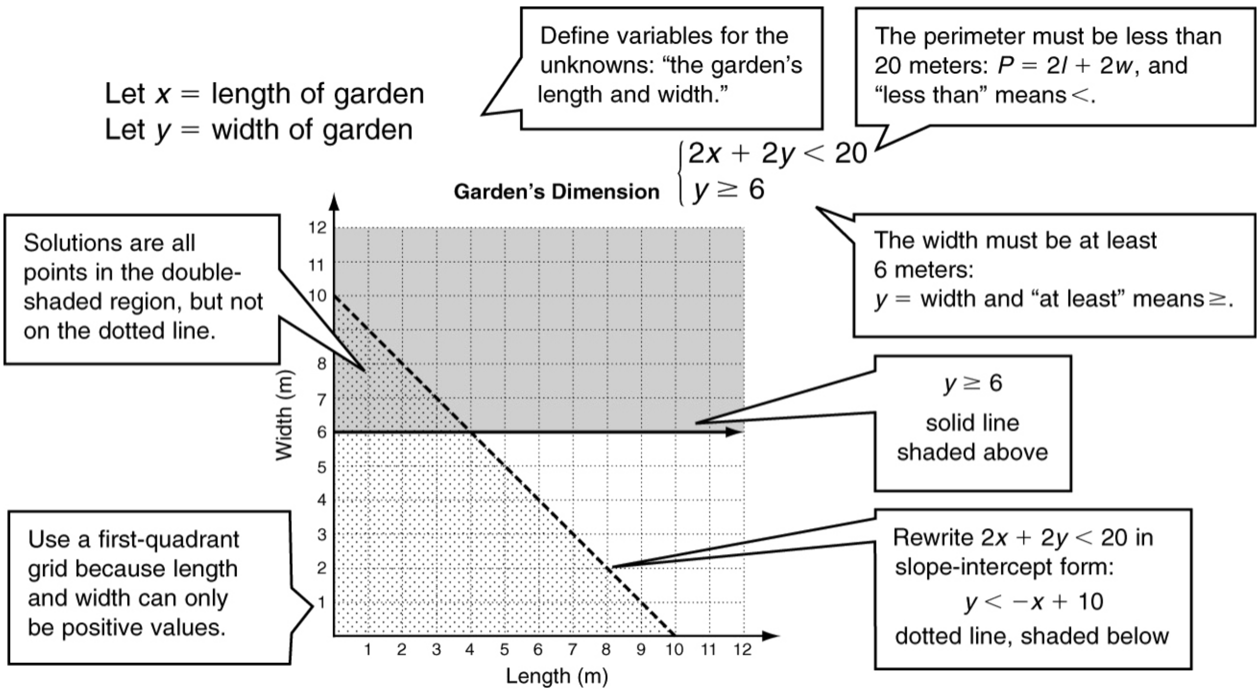
Example 4

To solve a real-world problem involving inequalities, there is a lot of information that must be analyzed. Study the example below.

The perimeter of a garden must be less than 20 meters.
The width must be at least 6 meters.
Show all possible combinations of the garden's length and width.
Identify two possible solutions.

Let x = length of garden
Let y = width of garden

$$\begin{cases} 2x + 2y < 20 \\ y \geq 6 \end{cases}$$



Possible solutions: $x = 2$ m, $y = 7$ m, and $x = 1$ m, $y = 8$ m

Manuel is going to build a rabbit cage with a rectangular base.
The perimeter can be no greater than 30 feet.
The length must be greater than 8 feet.
Let x = length and y = width.

A. Write a system of linear inequalities to describe the possible cage sizes.

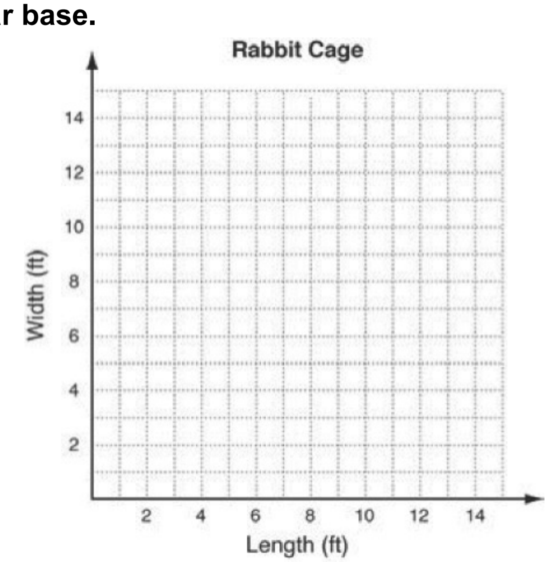
____

B. Graph the system to show possible dimensions.

C. Give two possible dimensions for the cage.

____

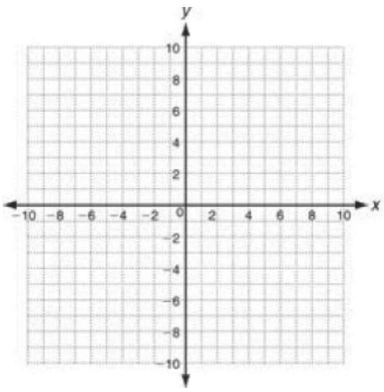
____



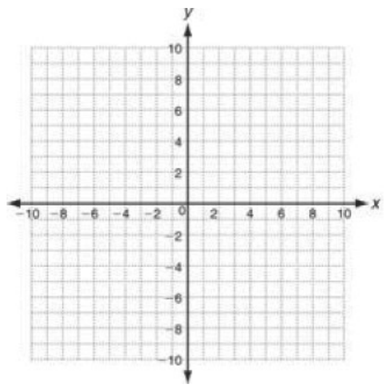
SELF CHECK

Solve each system of linear inequalities by graphing. Check your answer by testing an ordered pair from each region of your graph.

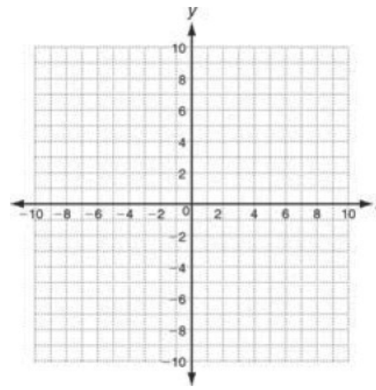
1. $\begin{cases} y > x - 3 \\ y \geq -x + 6 \end{cases}$



2. $\begin{cases} y < x \\ y > -2x + 1 \end{cases}$

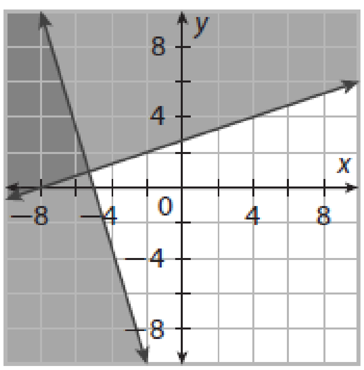


3. $\begin{cases} y > 2x - 2 \\ y \leq 2x + 3 \end{cases}$

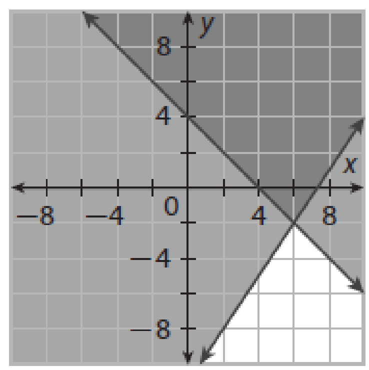


Determine if the given point is a solution of the system of inequalities. If not, find a point that is.

4. $(-9, 4)$



5. $(6, -2)$



6.

Match the inequality with the correct boundary line. Answers may be used more than once.

a. $y = 3x$ ________ $-x + 3y \leq 0$

b. $y = \frac{1}{3}x$ ________ $y > -x + \frac{1}{2}$

c. $y = x - 0.5$ ________ $y \leq \frac{1}{3}x$

d. $y = -x + \frac{1}{2}$ ________ $\frac{2}{3} + \frac{1}{3}y \geq x$

e. $y = 3x - 2$ ________ $-y > x - 0.5$

f. $y = x$ ________ $\frac{1}{3}y \geq x$

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For each inequality, write the equation of the corresponding line in slope-intercept form. Then state whether you shade above or below the line to graph the inequality. The first one is done for you.

1. $2x + y < 4$ 2. $y \geq 3x - 6$ 3. $4x - y \leq 7$

$y = -2x + 4$; below ______________________ ______________________

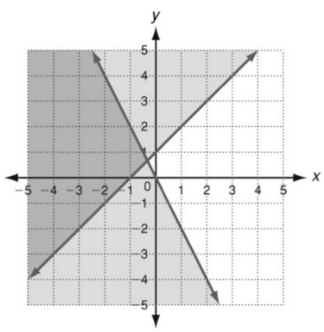
Tell whether the ordered pair (3, 2) is a solution of the given system. The first one is done for you.

4. $\begin{cases} y < 2x - 5 \\ y > -x + 2 \end{cases}$ 5. $\begin{cases} x + y \leq 5 \\ 3x + 2y > 10 \end{cases}$ 6. $\begin{cases} x < 3y - 2 \\ y > 3x - 7 \end{cases}$

no ______________________ ______________________

Graph the system of linear inequalities. a. Give two ordered pairs that are solutions. b. Give two ordered pairs that are not solutions. The first one is done for you.

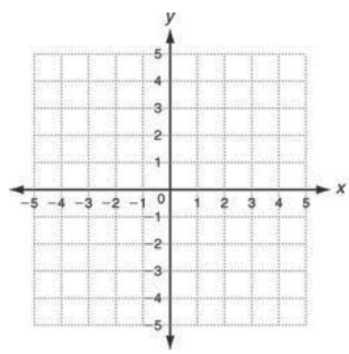
7. $\begin{cases} y \geq x + 1 \\ y \leq -2x \end{cases}$



a. **(–1, 0) and (–3, 2)**

b. **(0, –3) and (4, 0)**

8. $\begin{cases} y < 2x + 4 \\ y > x - 1 \end{cases}$



a. ______________________

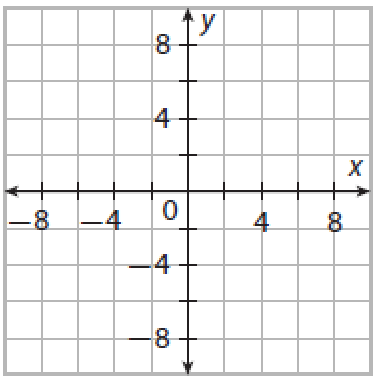
b. ______________________

Solve.

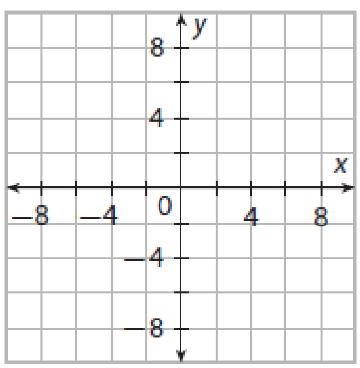
9. Coach Jules bought more than five bats. Some were wood and some were composite. The wood bats cost \$49 each and the composite bats cost \$100 each. Coach Jules spent less than \$400. Write the system of equations that could be used to represent this situation. Let *w* stand for wood bats and *c* stand for composite bats.

10. Graph

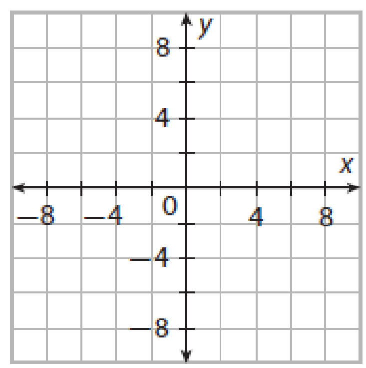
$$\begin{cases} y < \frac{1}{3}x - 6 \\ y \geq \frac{1}{3}x + 5 \end{cases}$$



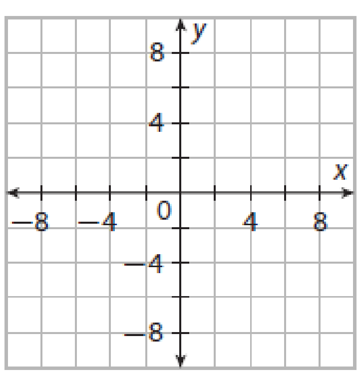
11. $\begin{cases} y \leq -\frac{3}{5}x \\ y > -x - 4 \end{cases}$



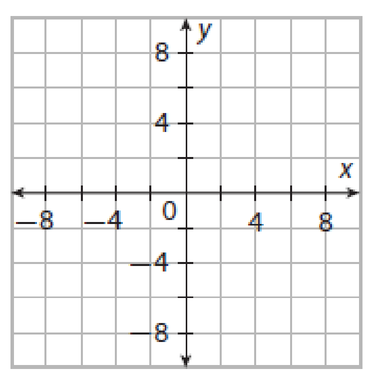
12. $\begin{cases} y \geq 2x + 6 \\ y < -\frac{1}{2}x - 1 \end{cases}$



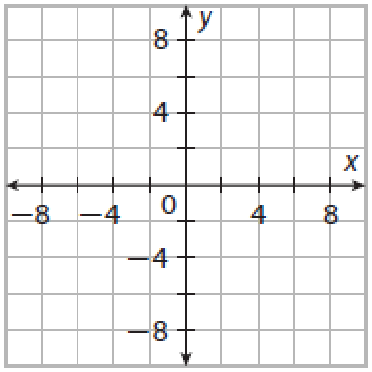
13. $\begin{cases} y \leq \frac{4}{5}x - 4 \\ y < 2x - 8 \end{cases}$



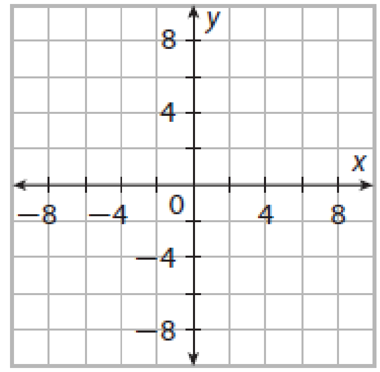
14. $\begin{cases} x \geq -6 \\ y < 3 \end{cases}$



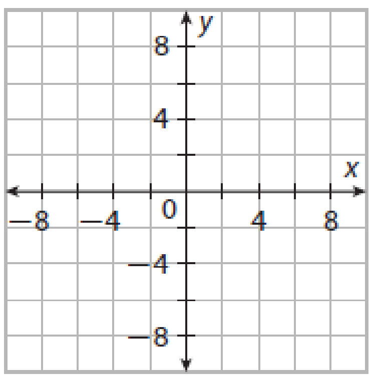
15. $\begin{cases} y \leq 3x + 6 \\ y < 3x - 8 \end{cases}$



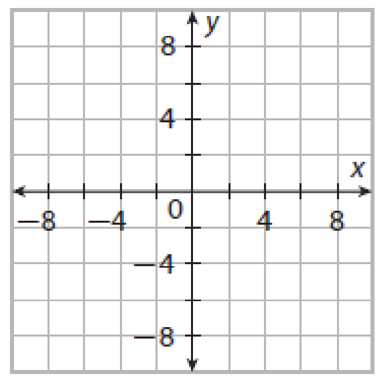
16. $\begin{cases} y \geq \frac{2}{5}x + 4 \\ y \leq \frac{2}{5}x - 6 \end{cases}$



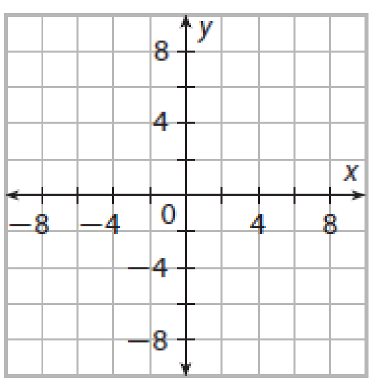
17. $\begin{cases} y \geq \frac{5}{4}x - 6 \\ y \geq \frac{5}{4}x \end{cases}$



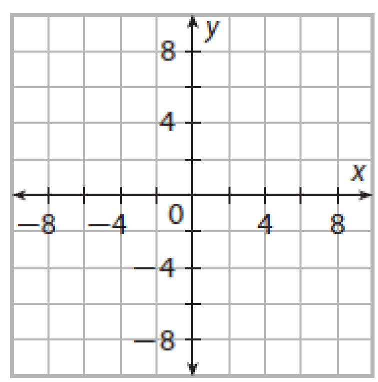
18. $\begin{cases} y \geq -\frac{3}{2}x - 3 \\ y \leq -\frac{3}{2}x + 10 \end{cases}$



19. $\begin{cases} x < 6 \\ x \geq -3 \end{cases}$

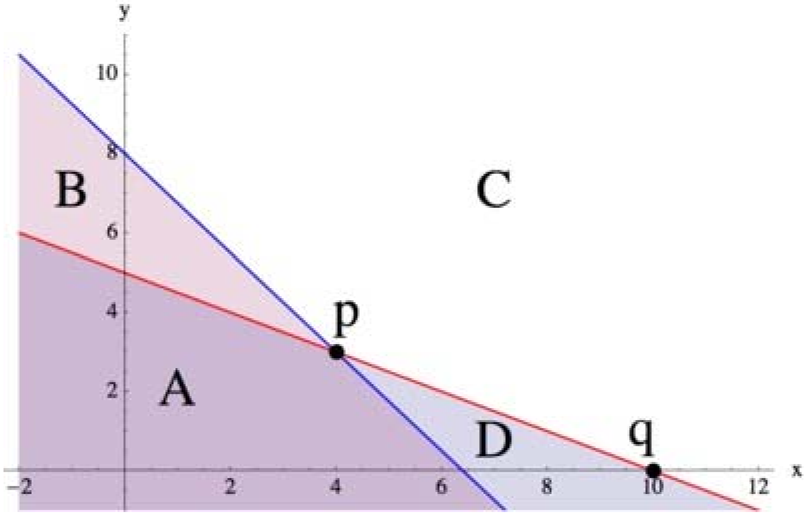


20. $\begin{cases} y \geq \frac{9}{4}x - 1 \\ y < \frac{9}{4}x - 9 \end{cases}$



Systems of Linear Inequalities (Illustrative Mathematics)

Given below are the graphs of two lines, $y = -0.5x + 5$ and $y = -1.25x + 8$, and several regions and points are shown. Note that C is the region that appears completely white in the graph.



a. For each region and each point, write a system of equations or inequalities, using the given two lines, that has the region or point as its solution set and explain the choice of $\leq$,$\geq$, or $=$ in each case. (You may assume that the line is part of each region.)

b. The coordinates of a point within a region have to satisfy the corresponding system of inequalities. Verify this by picking a specific point in each region and showing that the coordinates of this point satisfy the corresponding system of inequalities for that region.

c. In the previous part, we checked that specific coordinate points satisfied our inequalities for each region. Without picking any specific numbers, use the same idea to explain how you know that all points in the 3rd quadrant must satisfy the inequalities for region A

A2.U6.C3.H.05.HW.InequalitySystems

Name ________________________________ Period _______ Date ___________

Tell whether the ordered pair is a solution of the given system.

1. $(2, -2);\begin{cases} y < x - 3 \\ y > -x + 1 \end{cases}$

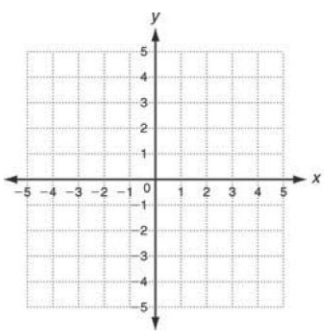
2. $(2, 5);\begin{cases} y > 2x \\ y \geq x + 2 \end{cases}$

3. $(1, 3);\begin{cases} y \leq x + 2 \\ y > 4x - 1 \end{cases}$

____________________ ____________________ ____________________

Graph the system of linear inequalities. a. Give two ordered pairs that are solutions. b. Give two ordered pairs that are not solutions.

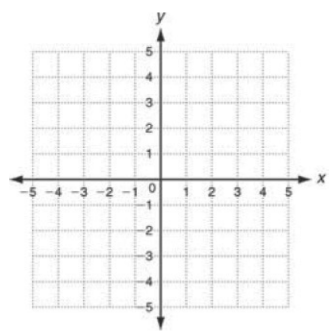
4. $\begin{cases} y \leq x + 4 \\ y \geq -2x \end{cases}$



a. ___________________

b. ___________________

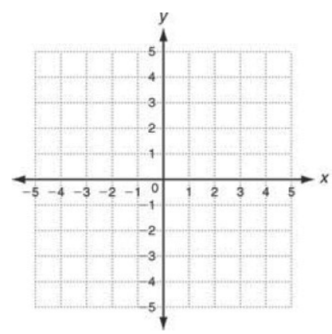
5. $\begin{cases} y \leq \dfrac{1}{2}x + 1 \\ x + y < 3 \end{cases}$



a. ___________________

b. ___________________

6. $\begin{cases} y > x - 4 \\ y < x + 2 \end{cases}$



a. ___________________

b. ___________________

7. Charlene makes \$10 per hour babysitting and \$5 per hour gardening. She wants to make at least \$80 a week, but can work no more than 12 hours a week.

   a. Write a system of linear equations.

   ____

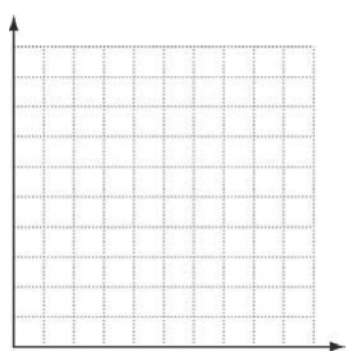
   b. Graph the solutions of the system.

   c. Describe all the possible combinations of hours that Charlene could work at each job.

   ____

   ____

   d. List two possible combinations. ____



GUIDED NOTES

Real-world situations can often be modeled by systems of equations or inequalities. This is sometimes referred to as linear programming.

Mrs. Hathaway bought a total of 12 items made up of some sticky notes and some pens. The sticky notes cost $4 each and the pens cost $2 each. She spent a total of $40 on all items. How many pens and how many sticky notes did she buy?

Organize the information.

	Sticky Notes	**Pens**	**Total**
Number of Items	n	p	12
Cost	$4n$	$2p$	40

Write two equations. Use the information in each row of the chart.

Number of Items	n	p	12	⇒	$n+p=12$
Cost	$4n$	$2p$	40	⇒	$4n+2p=40$

Write each equation in slope-intercept form.

$n+p=12$

$n=-p+12$

$4n+2p=40$

$4n=-2p+40$

$n=-\frac{1}{2}p+10$

Set the equations equal to each other and solve.

$-p+12=-\frac{1}{2}p+10$

$12=\frac{1}{2}p+10$

$2=\frac{1}{2}p$

$4=p$

She bought 4 pens.

$n+p=12$

$n+4=12$

$n=8$

She bought 8 sticky notes.

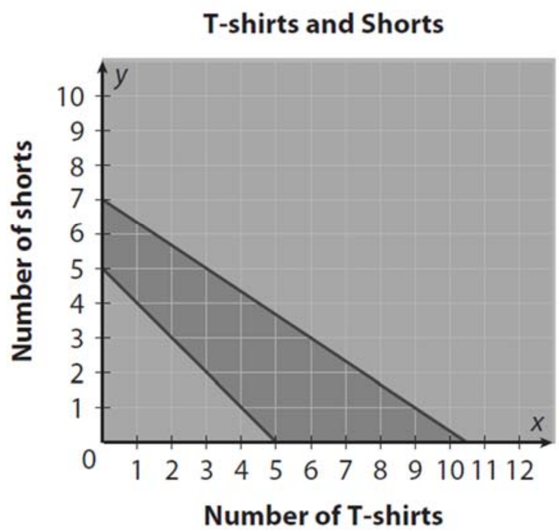
Sue is buying T-shirts and shorts. T-shirts cost \$14 and shorts cost \$21. She plans on spending no more than \$147 and buy at least 5 items. Show and describe all combinations of the number of T-shirts and shorts she could buy.

First write the system. Let x represent the number of T-shirts, and let y represent the number of shorts.

$x + y \geq 5$ She wants to buy at least 5 items.

$14x + 21y \leq 147$ She wants to spend no more than \$147.

Graph the system of inequalities: $\begin{cases} x + y \geq 5 \\ 14x + 21y \leq 147 \end{cases}$



The possible solutions are where the shaded regions overlap. So, a possible solution is 5 T-shirts and 2 shorts. Substitute this value into the inequalities to make sure it is a reasonable solution.

$$\begin{cases} x + y \geq 5 \\ 14x + 21y \leq 147 \end{cases} \rightarrow \begin{cases} 5 + 2 \overset{?}{\geq} 5 \\ 14(5) + 21(2) \overset{?}{\leq} 147 \end{cases} \rightarrow \begin{cases} 7 \geq 5 \\ 112 \leq 147 \end{cases}$$

The result is two inequalities that are true, so this is a reasonable answer.

Example!

Problem 1

Marcel bought a total of 20 markers and pens. Markers cost \$2.50 each and pens cost \$1.50 each. Marcel spent a total of \$42.00. How many markers and how many pens did he buy?

Write two equations.

Markers + pens = 20 items $\Longrightarrow$ $m + p = 20$

Markers cost + pens cost = \$42 total cost $\Longrightarrow$ $2.50m + 1.50p = 42$

You have m and $2.50m$. Multiply m by –2.50 to make them opposites.

$-2.50(m + p = 20) \Longrightarrow -2.50m + -2.50p = -50$

Add the equations:

$$\begin{array}{r} 2.50m + 1.50p = 42 \\ \underline{-2.50m - 2.50p = -50} \\ 0 - 1p = -8 \\ p = 8 \end{array}$$

$p = 8 \Longrightarrow$ He bought 8 pens.

Find the number of markers:

$m + p = 20$
$m + 8 = 20$
$m = 12$ $\Longrightarrow$

Check the cost:
$2.50m + 1.50p \stackrel{?}{=} 42$
$2.50(12) + 1.50(8) \stackrel{?}{=} 42$
$30 + 12 = 42$
He bought 12 markers.

Example 1 Tell how to make the x-variable opposites for each system of equations.

A) $x + y = 22$
$4x + 8y = 38$

____

B) $x + y = 7$
$3.2x + 1.8y = 13$

____

C). $-1.3x - 4y = -18$
$x + y = 5$

____

Example 2 Write a system of equations and solve the problem.

Larry bought 15 pairs of dress socks and sports socks. Dress socks cost $7 each pair and sports socks cost $4 each pair. He spent a total of $75. How many pairs of dress socks and sports socks did he buy?

Equations: ____

Solution: ____

Example 3

John has to buy two different kinds of rope. Rope A costs \$0.60 per foot and Rope B costs \$0.90 per foot. John needs to buy at least 15 feet of rope, but he wants to spend no more than \$18. Show and describe all combinations of the number of feet of each type of rope John can buy.

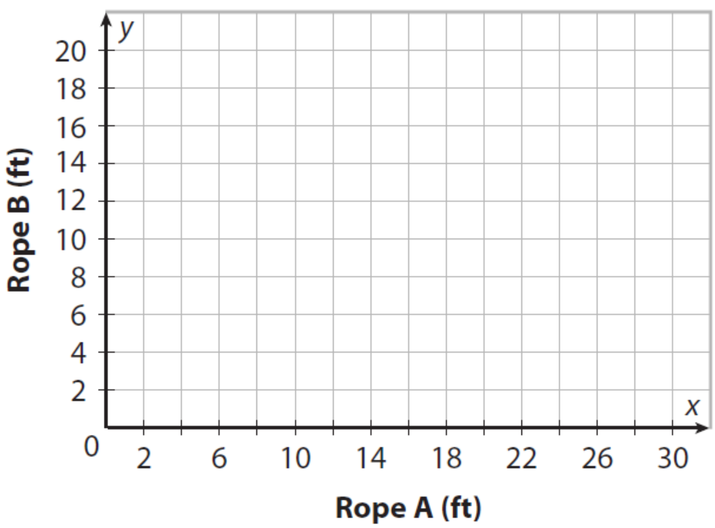
First write the system. Let x represent the amount of Rope A, and let y represent the amount of Rope B.

$\square + \square \geq 15$

$\square + 0.9y \; \square \; 18$

Graph the system.

Buying Rope



Describe the solutions to the system.

The possible solutions are ______________________

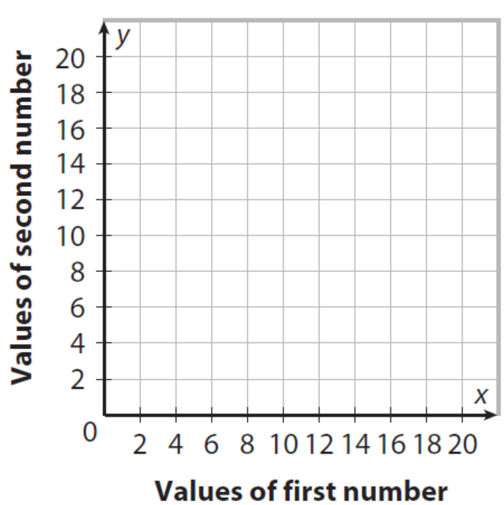
____

____

____

Example 4

The sum of two numbers is at least 8, and the sum of one of the numbers and 3 times the second number is no more than 15.



SELF CHECK

Bill has 51 dimes and nickels. The total value of the coins is $4.25.
How many dimes and how many nickels does he have?

Make a chart.

	Nickels	Dimes	Total
Number of Coins	n	d	51
Value of Coins			$4.25

The problem does not include the value of each nickel or the value of each dime.

Think: The value of 1 nickel = 5¢, or $0.05.

The value of 1 dime = 10¢, or $0.10.

Use the values to complete the chart.

Write an equation for each row of the chart.

Number of Coins	n	d	51

$n + d = 51$

Value of Coins			$4.25

$0.05n + 0.10d = 4.25$

Use the values to write the terms for the equation.

Solve as usual.

$-20(0.05n + 0.10d = 4.25)$
$-n - 2d = -85$

$n + d = 51$
$-n - 2d = -85$
$-d = -34$, or $d = 34$

He has 34 dimes.

$n + 34 = 51$
$n = 17$

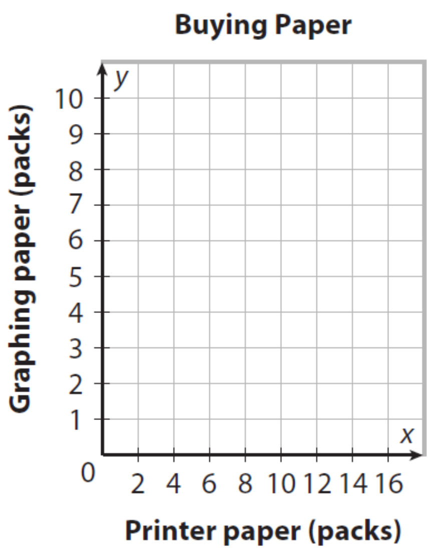
He has 17 nickels.

1. Lauren has 85 quarters and dimes. The total value is $16.90.
   How many quarters and how many dimes does she have?

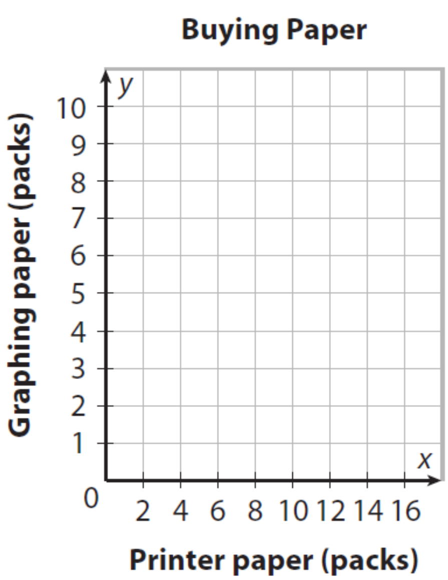
2. Mike has 25 bills. Some are $5 bills and some are $20 bills. The total value is $335.
   How many $5 bills and how many $20 bills does he have?

3. Write a system of inequalities for the given situation and graph the system. Then determine if the point (8, 4) is a solution to the system.

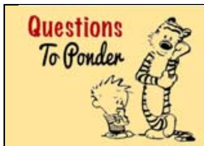
A student has to buy graph paper and printer paper. The printer paper costs $2 a pack, while the graphing paper costs $3 a pack. She wants to buy at least 6 packs of paper but wants to spend at most $27.



Now assume that she wants to buy at least 7 packs and will spend at most $30.



A2.U6.C3.**I.02.Notes.LinearProgramming**



Is there anything special about the vertices of the shape formed by the solution area?
Is it possible for a system of two linear inequalities to only have one solution?
Why can't a system of inequalities be solved using the same methods as solving systems of equations?
When writing a system of equations or inequalities from a situation, how do you know that you have possibly written the system correctly?

Name ____ Period ___________ Date ___________

Solve each problem. The first one is started for you.

1. A student bought some markers and pads of paper. The markers cost \$2 each and the pads of paper cost \$4 each. He bought 8 items in all and spent \$26. How many markers and how many pads of paper did he buy?

   Let m = ___**markers**___ and let p = ___**pads of paper**___

   Equations: ___ $\mathbf{2m + 4p = 26,\ m + p = 8}$ ___

   Multiply $m + p = 8$ by –2 to make opposite coefficients:

   $-2(m + p = 8)$ ______________________________

   Add: $2m + 4p = 26$ He bought _____ pads of paper.
   $\underline{-2m - 2p = -16}$
   _____ $\mathbf{= 10}$

   $m + 5 = 8$ $\mathbf{p =}$ _____ He bought ______ markers.

   m = __________

2. A student bought some music CDs and some movie DVDs. The CDs cost \$9 each and the DVDs cost \$17 each. He bought 7 items in all for \$87. How many CDs and how many DVDs did he buy?

   Equations: __________________________________

   Solution: ____________________________________

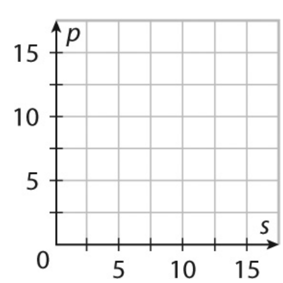
Write a system of inequalities and graph them to solve the problem. The work is started for you.

3. Alie needs to buy at least 12 candles. Plain candles sell for \$4 each and scented candles sell for \$7 each. She can spend no more than \$57. Give one possible solution.

   Inequalities: ___ $\mathbf{4p + 7s \le 57}$ ___

   ____

   Possible solution: ____________________



4. Tia has 25 china figures in her collection. The horse figures cost $2 each, and the cat figures cost $1 each. She paid $39 for all the figures in the collection. How many horses and how many cats does she have?

Equations: ___________________________________

Solution: ___________________________________

5. Mr. Wallace has 32 models of antique cars. The Hupmobile models cost $5 each, and the Duesenberg models cost $18 each. He paid a total of $264 for all the models. How many Hupmobile models and how many Duesenberg models does he have?

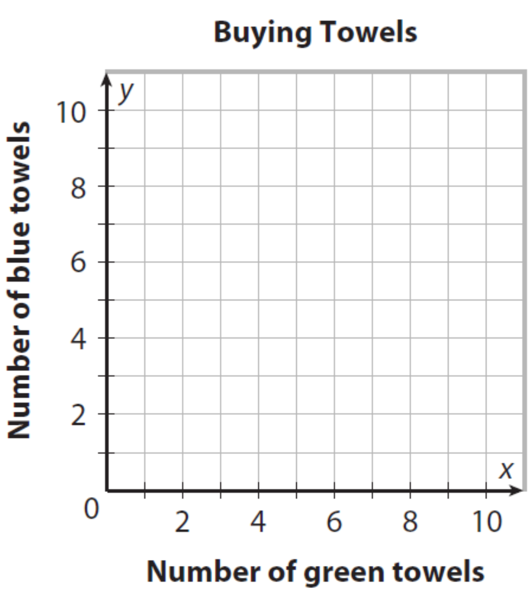
Equations: ___________________________________

Solution: ___________________________________

Write the system of inequalities that represents the situation. Then graph the system and describe the solutions. Give one possible solution.

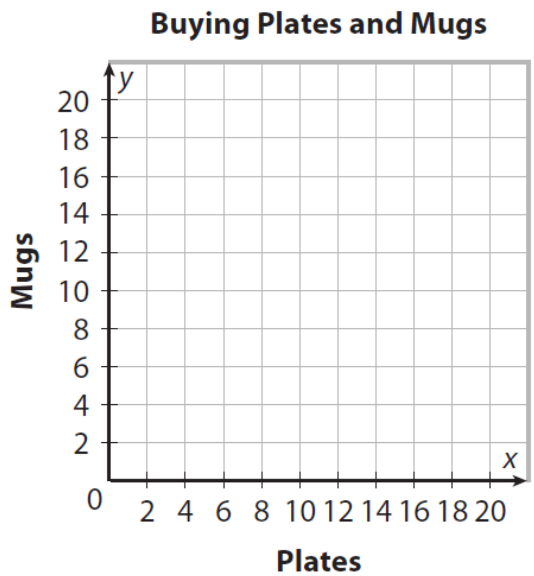
6.

Angelique is buying towels for her apartment. She finds some green towels that cost $8 each and blue towels that cost $10 each. She wants to buy at least 4 towels but doesn't want to spend more than $70. How many of each towel can she purchase?



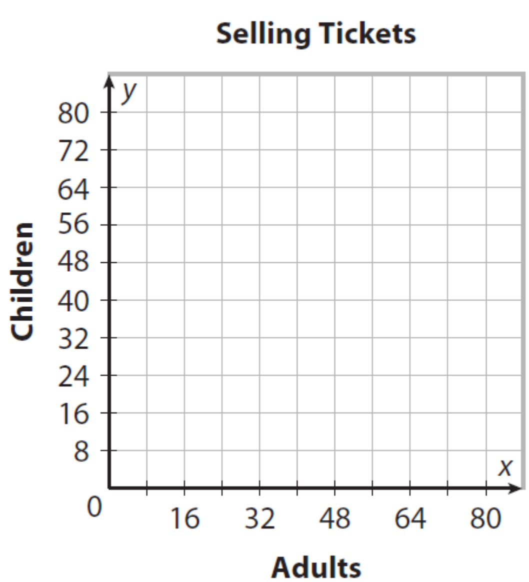
7.

Katie is purchasing plates and mugs for her house. She would like to buy at least 8 items. Determine the possibilities if the plates cost $8 each and the mugs cost $7 each, and she plans to spend no more than $112.



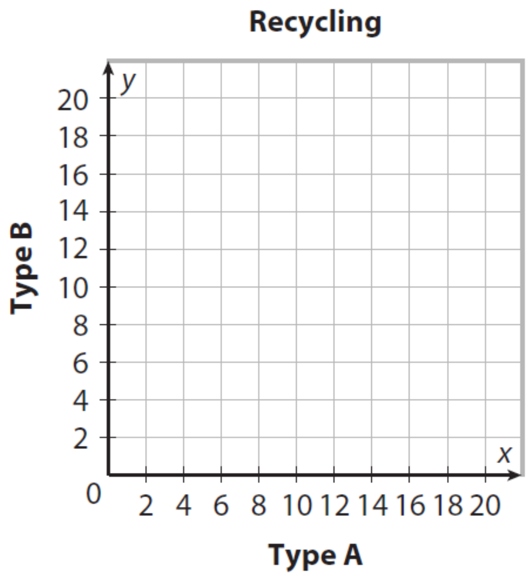
8.

Christine is selling tickets at a museum. She knows that she has sold at least 40 tickets. The adult tickets cost 14 dollars and the children's tickets cost 12 dollars. If she knows she has sold no more than $720 worth of tickets, what are the possible combinations?



9.

Mike is bringing cans and bottles to a recycling center. For a type A can or bottle he gets 5 cents, and for a type B can or bottle he gets 10 cents. He knows that he has redeemed at least 11 cans but has no more than 95 cents. What are the possible combinations?

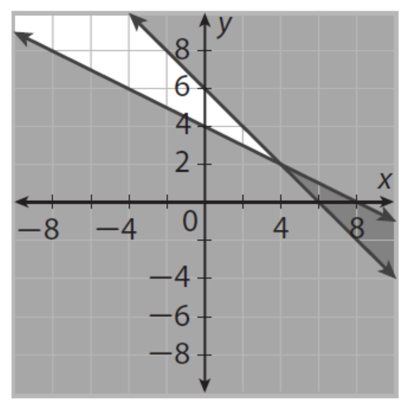


10.

Explain the Error A student is given the following system. He graphs the system as shown and determines that a solution is (7, 0). Where did the student go wrong? What should the correct answer be?

$x + y = 6$

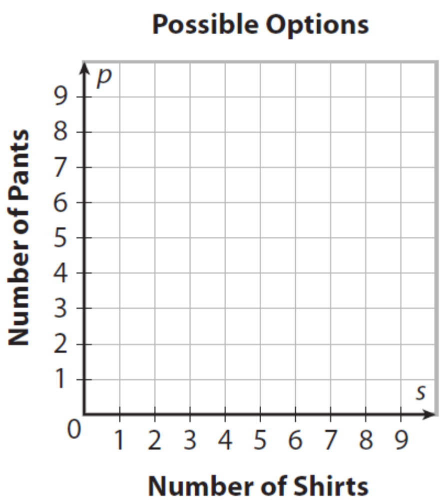
$x + 2y = 8$



There is an *implied* goal here. Sometimes, it's not implicitly stated. So think about this – when you have a set budget with which to shop, what is your GOAL?

Graphing Systems of Linear Inequalities with Constraints- Application

Amy is at the store to buy shirts and pants. The shirts cost $40 each and the pants cost $50 each. She plans to spend no more than $400 and buy at least 5 items. Find a possible combination of shirts and pants she can buy. How do you know this is a solution? What are two possible ways to show that this is a solution?



Name ________________________________ Period ________ Date__________

Write a system of equations to solve each problem.

SHOW YOUR WORK ON A SEPARATE PIECE OF PAPER FOR CREDIT!

1. For a small party of 12 people, the caterer offered a choice of a steak dinner for $12.00 per meal or a chicken dinner for $10.50 per meal. The final cost for the meals was $138.00. How many of each meal was ordered?

   Equations: __________________________________

   Solution: ____________________________________

2. A clubhouse was furnished with a total of 9 couches and love seats so that all 23 members of the club could be seated at once. Each couch seats 3 people and each love seat seats 2 people. How many couches and how many love seats are in the clubhouse?

   Equations: __________________________________

   Solution: ____________________________________

3. A small art museum charges $5 for an adult ticket and $3 for a student ticket. At the end of the day, the museum had sold 89 tickets and made $371. How many student tickets and how many adult tickets were sold?

   Equations: __________________________________

   Solution: ____________________________________

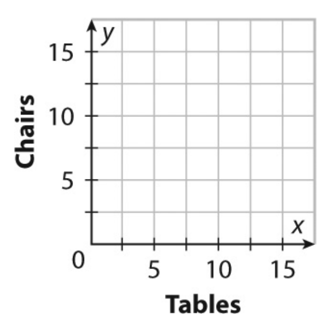
4. Cassie has a total of 110 coins in her piggy bank. All the coins are quarters and dimes. The coins have a total value of $20.30. How many quarters and how many dimes are in the piggy bank?

   Equations: __________________________________

   Solution: ____________________________________

Write a system of inequalities and graph them to solve the problem.

5. Jack is buying tables and chairs for his deck party. Tables cost $25 and chairs cost $15. He plans to spend no more than $285 and buy at least 16 items. Show and describe the solution set, and suggest a reasonable solution to the problem.



   Equations: __________________________________

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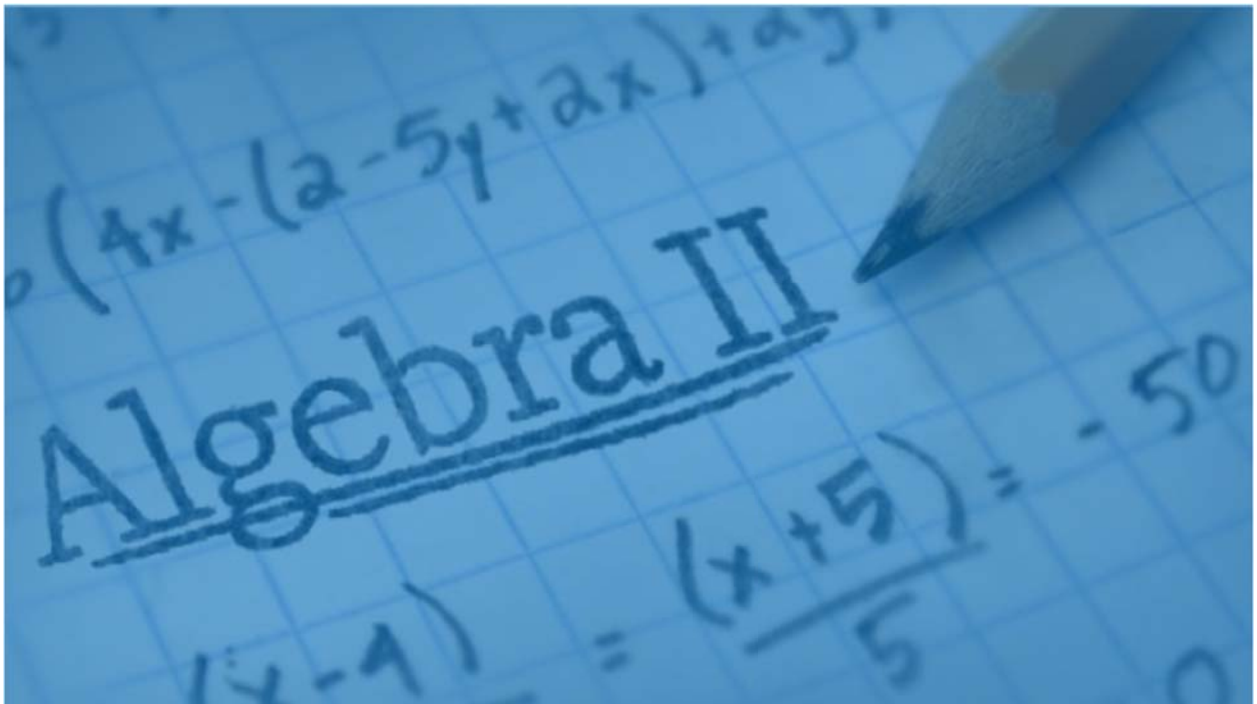


Unit 7

Inferences and Conclusions from Data

Algebra 2

Unit 7: Inferences and Conclusions from Data

*Note: Unit 7, Lessons F and G are not included in workbook as of Fall 2019-2020. Use state tasks located on SharePoint, these lessons may be updated at a later date.

A2.U7.C1.**A.01.vocab.StudyDesign**

Term	Definition	Notation	Diagram/Visual
Population			
Parameter			
Sample			
Statistics			
Mean of Population			
Mean of Sample			
Standard Deviation of Population*			
Standard Deviation of Sample*			

*note: students do not have to calculate or work with standard deviation. This is just a notation and basic explanation for now.

A2.U7.C1.**A.01.vocab.StudyDesign**

Term	Definition	Notation	Diagram/Visual
Variance			
Unit			
Variables			
Categorical Variable			
Qualitative Variable			
Numerical Variable			
Quantitative Variable			
Explanatory Variable			

Algebra 2 Unit # 7 Concept# 1

A2.U7.C1.**A.01.vocab.StudyDesign**

Term	Definition	Notation	Diagram/Visual
Response Variable			
Independent Variable			
Dependent Variable			
Predictor Variable			
Outcome Variable			
Observational Study			
Experimental Study			
Sampling Error			

A2.U7.C1.**A.01.vocab.StudyDesign**

Term	Definition	Notation	Diagram/Visual
Bias			
Simple Random Sampling			
Systematic Sampling			
Stratified Random Sampling			
Clustered Sampling			
Non-Probability Sampling Methods			
Convenience Sampling			
Sampling Bias vs. Response Bias			

A2.U7.C1.**A.01.vocab.StudyDesign**

Term	Definition	Notation	Diagram/Visual
Voluntary Response Sampling			
Incorrect Sampling Frame			
Undercoverage			
Size Bias			
Voluntary Response Bias			
Non-Response Bias			
Questionnaire Bias			
Incorrect Response Bias			

A2.U7.C1.**A.01.vocab.StudyDesign**

Term	Definition	Notation	Diagram/Visual
Randomization			
Control			
Replication			
Claim			

INTRODUCTION, STUDY DESIGNS

You will answer questions about the information on the next two pages about Galapagos Tortoises.

The Galapagos Tortoises

In order to learn some basic vocabulary of statistics and learn how to distinguish between different types of variables, we will use the example of information about the Giant Galapagos Tortoise.

Approximating the Distribution of the Galapagos Tortoises

The Galapagos Islands, off the coast of Ecuador in South America, are famous for the amazing diversity and uniqueness of life they possess. One of the most famous Galapagos residents is the Galapagos Giant Tortoise, which is found nowhere else on earth. Charles Darwin's visit to the islands in the 19th Century and his observations of the tortoises were extremely important in the development of his theory of evolution.

The tortoises lived on nine of the Galapagos Islands, and each island developed its own unique species of tortoise. In fact, on the largest island, there are four volcanoes, and each volcano has its own species. When first discovered, it was estimated that the tortoise **population** of the islands was around 250,000. Unfortunately, once European ships and settlers started arriving, those numbers began to plummet. Because the tortoises could survive for long periods of time without food or water, expeditions would stop at the islands and take the tortoises to sustain their crews with fresh meat and other supplies for the long voyages. Also, settlers brought in domesticated animals like goats and pigs that destroyed the tortoises' habitat. Today, two of the islands have lost their species, a third island has no remaining tortoises in the wild, and the total tortoise population is estimated to be around 15,000. The good news is there have been massive efforts to protect the tortoises. Extensive programs to eliminate the threats to their habitat, as well as breed and reintroduce populations into the wild, have shown some promise.

The following chart shows the approximate distribution of Giant Galapagos Tortoises in 2004,
Estado Actual De Las Poblaciones de Tortugas Terrestres Gigantes en las Islas Galápagos, Marquez, Wiedenfeld, Snell, Fritts, MacFarland, Tapia, y Nanjoa, Scologia Aplicada, Vol. 3, Num. 1,2, pp. 98 11.

Island or Volcano	Species	Climate Type	Shell Shape	Estimate of Total Population	Population Density (per km2)	Number of Individuals Repatriated∗
Wolf	becki	semi-arid	intermediate	1139	228	40
Darwin	microphyes	semi-arid	dome	818	205	0
Alcedo	vanden- burghi	humid	dome	6,320	799	0
Sierra Negra	guntheri	humid	flat	694	122	286
Cerro Azul	vicina	humid	dome	2.574	155	357
Santa Cruz	nigrita	humid	dome	3,391	730	210
Española	hoodensis	arid	saddle	869	200	1,293
San Cristóbal	chathamen- sis	semi-arid	dome	1,824	559	55
Santiago	darwini	humid	intermediate	1,165	124	498
Pinzón	ephippium	arid	saddle	532	134	552
Pinta	abingdoni	arid	saddle	1	Does not apply	0

∗Repatriation is the process of raising tortoises and releasing them into the wild when they are grown to avoid local predators that prey on the hatchlings.

WATCH THIS!
The Importance of Statistics Video:
http://tiny.cc/StatsImportance
You can use the QR code at right, as well→

Part 1: POPULATION VS. SAMPLE

We have already defined a population as the total group being studied. Most of the time, it is extremely difficult or very costly to collect all the information about a population. In the Galapagos, it would be very difficult and perhaps even destructive to search every square meter of the habitat to be sure that you counted every tortoise. In an example closer to home, it is very expensive to get accurate and complete information about all the residents of the United States to help effectively address the needs of a changing population. This is why a complete counting, or *census*, is only attempted every ten years. Because of these problems, it is common to use a smaller, representative group from the population, called a **sample**.

You may recall the tortoise data included a variable for the estimate of the population size. This number was found using a sample and is actually just an approximation of the true number of tortoises. If a researcher wanted to find an estimate for the population of a species of tortoises, she would go into the field and locate and mark a number of tortoises. She would then use statistical techniques that we will discuss later in this text to obtain an estimate for the total number of tortoises in the population. In statistics, we call the actual number of tortoises a **parameter**. Any number that describes the individuals in a sample (length, weight, age) is called a **statistic**. Each statistic is an estimate of a parameter, whose value may or may not be known.

GUIDED NOTES

When dealing with statistical data, it is important to distinguish between "population" data sets and "sample" data sets.

Definition: A population data set contains all members of a specified group (the entire list of possible data values). [Utilizes the count n in formulas.]
Example: The population may be "ALL people living in the US."

Definition: A sample data set contains a part, or a subset, of a population. The size of a sample is always less than the size of the population from which it is taken. [Utilizes the count n - 1 in formulas.]
Example: The sample may be "SOME people living in the US."

It is important to know in data calculations if you are working with an entire population (where you have all of the possible data), or if you are working with only a sample (a part) of the data. In addition, if you are using a sample of the data, you need to know if you will be making generalizations about the entire population, based upon this sample.
Note: When working with "sample data sets", statisticians use n for the number of data entries and $\bar{x}$ for the mean, however, when working with "population data sets", they use N for the number of data entries and μ for the mean. In Algebra 2, to avoid confusion and to coordinate with the notations used by the TI-84+ calculators, we will be using n for the number of data entries and $\bar{x}$ for the mean for both population and sample data sets.
In statistics, numerical values used are either PARAMETERS or STATISTICS.

Example! **Let's take a look at an example dealing with average (mean), to see an application of "population" versus "sample". For (a) and (b) below, determine if you are going to calculate for a population or a sample.**

(a) Find the average of the heights of all fourteen-year-old boys in your Algebra class.
This task is only dealing with the heights of fourteen-year-old <u>boys in one specific class</u>. The intent is not to estimate the heights of all fourteen-year-old boys in the world. The "population" in this task is only the fourteen-year-old boys in your Algebra class. Since you have the entire population available for this situation, you will be finding the **population average (mean)**.
(b) Find the average of the heights of all fourteen-year-old boys in the world.
In this situation, the <u>population is extremely large. There is actually no way of obtaining all of the data in the population. You simply will not have all of the data available for your use.</u> You will need to use a sample of the population. It will be necessary to "estimate" the population's average based upon the average of a sample of the population. You will be finding the **sample average (mean).**

SELF CHECK Population or Sample? Give it a try:

Some questions will clearly state whether you are working with a population or a sample. If no statement is present, ask yourself if the statistical findings will be used to describe a larger group. If the answer is yes, you are working with a sample. Real world statisticians primarily work with sample situations, since real-world data can be overwhelmingly large.

Directions: For the following problems, decide if the situation is dealing with a "population" data set, or with a "sample" data set. Explain your decision.

1. Mrs. Smith wants to do a statistical analysis on students' final examination scores in her math class for the past year. Should she consider her data to be a population data set or a sample data set?

2. A group of students surveys 100 students from their freshman class to determine the number of pets in each student's household. The group plans to compute statistical findings on their data and generalize these findings to the homes of all freshmen students. Should the group consider their data to be a population data set or a sample data set?

1. Population Data Set 2. Sample Data Set

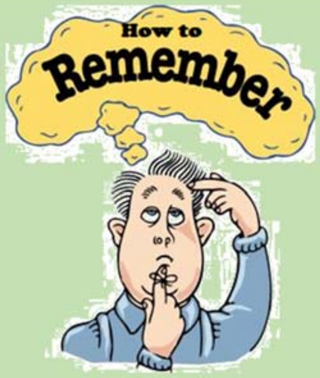
CAN YOU FILL IN THE BLANKS BELOW TO SUMMARIZE POPULATION AND SAMPLE?

I should use "population" when:

1. I know you have the ___________ population.
2. I have a sample of a __________ population, but I am only interested in this ______________,and I will not be _______________________my ___________________to the _____________ larger population.

I should use "sample" when:

1. I have a __________ of a ___________ population, and I wish to _________________________________ from this sample to the ______________ larger population from which this __________ was taken.
2. The sample will be used as an _________________of the population.



...the difference between Parameter and Statistic, their respective notations, and what they mean:

Think of a cow parachuting when you think of parameter. This cow is so far up in the sky that she can see the entire population in all directions. Also, remember that cows say moo, er….**mu (μ)**!

Parameter

- Parameter is the term used for numbers and variables of a population.
- μ refers to a population mean
- σ refers to the standard deviation* of a population
- Generally divided by (n), where division occurs.**

Statistic

- Statistic is the term used for numbers and variables of a sample.
- $\bar{x}$ refers to a sample mean
- s refers to the standard deviation* of a sample
- Generally divided by $(n-1)$, where division occurs.**

**You learned about Mean Absolute Deviation in Algebra 1 (remember, it was "the average distance from the mean" of a set of numbers), and it told you how spread out data is. We will extend that idea to the more useful and statistically predictive "standard deviation" later in this unit!*

***Note: The practice of dividing by $(n-1)$ (instead of (n)) when working with a sample of the entire population, produces a slight difference in the final calculation. This slight difference allows the sample to give a better mathematical estimate of the population. Think of dividing by $(n-1)$ (instead of (n)) in the sample as a means of "compensating" for the fact that we are working with a sample of the population, rather than with the entire population. It statistically gives the best estimate.*

For calculator info on population versus sample, go to this URL: **http://tiny.cc/TICalcInfo** or use this QR Code:

Part 2: Classifying Variables by Data Type and by Function in Study

BY DATA TYPE: Each member of the population is called a **unit**. In this example, the population is all Galapagos Tortoises, and the units are the individual tortoises. It is not necessary for a population or the units to be living things, like tortoises or people. For example, an airline employee could be studying the population of jet planes in her company by studying individual planes.

A researcher studying Galapagos Tortoises would be interested in collecting information about different characteristics of the tortoises. Those characteristics are called **variables.** Each column of the previous figure contains a **variable.** In the first column, the tortoises are labeled according to the island (or volcano) where they live, and in the second column, by the scientific name for their species. When a characteristic can be neatly placed into well-defined groups, or categories, that do not depend on order, it is called a ***categorical variable***, or ***qualitative variable.***

The last three columns of the previous figure provide information in which the count, or quantity, of the characteristic is most important. We are interested in the total number of each species of tortoise, or how many individuals there are per square kilometer. This type of variable is called a ***numerical variable,*** or ***quantitative variable***.

SELF CHECK Determine whether each of the variables **Climate Type, Shell Shape, Number of Tagged Individuals,** and **Number of Individuals Repatriated** are numerical or categorical variables.

Fill in the blanks:

Variable	Explanation	Type
Climate Type	Many of the islands and volcanic habitats have three distinct climate types.	
Shell Shape	Over many years, the different species of tortoises have developed different shaped shells as an adaptation to assist them in eating vegetation that varies in height from island to island.	
Number of Tagged Individuals	Tortoises were captured and marked by scientists to study their health and assist in estimating the total population.	
Number of Individuals Repatriated	There are two tortoise breeding centers on the islands. Through these programs, many tortoises have been raised and then reintroduced into the wild.	

Answer: Categorical, Categorical, Numerical, Numerical

WATCH THIS!
Sampling Video:
http://tiny.cc/sampling2
You can use the QR code at right, as well→

BY FUNCTION IN STUDY:

In some research studies one variable is used to predict or explain differences in another variable. In those cases, the **explanatory variable** is used to predict or explain differences in the **response variable**. In an experimental study, the explanatory variable is the variable that is manipulated by the researcher.

Explanatory Variable
Also known as the *independent* or *predictor variable*, it explains variations in the response variable; in an experimental study, it is manipulated by the researcher

Response Variable
Also known as the *dependent* or *outcome variable,* its value is predicted or its variation is explained by the explanatory variable; in an experimental study, this is the outcome that is measured following manipulation of the explanatory variable

Example! **Public Speaking Approaches** In the study below,which is (are) explanatory variable(s), and which is the response variable?

A public speaking teacher has developed a new lesson that she believes decreases student anxiety in public speaking situations more than the old lesson. She designs an experiment to test if her new lesson works better than the old lesson. Public speaking students are randomly assigned to receive either the new or old lesson; their anxiety levels during a variety of public speaking experiences are measured.
This experiment has **one** explanatory variable: the lesson received. The response variable is **anxiety level**.

SELF CHECK **Coffee Bean Origin** In the study below, (a) what type of study is this, (b) what is (are) explanatory variable(s), and (c) what is the response variable?

A researcher believes that the origin of the beans used to make a cup of coffee affects hyperactivity. He wants to compare coffee from three different regions: Africa, South America, and Mexico. Volunteers for the study are into three groups, and each will drink coffee from one of the three regions. Afterward, their levels of focus and activity is measured.

a) This study type is ______________________.
b) The **explanatory variable** is________________;
c) The **response variable** is____________________.

(a) Experimental **(b)** origin of coffee (3 levels – Africa, South America, and Mexico) **(c)** hyperactivity level

SELF CHECK **Height and Age** In the study below, (a) what type of study is this, (b) what is (are) explanatory variable(s), and (c) what is the response variable?

A group of middle school students wants to know if they can use height to predict age. They take a random sample of 50 people at their school, both students and teachers, and record each individual's height and age.

a) This study type is ______________________.
b) The **explanatory variable** is________________;
c) The **response variable** is____________________.

(a) Observational **(b)** Height **(c)** Age

SELF CHECK **Gender and Height** In the research question below, (a) what type of study would answer this question, (b) what is (are) explanatory variable(s), and (c) what is the response variable?

Research question: Do third grade boys tend to be taller than third grade girls?

a) This study type is ______________________.
b) The **explanatory variable** is________________;
c) The **response variable** is____________________.

(a) Observational **(b)** Gender **(c)** Height

Part 3: Errors in Sampling

We have to accept that estimates derived from using a sample have a chance of being inaccurate. This cannot be avoided unless we measure the entire population. The researcher has to accept that there could be variations in the sample due to chance that lead to changes in the population estimate. A statistician would report the estimate of the parameter in two ways: as a **point estimate** (e.g., 915) and also as an **interval estimate**. For example, a statistician would report: “I am fairly confident that the true number of tortoises is actually between 561 and 1075.” This range of values is the unavoidable result of using a sample, and not due to some mistake that was made in the process of collecting and analyzing the sample. The difference between the true parameter and the statistic obtained by sampling is called **sampling error**. It is also possible that the researcher made mistakes in her sampling methods in a way that led to a sample that does not accurately represent the true population.

Determining Errors That May Have Occurred

What are some possible errors that could be involved in the study of the Galapagos tortoises?

The researcher could have picked an area to search for tortoises where a large number tend to congregate (near a food or water source, perhaps). If this sample were used to estimate the number of tortoises in all locations, it may lead to a population estimate that is too high.

This type of systematic error in sampling is called **bias**. Statisticians go to great lengths to avoid the many potential sources of bias. We will investigate this in more detail in a later chapter.

Example! Categorical or Quantitative?

Example 1

Indicate whether importance of political party affiliation to people (very, somewhat, or not very important) is a categorical or quantitative variable.

This is categorical data because the information collected will fall into one of the three categories: very, somewhat, or not very important.

Example 2

Indicate whether hours spent reading yesterday is a categorical or quantitative variable.

This is measured by numbers of hours, so it is quantitative data.

Example 3

Indicate whether the weights of adult men, in pounds is a quantitative or categorical variable.

This is measured in pounds, so it is quantitative data.

Example 4

Indicate whether favorite type of book (fiction, nonfiction) is a categorical or quantitative variable.

This is categorical data because the information collected will fall into one of the many categories: fiction, nonfiction, et cetera.

SELF CHECK **Review**

For 1-3, identify the population, the units, and each variable, and tell if the variable is categorical or quantitative.

1. A quality control worker with Sweet-Tooth Candy weighs every 100th candy bar to make sure it is very close to the published weight.
2. Doris decides to clean her sock drawer out and sorts her socks into piles by color.
3. A researcher is studying the effect of a new drug treatment for diabetes patients. She performs an experiment on 200 randomly chosen individuals with type II diabetes. Because she believes that men and women may respond differently, she records each person's gender, as well as the person's change in blood sugar level after taking the drug for a month.

For 4-6, indicate for each of the following characteristics of an individual whether the variable is categorical or quantitative (numerical):

4. Length of arm from elbow to shoulder (in inches)
5. Number of DVD's the person owns.
6. Feeling about own height (too tall, too short, about right)
7. In Physical Education class, the teacher has the students count off by two's to divide them into teams. Is this a categorical or quantitative variable?
8. A school is studying its students' test scores by grade. Explain how the characteristic 'grade' could be considered either a categorical or a numerical variable.
9. What are the best ways to display categorical and numerical data?
10. Is it possible for a variable to be considered both categorical and numerical?
11. How can you compare the effects of one categorical variable on another or one quantitative variable on another?

Part 3: Sampling Methods

Probability Sampling Methods

1. Simple random sampling

In this case each individual is chosen entirely by chance and each member of the population has an equal chance, or probability, of being selected. One way of obtaining a random sample is to give each individual in a population a number, and then use a table of random numbers to decide which individuals to include.[1] For example, if you have a sampling frame of 1000 individuals, labelled 0 to 999, use groups of three digits from the random number table to pick your sample. So, if the first three numbers from the random number table were 094, select the individual labelled "94", and so on.

As with all probability sampling methods, simple random sampling allows the sampling error to be calculated and reduces selection bias. A specific advantage is that it is the most straightforward method of probability sampling. A disadvantage of simple random sampling is that you may not select enough individuals with your characteristic of interest, especially if that characteristic is uncommon. It may also be difficult to define a complete sampling frame and inconvenient to contact them, especially if different forms of contact are required (email, phone, post) and your sample units are scattered over a wide geographical area.

2. Systematic sampling

Individuals are selected at regular intervals from the sampling frame. The intervals are chosen to ensure an adequate sample size. If you need a sample size n from a population of size x, you should select every x/n^{th} individual for the sample. For example, if you wanted a sample size of 100 from a population of 1000, select every 1000/100 = 10^{th} member of the sampling frame.

Systematic sampling is often more convenient than simple random sampling, and it is easy to administer. However, it may also lead to bias, for example if there are underlying patterns in the order of the individuals in the sampling frame, such that the sampling technique coincides with the periodicity of the underlying pattern. As a hypothetical example, if a group of students were being sampled to gain their opinions on college facilities, but the Student Record Department's central list of all students was arranged such that the sex of students alternated between male and female, choosing an even interval (e.g. every 20^{th}student) would result in a sample of all males or all females. Whilst in this example the bias is obvious and should be easily corrected, this may not always be the case.

3. Stratified Random sampling

In this method, the population is first divided into subgroups (or strata) who all share a similar characteristic. It is used when we might reasonably expect the measurement of interest to vary between the different subgroups, and we want to ensure representation from all the subgroups. For example, in a study of stroke outcomes, we may stratify the population by sex, to ensure equal representation of men and women. The study sample is then obtained by taking equal sample sizes from each stratum. In stratified random sampling, it may also be appropriate to choose non-equal sample sizes from each stratum. For example, in a study of the health outcomes of nursing staff in a county, if there are three hospitals each with different numbers of nursing staff (hospital A has 500 nurses, hospital B has 1000 and hospital C has 2000), then it would be appropriate to choose the sample numbers from each hospital *proportionally* (e.g. 10 from hospital A, 20 from hospital B and 40 from hospital C). This ensures a more realistic and accurate estimation of the health outcomes of nurses across the county, whereas simple random sampling would over-represent nurses from hospitals A and B. The fact that the sample was stratified should be taken into account at the analysis stage. Stratified random sampling improves the accuracy and representativeness of the results by reducing sampling bias. However, it requires knowledge of the appropriate characteristics of the sampling frame (the details of which are not always available), and it can be difficult to decide which characteristic(s) to stratify by.

4. Clustered sampling

In a clustered sample, subgroups of the population are used as the sampling unit, rather than individuals. The population is divided into subgroups, known as clusters, which are randomly selected to be included in the study.

Clusters are usually already defined, for example individual GP practices or towns could be identified as clusters. In single-stage cluster sampling, all members of the chosen clusters are then included in the study. In two-stage cluster sampling, a selection of individuals from each cluster is then randomly selected for inclusion. Clustering should be taken into account in the analysis. The General Household survey, which is undertaken annually in England, is a good example of a (one-stage) cluster sample. All members of the selected households (clusters) are included in the survey.[1]

Cluster sampling can be more efficient that simple random sampling, especially where a study takes place over a wide geographical region. For instance, it is easier to contact lots of individuals in a few GP practices than a few individuals in many different GP practices. Disadvantages include an increased risk of bias, if the chosen clusters are not representative of the population, resulting in an increased sampling error.

Non-Probability Sampling Methods

1. Convenience sampling

Convenience sampling is perhaps the easiest method of sampling, because participants are selected based on availability and willingness to take part. Useful results can be obtained, but the results are prone to significant bias, because those who volunteer to take part may be different from those who choose not to (volunteer bias), and the sample may not be representative of other characteristics, such as age or sex. Note: volunteer bias is a risk of all non-probability sampling methods.

2. Quota sampling

This method of sampling is often used by market researchers. Interviewers are given a quota of subjects of a specified type to attempt to recruit. For example, an interviewer might be told to go out and select 20 adult men, 20 adult women, 10 teenage girls and 10 teenage boys so that they could interview them about their television viewing. Ideally the quotas chosen would proportionally represent the characteristics of the underlying population.

Whilst this has the advantage of being relatively straightforward and potentially representative, the chosen sample may not be representative of other characteristics that weren't considered (a consequence of the non-random nature of sampling). [2]

3. Judgement (or Purposive) Sampling

Also known as selective, or subjective, sampling, this technique relies on the judgement of the researcher when choosing who to ask to participate. Researchers may implicitly thus choose a "representative" sample to suit their needs, or specifically approach individuals with certain characteristics. This approach is often used by the media when canvassing the public for opinions and in qualitative research.

Judgement sampling has the advantage of being time-and cost-effective to perform whilst resulting in a range of responses (particularly useful in qualitative research). However, in addition to volunteer bias, it is also prone to errors of judgement by the researcher and the findings, whilst being potentially broad, will not necessarily be representative.

4. Snowball sampling

This method is commonly used in social sciences when investigating hard-to-reach groups. Existing subjects are asked to nominate further subjects known to them, so the sample increases in size like a rolling snowball. For example, when carrying out a survey of risk behaviours amongst intravenous drug users, participants may be asked to nominate other users to be interviewed.

Snowball sampling can be effective when a sampling frame is difficult to identify. However, by selecting friends and acquaintances of subjects already investigated, there is a significant risk of selection bias (choosing a large number of people with similar characteristics or views to the initial individual identified).

Part 4: Bias in Sampling, Bias in Responding

Bias in Samples and Surveys

The term most frequently applied to a non-representative sample is bias. Bias has many potential sources. It is important when selecting a sample or designing a survey that a statistician make every effort to eliminate potential sources of bias. In this section, we will discuss some of the most common types of bias. While these concepts are universal, the terms used to define them here may be different than those used in other sources.

Sampling Bias: In general, sampling bias refers to the methods used in selecting the sample. The sampling frame is the term we use to refer to the group or listing from which the sample is to be chosen. If you wanted to study the population of students in your school, you could obtain a list of all the students from the office and choose students from the list. This list would be the sampling frame.

1. **Incorrect Sampling Frame** If the list from which you choose your sample does not accurately reflect the characteristics of the population, this is called incorrect sampling frame. A sampling frame error occurs when some group from the population does not have the opportunity to be represented in the sample.
   Recognizing an Incorrect Sampling Frame: Surveys that are often done over the telephone will have an incorrect sampling frame. You could use the telephone book as a sampling frame by choosing numbers from the telephone book. However, in addition to the many other potential problems with telephone polls, some phone numbers are not listed in the telephone book. Also, if your population includes all adults, it is possible that you are leaving out important groups of that population. For example, many younger adults in particular tend to use only their cell phones or computer-based phone services and may not even have traditional phone service. Even if you picked phone numbers randomly, the sampling frame could be incorrect, because there are also people, especially those who may be economically disadvantaged, who have no phone. There is absolutely no chance for these individuals to be represented in your sample. A term often used to describe the problems when a group of the population is not represented in a survey is **undercoverage**.
   Undercoverage can result from all of the different sampling biases.
   One of the most famous examples of sampling frame error occurred during the 1936 U.S. presidential election. The Literary Digest, a popular magazine at the time, conducted a poll and predicted that Alf Landon would win the election that, as it turned out, was won in a landslide by Franklin Delano Roosevelt. The magazine obtained a huge sample of ten million people, and from that pool, 2 million replied. With these numbers, you would typically expect very accurate results. However, the magazine used their subscription list as their sampling frame. During the depression, these individuals would have been only the wealthiest Americans, who tended to vote Republican, and left the majority of typical voters under-covered.
2. **Size Bias** If one particular subgroup in a population is likely to be over-represented or under-represented due to its size, this is sometimes called size bias. If we chose a state at random from a map by closing our eyes and pointing to a particular place, larger states would have a greater chance of being chosen than smaller ones. As another example, suppose that we wanted to do a survey to find out the typical size of a student's math class at a school. The chances are greater that we would choose someone from a larger class for our survey. To understand this, say that you went to a very small school where there are only four math classes, with one class having 35 students, and the other three classes having only 8 students. If you simply choose students at random, it is more likely you will select students for your sample who will say the typical size of a math class is 35, since there are more students in the larger class.
   Recognizing Size Bias: A person driving on an interstate highway tends to say things like, "Wow, I was going the speed limit, and everyone was just flying by me." The conclusion this person is making about the population of all drivers on this highway is that most of them are traveling faster than the speed limit. This may indeed be true, but let's say that most people on the highway, along with our driver, really are abiding by the speed limit. In a sense, the driver is collecting a sample, and only those few who are close to our driver will be included in the sample. There will be a larger number of drivers going faster in our sample, so they will be over-represented. As you may already see, these definitions are not absolute, and often in a practical example, there are many types of overlapping bias that could be present and contribute to overcoverage or undercoverage. We could also cite incorrect sampling frame or convenience bias as potential problems in this example.

Response Bias

The term response bias refers to problems that result from the ways in which the survey or poll is actually presented to the individuals in the sample.

1. **Voluntary Response Bias:** Television and radio stations often ask viewers/listeners to call in with opinions about a particular issue they are covering. The websites for these and other organizations also usually include some sort of online poll question of the day. Reality television shows and fan balloting in professional sports to choose all-star players make use of these types of polls as well. All of these polls usually come with a disclaimer stating that, "This is not a scientific poll." While perhaps entertaining, these types of polls are very susceptible to voluntary response bias. The people who respond to these types of surveys tend to feel very strongly one way or another about the issue in question, and the results might not reflect the overall population. Those who still have an opinion, but may not feel quite so passionately about the issue, may not be motivated to respond to the poll. This is especially true for phone-in or mail-in surveys in which there is a cost to participate. The effort or cost required tends to weed out much of the population in favor of those who hold extremely polarized views. A news channel might show a report about a child killed in a drive-by shooting and then ask for people to call in and answer a question about tougher criminal sentencing laws. They would most likely receive responses from people who were very moved by the emotional nature of the story and wanted anything to be done to improve the situation. An even bigger problem is present in those types of polls in which there is no control over how many times an individual may respond.

2. **Non-Response Bias:** One of the biggest problems in polling is that most people just don't want to be bothered taking the time to respond to a poll of any kind. They hang up on a telephone survey, put a mail-in survey in the recycling bin, or walk quickly past an interviewer on the street. We just don't know how much these individuals' beliefs and opinions reflect those of the general population, and, therefore, almost all surveys could be prone to non-response bias.
3. **Questionnaire Bias:** Questionnaire bias occurs when the way in which the question is asked influences the response given by the individual. It is possible to ask the same question in two different ways that would lead individuals with the same basic opinions to respond differently. Consider the following two questions about gun control.
   "Do you believe that it is reasonable for the government to impose some limits on purchases of certain types of weapons in an effort to reduce gun violence in urban areas?"
   "Do you believe that it is reasonable for the government to infringe on an individual's constitutional right to bear arms?"
   A gun rights activist might feel very strongly that the government should never be in the position of limiting guns in any way and would answer no to both questions. Someone who is very strongly against gun ownership, on the other hand, would probably answer yes to both questions. However, individuals with a more tempered, middle position on the issue might believe in an individual's right to own a gun under some circumstances, while still feeling that there is a need for regulation. These individuals would most likely answer these two questions differently.
   You can see how easy it would be to manipulate the wording of a question to obtain a certain response to a poll question. Questionnaire bias is not necessarily always a deliberate action. If a question is poorly worded, confusing, or just plain hard to understand, it could lead to non-representative results. When you ask people to choose between two options, it is even possible that the order in which you list the choices may influence their response!
4. **Incorrect Response Bias:** A major problem with surveys is that you can never be sure that the person is actually responding truthfully. When an individual intentionally responds to a survey with an untruthful answer, this is called incorrect response bias. This can occur when asking questions about extremely sensitive or personal issues. For example, a survey conducted about illegal drinking among teens might be prone to this type of bias. Even if guaranteed their responses are confidential, some

teenagers may not want to admit to engaging in such behavior at all. Others may want to appear more rebellious than they really are, but in either case, we cannot be sure of the truthfulness of the responses.

Recognizing this Bias: Because the dangers of donated blood being tainted with diseases carrying a negative social stereotype increased in the 1990's, the Red Cross has recently had to deal with incorrect response bias on a constant and especially urgent basis. Individuals who have engaged in behavior that puts them at risk for contracting AIDS or other diseases have the potential to pass these diseases on through donated blood4. Screening for at-risk behaviors involves asking many personal questions that some find awkward or insulting and may result in knowingly false answers. The Red Cross has gone to great lengths to devise a system with several opportunities for individuals giving blood to anonymously report the potential danger of their donation.

In using this example, we don't want to give the impression that the blood supply is unsafe. According to the Red Cross, "Like most medical procedures, blood transfusions have associated risk. In the more than fifteen years since March 1985, when the FDA first licensed a test to detect HIV antibodies in donated blood, the Centers for Disease Control and Prevention has reported only 41 cases of AIDS caused by transfusion of blood that tested negative for the AIDS virus. During this time, more than 216 million blood components were transfused in the United States. The tests to detect HIV were designed specifically to screen blood donors. These tests have been regularly upgraded since they were introduced. Although the tests to detect HIV and other blood-borne diseases are extremely accurate, they cannot detect the presence of the virus in the 'window period' of infection, the time before detectable antibodies or antigens are produced. That is why there is still a very slim chance of contracting HIV from blood that tests negative. Research continues to further reduce the very small risk." Source: The American Red Cross

Study Design We're Watching You Task

Name ____ Date_____________________

There are two approaches to collecting data in statistics – observational studies and experiments. In observational studies, researchers observe characteristics from samples of an existing population and use the information collected to make inferences about the population. In an observational study, the researcher gathers data without trying to influence responses or imposing any controls on the situation. In an experiment, researchers gather data by imposing a treatment and observing responses.

- There are several key steps involved in designing an observational study.

  1. Determine the focus of the study. What is the variable of interest? What information is needed to answer the main question of interest?
  2. Develop a plan to collect data. How will subjects be observed?
  3. Determine the most appropriate sampling method and select the sample.
  4. Collect the data.
  5. Describe and interpret the data using appropriate statistical procedures and graphs.
  6. Report the findings of the study.

The basic principles of experimental design are
1. **Randomization** – Experimental units/subjects should be randomly assigned to treatment groups;
2. **Control** - Experimenters need to control any lurking variables, generally by comparing multiple treatment groups;
3. **Replication** – The experiment should involve many experimental units/subjects.

Use the information above as well as what you have learned in class to explore the following situations.
1. A local community has just installed red light cameras at its busiest intersection. The police department hopes that the cameras will encourage drivers to be more careful and that incidents of drivers running red lights at this intersection will decrease. Design an observational study that the police department could use to determine if the installation of the traffic light has had the deserved effect.

a. What is the focus of the study? What is the variable of interest?	**b.** Determine the data collection plan.
c. Funds are limited and there are only a few days to conduct the study. What is the most appropriate sampling method?	**d.** The police chief also wonders if there is a difference in driver behavior at different times of day. Incorporate this concern into your sampling method.

2. A few years ago, a study was conducted at Johns Hopkins hospital in Boston to see if exposure to ultrasound could affect the birth weight of a baby. Investigators followed unborn babies and their mothers until their birth and notes their

birth weight. A comparison was made between the birth weight of those babies exposed to ultrasound and those babies not exposed to ultrasound. Whether an ultrasound was used on the baby was a decision made by the mother's doctor, based on medical justification. Was this study an experiment or an observational study? Explain. List any possible confounding variables in this study. *(source: Chris Franklin, University of Georgia)*

3. Suppose the faculty of a Statistics department at a large university wanted to look at how students in the introductory Statistics courses might perform on exams under different environmental conditions. They decided to consider the effect of the size of the classroom (a smaller classroom where there are just enough seats for the students versus a large classroom where the students can spread out with an empty seat between each student). When the next exam is given in one section of the introductory Statistics course, 60 students will be randomly assigned to one of the treatments. The scores on the exam will then be compared. *(source: Chris Franklin, University of Georgia)*

a. Is this study an experiment or observational study? Explain.

b. Name the explanatory variable.

c. Name the response variable.

d. How many treatments will this study compare? Name the treatments

e. Diagram a completely randomized design for this study.

4. You want know if talking on a hands-free cell phone distracts drivers. Forty college students “drove” in a simulator equipped with a hands free cell phone. The car ahead brakes: how quickly does the subject respond?

a. What are the experimental units?

b. What is the explanatory variable?

c. What are the treatments?

d. What is the response variable?

e. Outline the design of the *above experiment.*

5. You want to determine the best color for attracting cereal leaf beetles to boards on which they will be trapped. You will compare four colors: blue, green, white and yellow. You plan to count the number of beetles trapped. You will mount one board on each of 16 poles evenly spaced in a square field.

a. What are the experimental units?

b. What is the explanatory variable?

c. What are the treatments?

d. What is the response variable?

e. Outline the design of the above experiment.

STUDY DESIGN And You Believed That?! Learning Task

Name ____ Date_____________________

A solid knowledge of statistical procedures will help you be an educated consumer of information. Every day, we are confronted by a news report citing a new relationship researchers have discovered, a claim being made in advertising, or other data-driven statement. Beginning with this task, we will examine the questions you should keep in mind when analyzing such claims.

1. Read the article below.

Facebook use linked to less textbook time

By Mary Beth Marklein, USA TODAY (April 13, 2009)

Does Facebook lead to lower grades? Or do college students with lower grades use Facebook more than their higher-achieving peers?

A study of 219 students at Ohio State University being presented at a conference this week doesn't answer those questions definitively. But it suggests a link between the social networking site and academic performance.

Students who said they used Facebook reported grade-point averages between 3.0 and 3.5; those who don't use it said they average 3.5 to 4.0. Also, Facebook users said they studied one to five hours a week, vs. non-users' 11 hours or more.

Ohio State doctoral student Aryn Karpinski, who conducted the research with graduate student Adam Duberstein, says the study is too narrow to conclude that Facebook and academics don't mix.

"It cannot be stated (that) Facebook use causes a student to study less or get lower grades," she says. "I'm just saying that they're related somehow, and we need to look into it further." Of the 68% of students who said they used Facebook, 65% accessed the site daily or multiple times daily.

Karpinski says 79% of Facebook users believe it has no impact on their academics; some say it helps them form study groups.

She says faculty ought to consider harnessing it as a learning tool. Yet a preliminary peek at a second survey suggests "a lot of faculty … didn't even know what Facebook is," she says.

1. What claim is being made? What is special about the claim?

Bias in Research and Studies

The first question raised when evaluating the believability of a claim is whether or not the questions and procedures were designed in such a way as to eliminate bias. It is critical for statisticians and researchers to avoid leading questions and questions that are vague or contain confusing wording.
For example, asking someone each of the following questions may illicit different responses even though all three questions address the same topic.

- "Is it really possible for a person to still believe that wearing a seat belt is not completely necessary?"
- "Is wearing a seat belt necessary for the complete safety of all passengers?"
- "Wearing a seat belt is currently required by state law. Do you agree with this law?"

2. How would you answer each of these questions? Did the wording of the questions influence your responses?

3. Refer to the article at the beginning of the task. Write two unbiased questions related to the article that researchers might have asked the subjects of the study.

Another possible source of bias in studies is in the *sampling* technique. A sample is a subgroup of the population. It is important that researchers use unbiased samples. In order to have an unbiased sample, the sample must be selected at random. There are many types of random samples. The most common are the following.

- A *simple random sample*, in which every possible sample of the same size has the same chance of being selected. This can be accomplished by assigning every member of the population a distinct number and then using a random number generator or table to select members of the sample.
- A *systematic sample*, in which every member of population is assigned a number or put in order and then members of the sample are selected at set intervals, for example every tenth member is selected for the sample.
- A *stratified random sample*, in which members of the population are grouped by a specific characteristic and then members from each group, or strata, are selected using a simple random sample procedure.
- A *cluster sample*, in which the researcher identifies pre-existing groups, or clusters, within the population and then randomly selects a set numbers of these clusters as the sample. In this case, every member of the selected cluster is a part of the sample.

There are also sampling methods that create bias in the study. Types of these methods are the following.

- *convenience sampling* – choosing individuals who are easiest to reach (asking the first ten people who walk by)
- *voluntary response sampling*- consists of people who choose themselves by responding to a general appeal (asking radio listeners to call in to share responses or vote on a particular issue or asking subjects to return a survey by mail or email).

Both convenience sampling and voluntary response lack the critical element of randomization.

4. Determine whether each study below has a source of bias. If there is a source of bias, describe the bias and why this bias makes the sample unrepresentative.

a) A medical company uses sick patients to test their competitors' drugs for side effects.

b) A medical company uses healthy patients to test their competitors' drugs for side effects.

c) A newspaper polls 9th grade students to determine if students are going into the Armed Forces after high school.

d) The Department of Education conducts and online poll that asks "Do you have internet service at home?"

e) A survey is mailed to voters in Houston County asking "Will you vote for the one cent sales tax increase in Houston County?"

f) A survey is mailed to voters in Warner Robins who make more than $150,000 a year asking "Will you vote for the one cent sales tax increase in Warner Robins?"

5. For each experiment, determine which sampling technique would be most appropriate. Then explain how you would obtain a sample and why the technique you chose was appropriate.

a) A company wants to decide who likes wheat bread more, men or women.

b) You want to estimate the number of people in your school who are vegetarian.

c) You want to determine whether 9th graders or 12th graders are more likely to be vegetarian.

d) The Department of Children Services wants to count the number of homeless children in a city. They only have enough counters to cover one-sixth of the city.

e) A manufacturer wants to test the taste of their frozen vegetables as the bags of vegetables come out of a freezer.

6. Referring to the article "Facebook use linked to less textbook time," what type of sampling technique do you believe the researchers may have used? Why?

7. Consider the student body of your high school to be the population for a study being conducted by the school newspaper. One of the newspaper students, Emma, is writing an article on study habits. She has carefully designed a survey of five questions. Is it reasonable to think that she can survey the entire student body? Why or why not?

8. Emma has decided to survey 50 students. She is trying to decide which type of random sample will be the most appropriate and easiest for her to complete successfully. For the following, explain how she could select each type of sample from the students at your school.

Simple Random:

Systematic:

Stratified Random:

Cluster:

If you were gathering information for this article, which of these samples would you use? Explain.

9. Another student on the staff thinks that Emma is making the assignment too difficult and suggested that she simply survey the students in her first period class. Would this be an appropriate sampling method? Explain.

10. Emma's friend, Marcus, is on the yearbook staff and is currently involved in the staff's effort to design this year's cover. The group wants to create a cover design that depicts the "typical" student from the school. In order to determine the typical student, they have decided to design a ten question survey focusing on physical characteristics, classes and extracurricular activities. The problem is that the students are having difficulty writing unbiased questions to gather the data they need. They have agreed that four questions should address physical characteristics, three should address classes taken, and two should address extracurricular activities. Pretend you are a member of this yearbook staff and write ten unbiased questions for this survey.

HOMEWORK

Statistical Studies & Definitions
1 Define. POPULATION: Define Parameter, show its Greek variables, and describe how it relates to population. 2. Define SAMPLE: Define Statistic, show its English variables, and describe how it relates to Sample. 3. List reasons why you might use a SAMPLE study instead of a POPULATION study? 4. A recent survey by the alumni of a major university indicated that the average salary of 8,500 of its 250,000 graduates was $123,000. Does this value describe a parameter or a statistic? WHY? 5. A survey of 976 American households found that 32% of the households own two cars. Identify the population and the sample. SAMPLE: POULATION:

For # 6– 8 Identify each of the following data sets as either: (P) Population or (S) Sample

_______6. ….the age of a few randomly selected participants in a study about a race of runners

_______7. ….the annual salary of each full-time teacher in a study about Perry High School

_______8. ….a survey of 750 Georgia homeowners in a study about all of Georgia's homeowners.

For # 9 – 11 Identify each of the following numerical values as either: (P) Parameter or (S) Statistic

_______9. …. of a company's employees the opinion of just those that were there on time one morning about what they thought of a new training program.

_______10. …. in a study about a small company of 25 employees, the range of their employee's salaries

_______11. …. in a study about the value of American homes in 2012, the average decrease of all the homes sold in Houston.

TYPES OF SAMPLES – Use your notes, and define:

12. Random sample:

13. Stratified sample:

14. Cluster sample:

15. Systematic sample:

16. Convenience sample:

Choose which sampling technique is used.

(R) Random (STR) Stratified (CLS) Cluster (CON) Convenience (SYS) Systematic

_______ 17. There are 250 seventh graders and 300 eighth graders at Generic Middle School. We ask 45 seventh graders and 50 eighth graders how many siblings they have to compare the two groups.

_______ 18. I ask all freshmen, no sophomores, no juniors, and all seniors if they prefer Vanilla or Cherry Coke (these four groups are my only four groups) to create a study of what should be in the vending machines.

_______ 19. I ask everyone in my 5th period class who has more than one computer at home in a study about all of my students for the year.

_______ 20. I collect data from every 15th student on my list of the entire school population.

_______ 21. After using a random number table to generate two-digit numbers, I decide on 10 people to choose from the population.

22. Rank the sampling types in order from what would usually be the WORST to BEST representation of a POPULATION. Provide brief explanations (especially if the ranking depends on the study).

(R) Random (STR) Stratified (CLS) Cluster (CON) Convenience (SYS) Systematic

Define the following TYPES of STUDIES and DATA COLLECTION METHODS

1. Observational:

2. Experimental:

   Treatment Group:

   Control Group & Placebo:

3. Simulations:

4. Census:

5. Sampling:

Choose the type of Study that is most likely to be used (each is used just once).

(E) Experimental (SIM) Simulation (C) Census (SMP) Sampling (O) Observational

________ 6. You want to know how many pets the teachers at Phoenix High School own.

________ 7. A drug is given to 15 patients and a placebo to another group to determine its effect on an illness.

________ 8. You are doing a study at a mall in which you are counting the number of men that wash their hands after using the restroom.

________ 9. You want to know the g-forces a person would experience during a fall from a 90 foot high bridge into a lake.

________ 10. You need data on the average number of hours worked per week by an American teenager with a part-time job.

11. Define Data Types.

a. Qualitative:

b. Quantitative:

For numbers 12 - 20 choose (QL) Qualitative or (QN) Quantitative

____12. The colors of automobiles on a used car lot.

____13. The number of seats in a movie theater

____14. Numbers on shirts of a girls' soccer team.

____15. Ages of the students at North High School.

____16. The temperatures of 30 refrigerators.

______17. The amount of fat grams of 24 different cookies.

____18. The years the Olympics were held in the United States.

____19. Marriage status (married, single, divorced).

____20. Social Security Numbers of the employees of a school.

11. Bias:

- Sampling Bias:

Which would most likely be the best representative sample and which would be the worst sample to use in determining the voting preference for the next president of the U.S. in the city of Warner Robins?

A. A reporter asks everyone in front of the court house who they plan on voting for and keeps a record.

B. An analyst gets a spreadsheet list from public records of a telephone number of each resident of the city and has the computer randomly sort the list and calls the first 100 residents to ask their preference.

C. A surveyor leaves a survey at the front of all of the restaurants in the city to ask customers their preference.

D. A surveyor asks all of the students at the local middle school their preference.

- Non Response Bias:

Explain why looking on the internet at reviews of a product may suffer from a Non-Response Bias.

- Response Bias:

Are there any concerns of Response Bias in the following survey questions?

- What is wrong with your current school?
- To improve education, should taxes be raised to fund building more schools?
- Why are teen-age drivers dangerous?
- How long does it take you to get to school?

A2.U7.C1.**B.01.vocab.ReviewOfCenterSpread**

Term	Definition	Notation	Diagram/Visual
Measures of Central Tendency			
Mean			
Median			
Mode			
Range			
Inference			
Line Graph			
Scatterplot			

A2.U7.C1.**B.01.vocab.ReviewOfCenterSpread**

Term	Definition	Notation	Diagram/Visual
Dot Plot			
Circle (Pie) Graph			
Bar Graph			
Histogram			
Stem-and-Leaf Plot			
Box-and-Whisker Plot			
Positive/Negative Correlation			
Outlier			

A2.U7.C1.**B.01.vocab.ReviewOfCenterSpread**

Term	Definition	Notation	Diagram/Visual
Quartile			
Q1,Q2,Q3			
IQR			

Measures of Central Tendency – Mean, Median, Mode, and Range

1) Have you ever been to an amusement park? Take a look at this dilemma.
An amusement park is designing a new section for children over 3 years old and under 8 years old. As part of their research, they used a survey of the heights and weights of a thousand children in that age group. Which measure of central tendency should they use to accommodate the greatest number of children on a roller coaster?

In the real world, there are many situations in which a large group of data is collected. In order to make sense of the data, we use a number of statistical measures. These measures help us to generalize a group of data, make inferences about it, and compare it with other groups of data.

Statistical measures include mean, median, mode and range. Depending on the situation, certain measures may be more helpful than others in interpreting data.

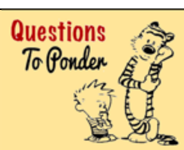
Let's look at these statistical measures.
The mean, median, and mode are three common measures of central tendency; they are three mathematical tools frequently used to analyze data.
The mean, commonly referred to as the average, is the sum of all the data items divided by the number of data items. The median is the middle number in a set of data that is ordered from lowest to highest. If there is an even number of data, we take the average of the middle two numbers to find the median. Finally, the mode is the number that occurs most often.

2) A manager at a small movie theater was analyzing the number of people who came to the movies during the week. Over nine days, he found the following data: 81, 89, 92, 85, 93, 62, 85, 105, and 90. Find the mean, median, and mode of the data.

a. First, let's find the mean. Remember that the mean is the same as the average.
Mean: add all of the data items and divide by the number of items.

b. Next, let's find the median.
Median: the middle number when the data is ordered from lowest to highest.
First reorder the data from least to greatest, then show the median.

c. Finally, let's find the mode.
The mode is the number that occurs most often.



3) Now let's go back to the dilemma from the beginning of the Concept. Which measure of central tendency should they use to accommodate the greatest number of children on a roller coaster?

Using Data Displays

Have you ever had a challenge trying on clothes? Well, take a look at this dilemma.
"I can't believe it!" Jacob exclaimed trying on his new long sleeved team shirt for the track team.
"What's the matter?" his friend Mattias asked.
"This shirt doesn't fit and this always happens to me. I am going to figure out why!" Jacob said taking off the shirt where the sleeves were too short once again.

After Jacob's anger had subsided, he started to think about this question. Was he the only one with this problem? Jacob decided to find out by measuring his peers' heights and arm lengths. He used inches and create a table like this:

Height (in)	52	53	56	58	59	62	64	65	66	67	68	69	70
Arm Length (in)	22	23	24	23	24	27	25	25	25.5	26	27	27	28

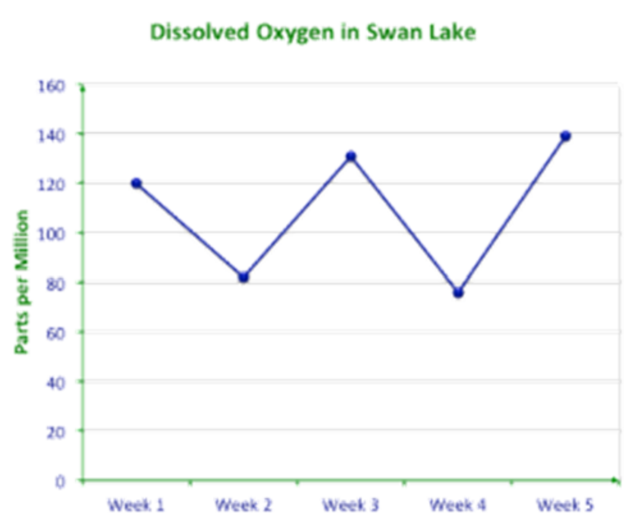
4) Now that Jacob has his data done, he needs to create a display. Which one should he create?

Think about this question throughout this Concept and in the end, you will help Jacob create the appropriate display for his data.

Data can exist in many forms. A frequent goal of collecting data is drawing conclusions based on the data. The best conclusions correspond with trends that the data shows. Depending on the data you have, certain types of displays are more appropriate or more effective than others. We must make good choices of displaying data in a logical way. Of course, in a world so full of data, it must be collected and organized carefully to aid in appropriate decision-making. Sometimes, two people look at the same graph and draw completely different conclusions. Graphs can show us many things but the conclusions that we draw based upon the graphs is oftentimes more a matter of opinion. The idea of graphs is, in part, to make inferences. Those inferences must be based on the data.

SELF CHECK

5) Take a look at this situation. Some scientists from the EPA were studying the amount of dissolved oxygen in a lake over several weeks. This graph was created by the data they found, and they came up with some conclusions below. Do you agree with their conclusions?



They studied the graph and came up with the following conclusions:

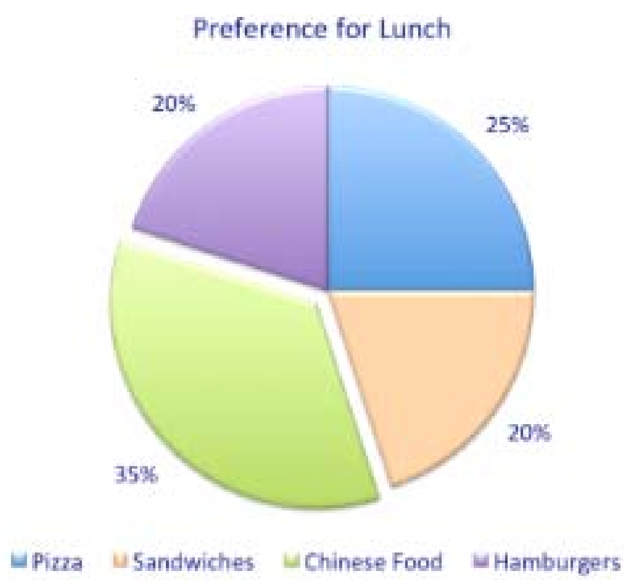
a. The amount of dissolved oxygen fluctuated over the 5 weeks.

b. The average amount of dissolved oxygen has been about 110 parts per million over 5 weeks.

c. The dissolved oxygen in Week 6 will be about 60 parts per million.

SELF CHECK

6) The boss at an office took a survey of people's preference for lunch because he wanted to treat the office to a lunch for the holidays. His data is shown below. Do you agree with his conclusions? Explain.



He ponders the following conclusions:

a. A lot of people like Chinese food.

b. Nobody likes Italian food.

c. If I order sandwiches, then 80% of the staff will be unhappy.

d. If I get some pizzas and some Chinese food, the majority will have their preference.

Using Data Displays

There are many ways to display data so how do you know which is the best way to display given data? Some choices are simply preferential but most types of data have types of displays that suit them best.

Types of Data

Two major types of data are categorical data and numerical data. Categorical data refers to data to which the independent variable is assigned a name, not a number. For example, you may take data based on the months May, June, July, and August or you may tally people based on males and females. Sometimes categories can be numbers that are used to name the categories. For example, players on a team are given numbers on their shirts. Those numbers are only used to clarify who is who. It would not make sense to use mathematical operations with the numbers. Generally, categorical data is simply tallied.

The second type of data is numerical. **Numerical data measures some characteristic of the variable. Examples of data that is measured numerically are time, height, weight, length, volume, density, force, etc. Anything that can be measured with a numerical system is numerical data.**

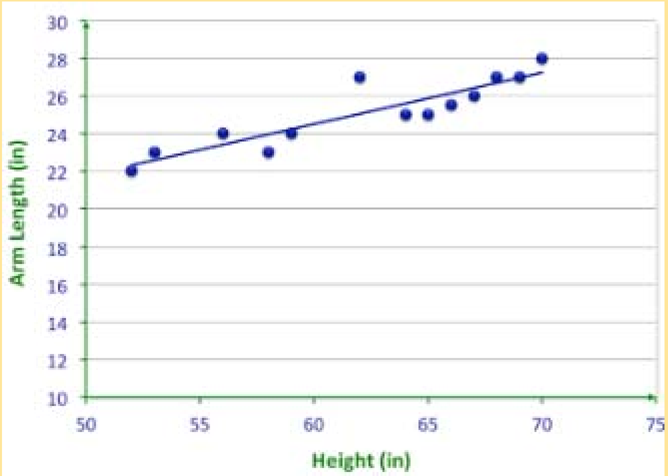
Types of Displays

We can use different data displays depending on the data. We can use line graphs, scatterplots, circle graphs, bar graphs, stem-and-leaf plots, box-and whisker plots, dot plots, and histograms.

- **Line graphs** are generally used to show change over time.
- **Scatterplots** are used to show a trend or a relationship (correlation) between to variables.
- **Dot Plots** are used to display the distribution of numerical values using dots, similar to a histogram.
- **Circle graphs** are best to show data that represents one whole or one hundred percent of something.
- **Bar graphs** are excellent for categorical data.
- **Histograms** are like bar graphs, but are used to display numerical data into numerical intervals, or bins.
- **Stem-and-leaf plots** are useful to represent ranges and can be used to illustrate ranges of two variables.
- **Box-and-whisker plots** are used to show how spread out data is and where the bulk of the data lies.

Questions To Ponder

7. Now let's go back to the dilemma from the beginning of the Concept.
By using a scatterplot, Jacob can compare the two variables, which are both numerical data, at once to see if there is a relationship. Graph the results. What conclusion can you draw from Jacob's measurements compared with the scatterplot below?



SELF CHECK

8. Answer each question about different data displays.

Example A

Which data display is best for categorical data?

Example B

Which data display is best for showing how data changes over time?

Example C

If I had data that was in the 10's, 20's, 40's, 50's and 60's, which display would be best?

A. Solution: Bar graph; B. Solution: Line graph; C. Solution: Scatterplot

DATA DISPLAYS – BOX AND WHISKER

GUIDED NOTES

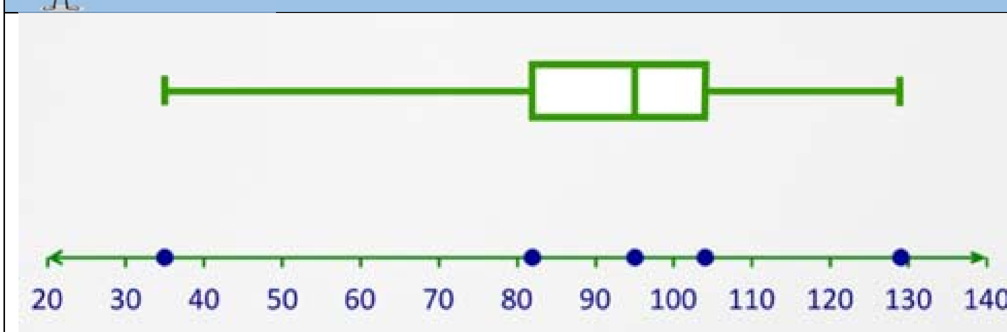
At times, it is useful to get a general idea of how data cluster together. You may recall from middle school that **box-and-whisker plots** display the distribution of data items along a number line. The data are divided into four equal parts, separated by points called **quartiles.** You can also see the smallest data point, the **extreme minimum**, and the largest data point, the **extreme maximum.**

A box-and-whisker plot is created by determining five points.

Step 1: place the data in order from smallest to largest.
Step 2: create a number line that shows the range of the data using equal intervals. Use the median as our middle point on the box-and-whisker plot and to split the data in half.
Step 3: Calculate the median of each half (the quartile). These separate the data into quarters.
Step 4: Use the highest datum and the lowest datum as endpoints, or extremes.
Step 5: Boxes are drawn between the quartiles and whiskers are drawn to the extremes.

The **interquartile range (IQR)** is the range between the first quartile and the third quartile. This shows you where the middle half of the data is. It can be calculated by subtracting the first quartile ($Q1$) from the third quartile ($Q3$)…mathematically, that would be $Q1 - Q3$.
Finally, the **outliers**, data items that are far away from the general trend, can be located as extremes that cause the whiskers to be exceptionally long. Data does not always have outliers. If there isn't a single point that is exceptionally far from other points, than an outlier doesn't exist. **Statistically, a data point is considered an outlier when it is more than 1.5* the IQR to the left of Q1 or to the right of Q3.**

Example! 1) Use the given box-and-whisker plot to identify the a) extremes, b) the median, c) the quartiles, d) the interquartile range, and e) the outliers (if any).



a) The extremes in this data set are approximately ____________________.
b) The median is approximately______________________.
c) The first quartile is approximately _______________and the third quartile approximately______________.
d) The interquartile range, then, is____________________.
e) Finally, the extreme minimum, _______________, does/does not *(circle one)* appear to be an outlier because____
____

As you know, outliers are points that are unusually large or small compared to the rest of the data. When we discuss measures of central tendency like mean, median, and mode, we must also remember that in the real world there are many exceptions. Sometimes when we consider data, we might choose to remove the outliers in order to draw better conclusions based on the data.

a) 35 and 129; b) 95; c) 82, 104; d) 104 – 82 or 22; e) 35, does, 1.5*the IQR is 33 units. Q1 – 33 = 82 – 33 = 49.
Anything lower than 49 is considered an outlier, so the minimum is definitely an outlier.

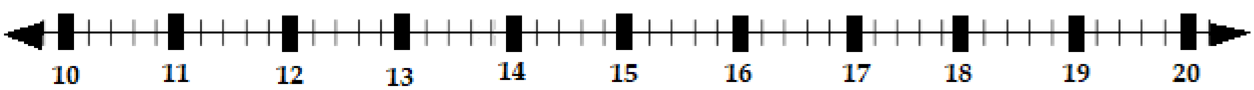
SELF CHECK

Shanda runs on her school's track team. They recently ran a 100 meter dash at a track meet and recorded official times. These are the results in seconds: 11.7, 10.8, 11.1, 10.9, 11.7, 11.6, 12.0, 19.6, 12.2, 11.6, 11.5, 11.6, 11.0, 12.0, 11.6, 11.5, 11.7, 11.3, 12.3, 10.1.

Shanda's time was 11.1 and she wants to know how she compares to the rest of her team. Use a box-and-whisker plot to help her figure this out.

2) Place the data in order below.

3) Identify the upper and lower extremes: Upper__________Lower____________.
4) Find the median: __________________.
5) Find the first and third quartiles: Q1_______ and Q3:_____________.
6) Draw the box-and-whisker plot that summarizes this data below:

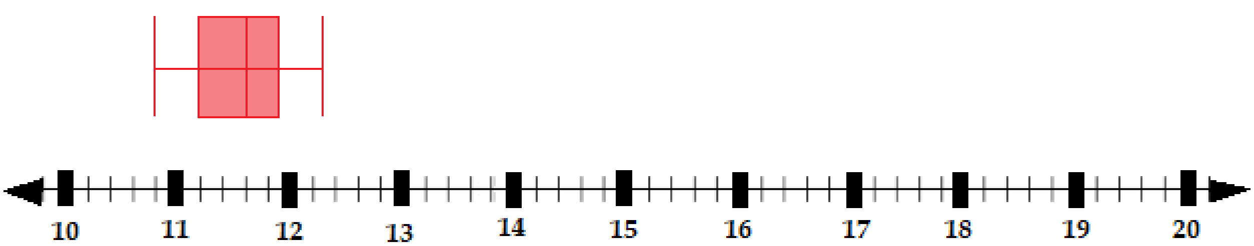


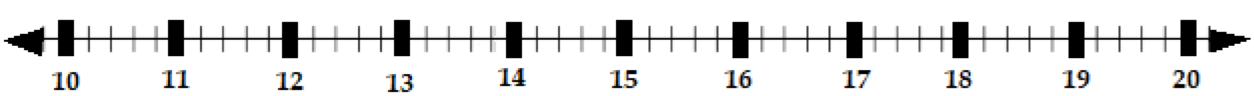
Questions To Ponder

When Shanda analyzes the box-and-whisker plot above, she finds that her time, 11.1 seconds, is barely less than the first quartile (remember in races you want the *lowest* time!). She knows that her friend, Teresa, is super fast. Shanda doesn't think she can realistically catch up to Teresa. Another teammate, Lisa, had fallen during the race, but got up and continued to the finish line. Shanda believes that neither Teresa nor Lisa's scores are useful in gauging her speed.

7) How might Shanda look at the data so that the results are not influenced by Teresa and Lisa's scores, which don't really help Shanda make conclusions about her own speed?

8) Recalculate Shanda's statistical measures without outliers, and create a new box and whisker plot below.





9) What happens to the data grouping when the outliers are removed? What does this mean for Shanda and her times?

SELF CHECK 9. Answer each question about different data displays.

Directions: Answer each question about data displays.

a. *What is considered numerical data?*

b. *What is considered categorical data?*

c. *If you were looking for a relationship between two values, would you use a scatterplot or a line graph?*

d. *If there was a relationship between the data would you have a positive correlation or a negative correlation?*

e. *The words positive correlation and negative correlation are associated with which type of data display?*

f. *If you had an outlier, then would you have a scatterplot or a box-and-whisker plot?*

g. *What is a quartile?*

h. *Which type of data display is a quartile associated with?*

i. *If you were watching a trend over time would you use a line graph or a scatterplot?*

j. *If you were comparing two trends and their results which data display would make the most sense?*

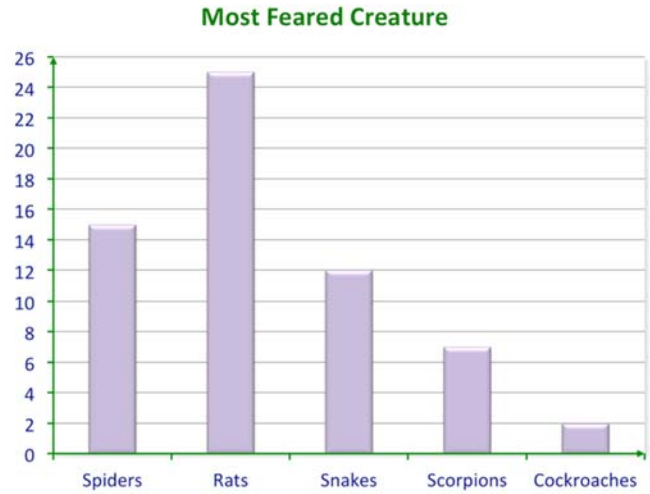
k. *What is the mean?*

l. *What is the median?*

m. *What is the mode?*

Directions: Answer each question.

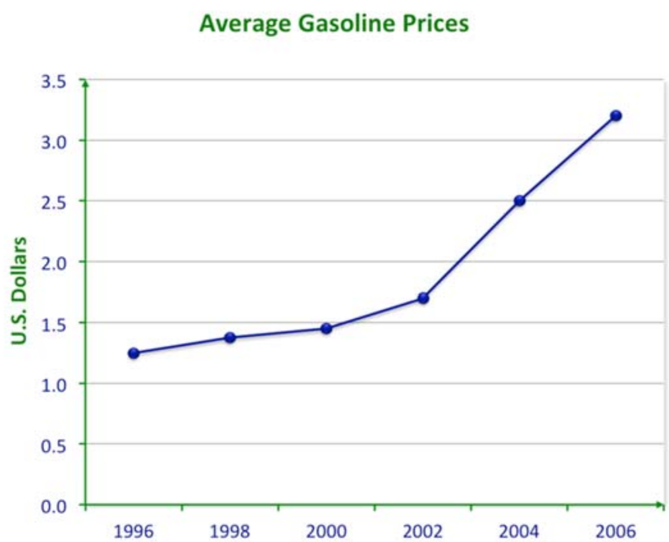
Do the conclusions fit the graph? Explain your reasoning.



n.

Conclusion 1: Rats are the most feared creature.

Conclusion 2: Rats are the most dangerous creature. Conclusion 3: Nobody is afraid of bats.



o.

Conclusion 1: Prices have increased every year for 10 years.

Conclusion 2: Prices of gasoline increased more rapidly after 2000. Conclusion 3: Prices will be even higher in 2008.

Explore More

Directions: Find the mean, median, mode and range by hand. Round all answers to the nearest tenths place. Notice that each answer has four answers.

13,18,24,21,16,24,14,17,24
1. Mean

2. Median

3. Mode

4. Range

116,137,120,75,98,98,137,139,139
5. Mean

6. Median

7. Mode

8. Range

22,24,25,30,32,34,37,22,22,38,40
9. Mean

10. Median

11. Mode

12. Range

123,150,163,150,163,150,180,200,201
13. Mean

14. Median

15. Mode

16. Range

Review of Center, Spread, and Data Displays

desmos

Students – go here, log in, and start the data displays task on Desmos.com.

http://tiny.cc/desmosstudentdata

Real Mathematics: Application

The Story…

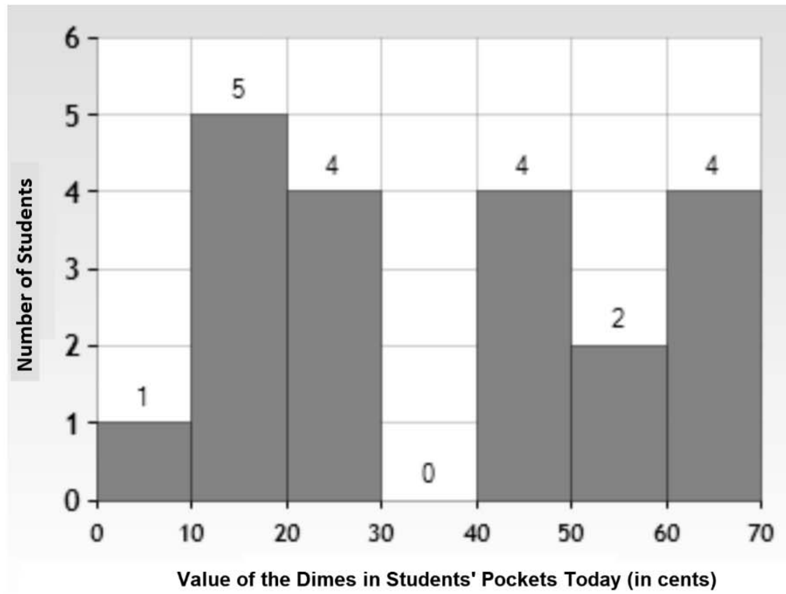
One day at school, a whole cache of DIMES was found in the ceiling!

It turns out, a clever rat named Rudy that had been living up there for years, had secretly stolen dimes from lockers, from under the coke machines, and from behind the bleachers in the gym. Strangely, the rat only liked dimes, not pennies, quarters, nickels, or paper bills. Since the rat had been discovered a month ago and was rehomed to a nice farm, his savings account was split between the different Algebra classes, and students got to pocket some of his dimes. Free money!

Students from each Algebra class were each polled to find out how much money, in dimes, they had been gifted by Rudy the Rat. Then they made different univariate data displays to represent their classes, but the displays got all mixed up together, and it's not clear which data display goes with which class anymore.

Your teacher will give you the mixed-up data displays, and your job is to match the correct ones together as a group, and explain how you know which data displays belong to the same class.

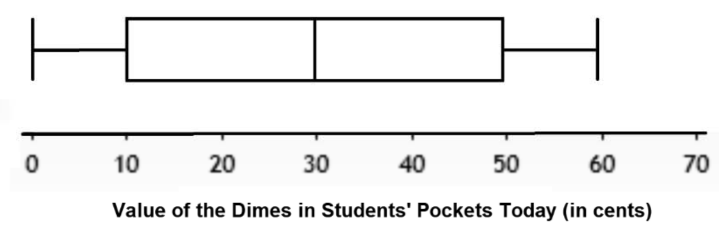
I



B

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	1
10	5
20	4
30	0
40	4
50	2
60	4

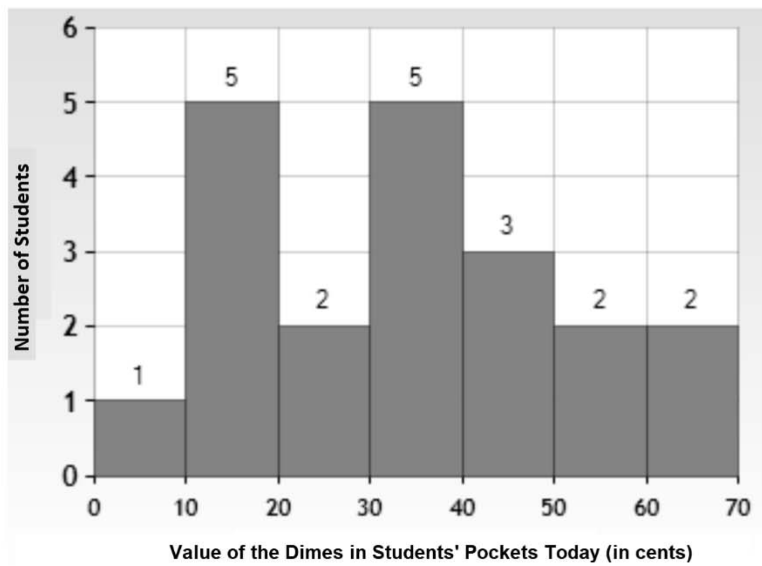
K



Y

MEAN:	31.5	RANGE:	60
MEDIAN:	30	M.A.D.:	18.5
MODE:	10	Q1:	10
MAX:	60	Q3:	50
MIN:	0	IQR:	40

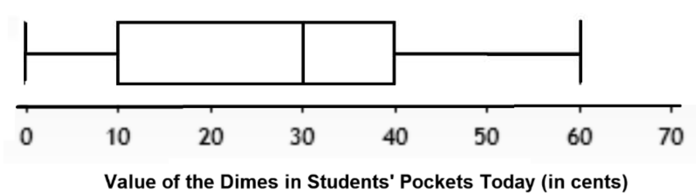
II



Y

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	1
10	5
20	2
30	5
40	3
50	2
60	2

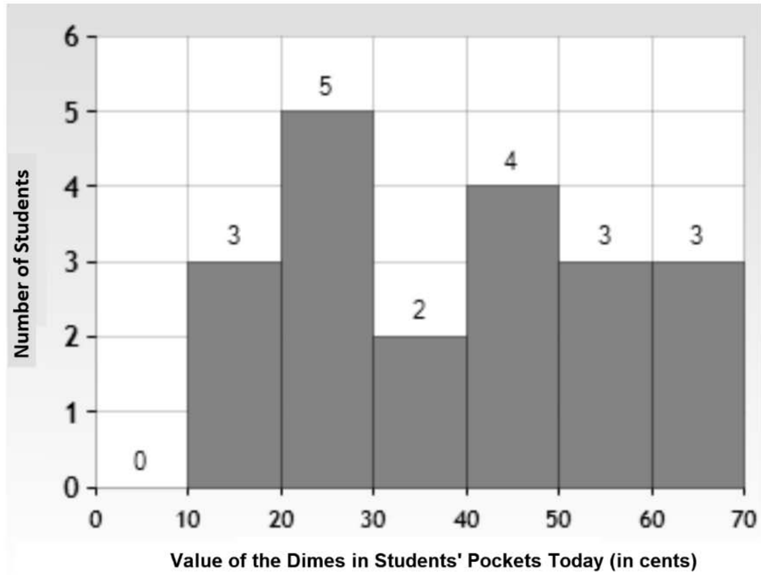
L



H

MEAN:	29	RANGE:	60
MEDIAN:	30	M.A.D.:	14.2
MODE:	10	Q1:	10
MAX:	60	Q3:	40
MIN:	0	IQR:	30

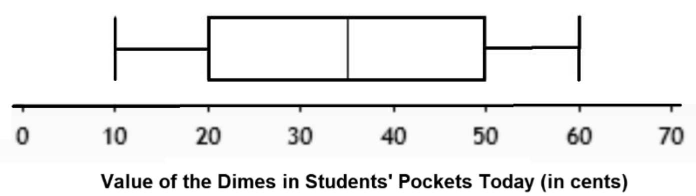
III



M

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	0
10	3
20	5
30	2
40	4
50	3
60	3

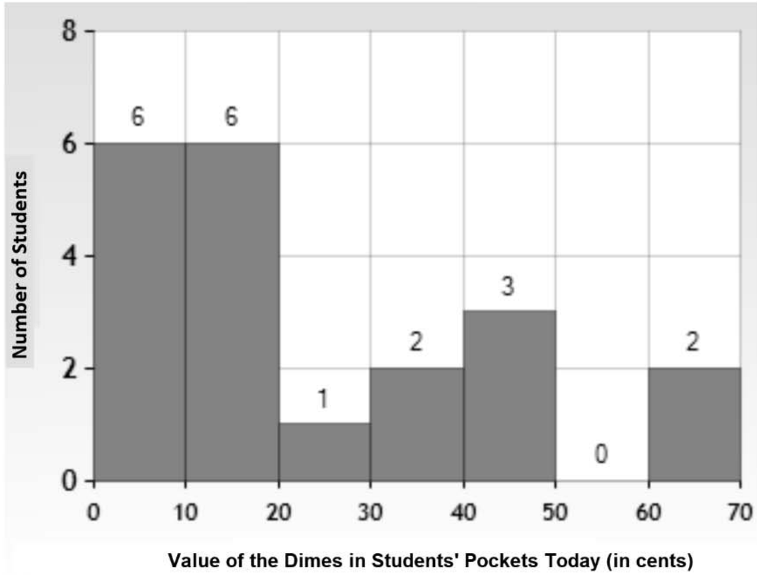
W



J

MEAN:	34	RANGE:	50
MEDIAN:	35	M.A.D.:	15
MODE:	20	Q1:	20
MAX:	60	Q3:	50
MIN:	10	IQR:	30

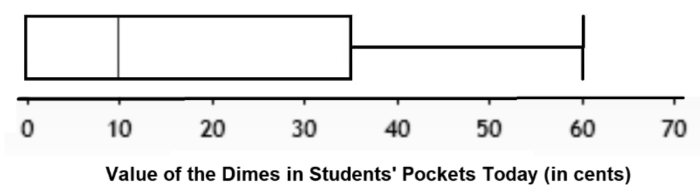
IV



G

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	6
10	6
20	1
30	2
40	3
50	0
60	2

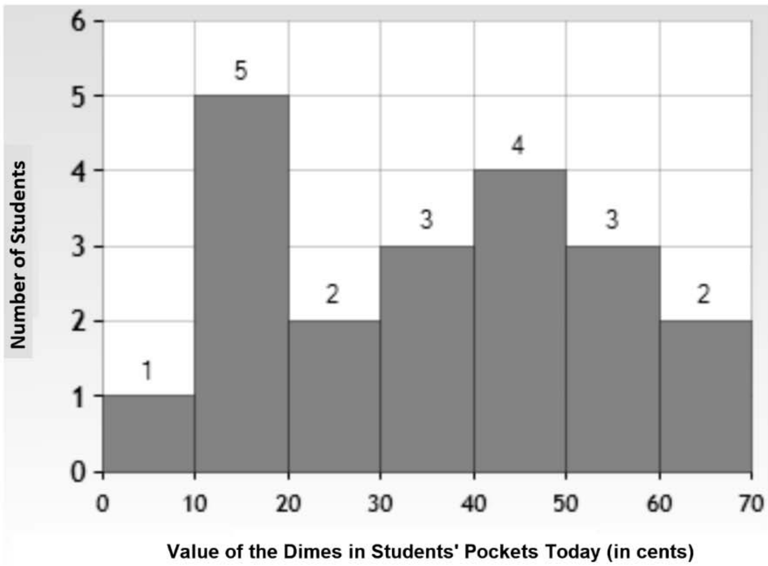
T



U

MEAN:	19	RANGE:	60
MEDIAN:	10	M.A.D.:	16.8
MODE:	0	Q1:	0
MAX:	60	Q3:	35
MIN:	0	IQR:	35

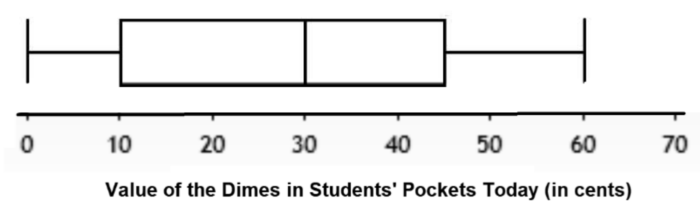
V



C

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	1
10	5
20	2
30	3
40	4
50	3
60	2

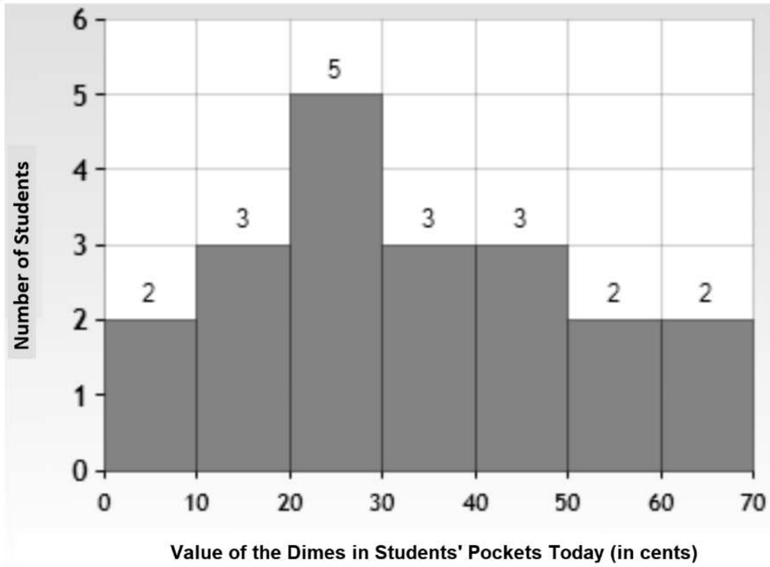
O



D

MEAN:	30.5	RANGE:	60
MEDIAN:	30	M.A.D.:	15.55
MODE:	10	Q1:	10
MAX:	60	Q3:	45
MIN:	0	IQR:	35

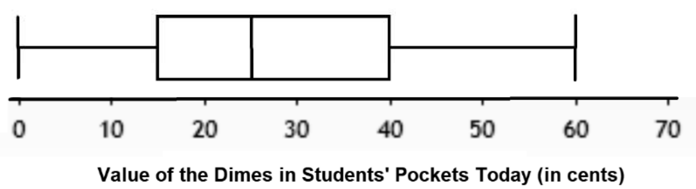
VI



A

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	2
10	3
20	5
30	3
40	3
50	2
60	2

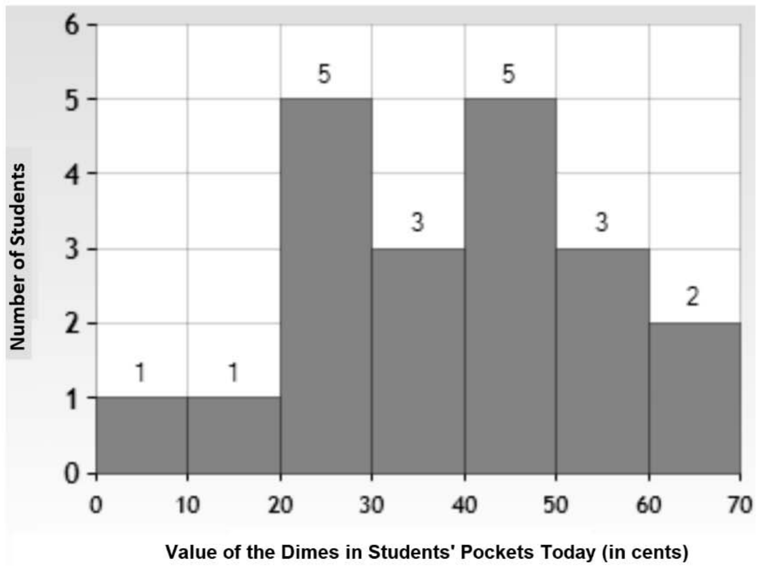
N



S

MEAN:	28	RANGE:	60
MEDIAN:	25	M.A.D.:	15
MODE:	20	Q1:	15
MAX:	60	Q3:	40
MIN:	0	IQR:	25

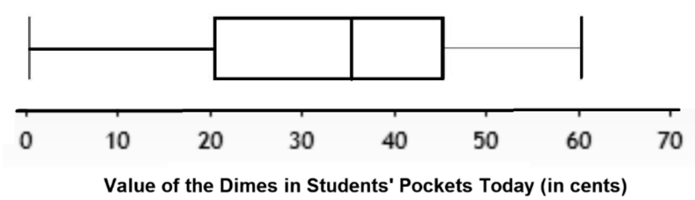
VII



X

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	1
10	1
20	5
30	3
40	5
50	3
60	2

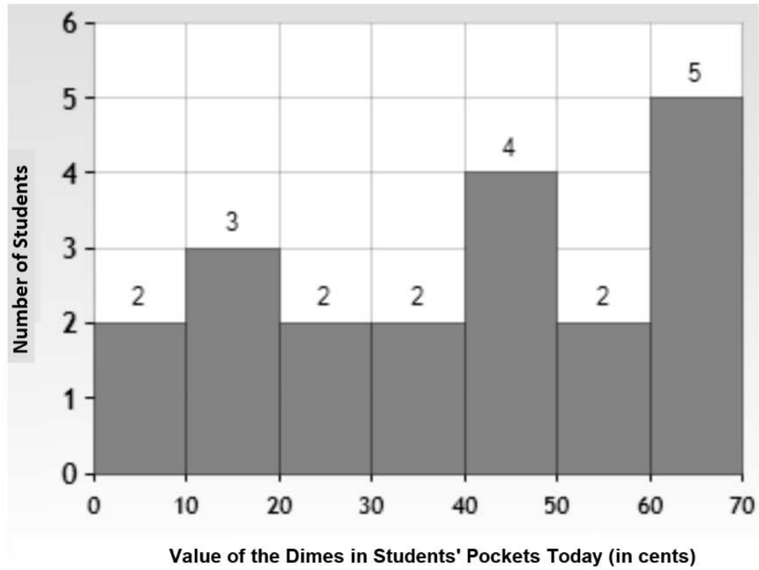
Q



E

MEAN:	33.5	RANGE:	60
MEDIAN:	35	M.A.D.:	13.5
MODE:	20	Q1:	20
MAX:	60	Q3:	45
MIN:	0	IQR:	25

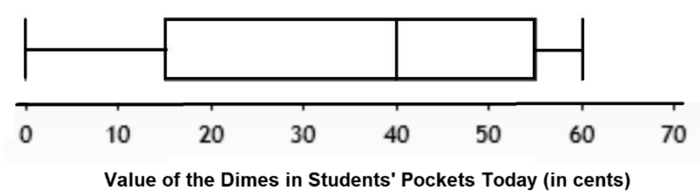
VIII



P

Frequency Table	
Value of Dimes (in Cents)	*Count*
0	2
10	3
20	2
30	2
40	4
50	2
60	5

F



R

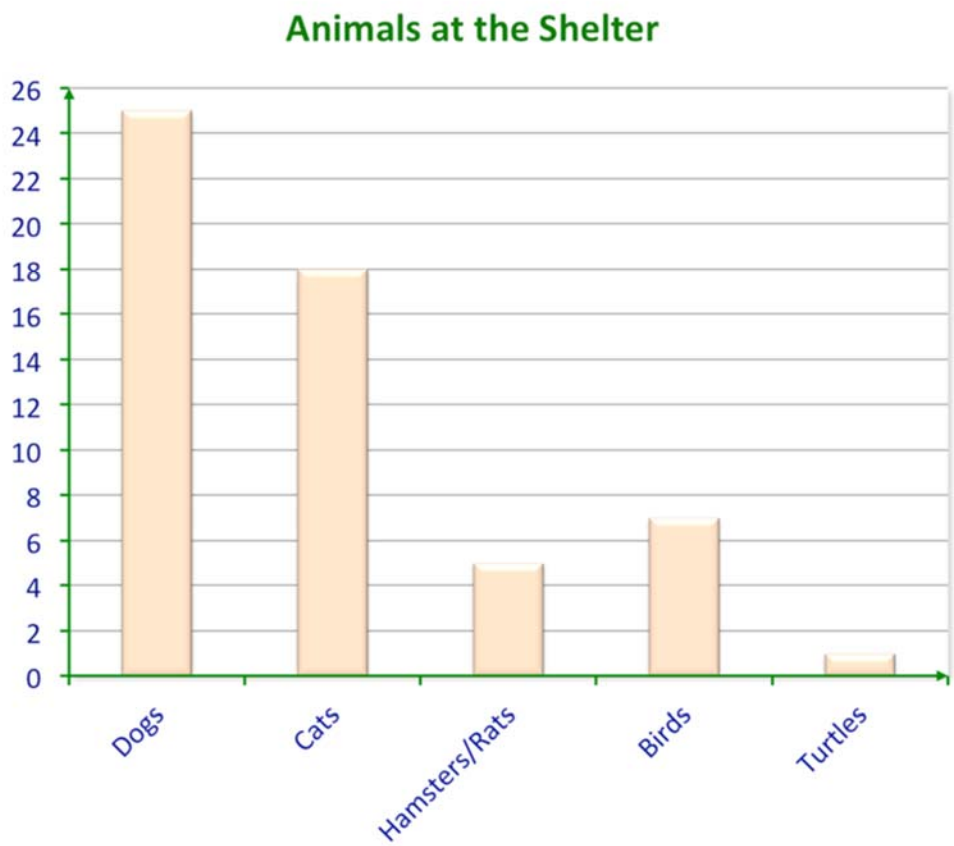
MEAN:	34.5	RANGE:	60
MEDIAN:	40	M.A.D.:	18.05
MODE:	60	Q1:	15
MAX:	60	Q3:	55
MIN:	0	IQR:	40

REVIEW OF CENTER/SPREAD: DATA DISPLAYS

Guided Practice

Here is one for you to try on your own.

A tally of the animals at a local shelter was taken so that children visiting on a field trip could see. Here are the results.



When children looked at the bar graph, they shouted out :
Bobby: "Nobody likes dogs!"
Lisa: "I didn't know that hamsters and rats were the same thing!"
Miguel: "Everyone must have taken all the turtles!"
Mona: "They must have mostly food for dogs and cats!"

How could a teacher respond? Were the conclusions of the children accurate?

A2.U7.C2.**C.01.vocab.MeanAbsDevStandardDevVariance**

Term	Definition	Notation	Diagram/Visual
Sigma			
Subscripted variables			
Standard Deviation of a Sample			
Variance of a Sample			
Standard Deviation of a Population			
Variance of a Population			

MEAN ABSOLUTE DEVIATION

Mean Absolute Deviation, Remember from Algebra 1, is how far, on average, all values are from the middle.
Calculating It
Find the mean of all values ... use it to work out distances ... then find the mean of those distances!
In three steps:

- 1. Find the mean of all values
- 2. Find the distance of each value from that mean (subtract the mean from each value, ignore minus signs)
- 3. Then find the mean of those distances

Deviation means different from the mean – you are calculating how different from the mean, on average, all of the data points are. SO when you think of “deviation,” just think **distance.**

Formula

- The formula (from the GA Milestones Algebra 1 Formula Sheet) is:

Mean Absolute Deviation

$$\frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

The sum of the distances between each data value and the mean, divided by the number of data values.

- **Σ** is Sigma, which means to sum up
- | | (the vertical bars) mean Absolute Value, -distance from zero, which is always positive
- x_i is each value (The subscript i counts the value – for instance x_1 is the first value in the list, x_2 is the second value in the list, and so on).
- n is the total number of data points you have

WATCH THIS!
Calculating Mean Absolute Deviation Video:
http://tiny.cc/KhanMAD
You can use the QR code at right, as well→

OR...

For calculator info on entering data and calculating Mean Absolute Deviation...

go to this URL:
http://tiny.cc/MADonTI84
or use this QR Code:

Example! **Find the Mean Absolute Deviation of 3, 6, 6, 7, 8, 11, 15, 16**
Solution on next page.

Step 1: Find the mean:

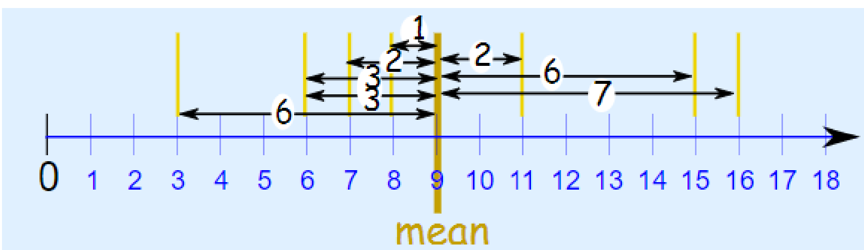
Mean, $\bar{x} = \dfrac{3+6+6+7+8+11+15+168}{8} = \dfrac{728}{8} = 9$

IN THIS EXAMPLE…
- n is the total number of data points you have - here, $n = 8$, so you divided 728 by 8 to find the mean.
- $\bar{x}$ is the mean (we just calculated that x **= 9**)

Step 2: Find the distance of each value from that mean…

Value	Distance from 9
3	6
6	3
6	3
7	2
8	1
11	2
15	6
16	7

…which looks like this:



(no minus signs because distance can't be negative!)

Step 3. Find the **mean of those distances**:

Mean Deviation = *6 + 3 + 3 + 2 + 1 + 2 + 6 + 7***8** = *30***8** = **3.75**

So, the **mean = 9**, and the **mean deviation = 3.75**
It tells us how far, on average, all values are from the middle.

In that example the values are, on average, 3.75 away from the middle.

SELF CHECK **Find the mean absolute deviation (MAD) of the data set**

The following table shows the number of classes that each teacher in the math department at Wilburton High School teaches.

Teacher	**Mr. Linn**	**Mrs. Ross**	**Mr. Riley**	**Ms. Moss**
Number of classes	3	7	4	2

M.A.D.=1.5

SELF CHECK **Compare the mean and the Mean Absolute Deviation of each teacher's test scores on test 1, and discuss what these mean in context. Which teacher did a better job teaching, and how do you know?**

Mr. Linn's students made the following scores:

79	88	70	72
85	81	75	88
73	84	70	79
87	80	85	67
83	79	75	73

Mr. Riley's students made the following scores:

98	63	93	60
70	93	60	99
68	66	90	55
90	94	68	79
93	70	99	65

Using the mean and MAD, tell who did a better job teaching and explain why you think so.

The mean scores of each class are the exact same (78.65), but the mean absolute deviation for each is very different. Mr. Linn's students' MAD was 5.42, meaning that the average kids' scores were within 5.42 points of the average. This means that almost every student in the class got approximately the same scores. Mr. Riley, whose average was the same as Mr. Linn's, had a great deal more variation in his scores. He had quite a few A's, but also several failing scores, as well. While an argument could be made for Mr. Riley, (some students may think he did better because he had a lot more A's, regardless of the students who failed), it should be pointed out that he had just as many students fail the test. Could it be that he alienates or ignores all but a handful of students in class? High variation in an educational setting is not generally a good thing. Mr. Linn's class, with a much lower MAD, is an indication of consistency in the class, and that "no child" was "left behind."

VARIANCE AND STANDARD DEVIATION

Standard Deviation

The Standard Deviation is a measure of how spread out numbers are.

Its symbol is **σ** (the greek letter sigma)

The formula is easy: it is the **square root** of the **Variance.** So now you ask, "What is the Variance?"

Variance

The Variance is defined as:

> The average of the **squared** differences from the Mean.

To calculate the variance follow these steps:

Work out the Mean (the simple average of the numbers)

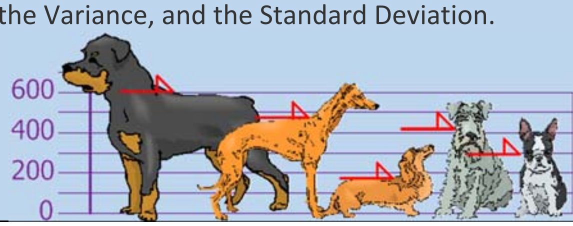
Then for each number: subtract the Mean and square the result (the squared difference).

Then work out the average of those squared differences. (Why Square?)

To calculate the standard deviation, follow these steps:

Square root the variance.

Example! Example: You and your friends have just measured the heights of your dogs (in millimeters). The heights (at the shoulders) are: 600mm, 470mm, 170mm, 430mm and 300mm. Find out the Mean, the Variance, and the Standard Deviation.

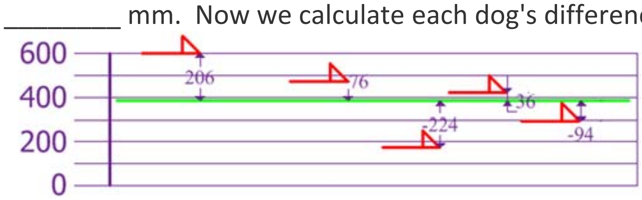


Your first step is to find the Mean:

Mean

$\bar{x}$ =

so the mean (average) height is ________ mm. Now we calculate each dog's difference from the Mean:



To calculate the Variance, take each difference, square it, and then average the result:

Variance

σ^2 =

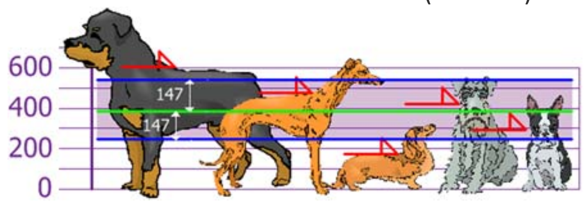
So the variance is 21704. And the **Standard Deviation is just the square root of Variance**, so:

Standard Deviation

σ

GUIDED NOTES

So…the good thing about the Standard Deviation is that it is useful. Now we can show which heights are within one Standard Deviation (147mm) of the Mean:



So, using the Standard Deviation we have a "standard" way of knowing what is normal, and what is extra large or extra small. Rottweilers **are** tall dogs. And Dachshunds **are** a bit short ... but don't tell them!

But ... there is a small change with Sample Data

Our example has been for a **Population** (the 5 dogs are the only dogs we are interested in).
But if the data is a **Sample** (a selection taken from a bigger Population), then the calculation changes!

When you have "N" data values that are:

- **The Population**: divide by **N** when calculating Variance (like we did)
- **A Sample**: divide by **N-1** when calculating Variance

All other calculations stay the same, including how we calculated the mean.

Example: if our 5 dogs are just a **sample** of a bigger population of dogs, we divide by **4 instead of 5** like this:

Sample Variance = 108,520 / **4** = **27,130**

Sample Standard Deviation = √27,130 = **164** (to the nearest mm)

Think of it as a "correction" when your data is only a sample.

Formulas

Here are the two formulas, explained at Standard Deviation Formulas if you want to know more – OR look up at what the symbols mean in MAD section above:

The "**Population** Standard Deviation": $$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2}$$

The "**Sample** Standard Deviation": $$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\overline{x})^2}$$

Looks complicated, but the important change is to
divide by **N-1** (instead of **N**) when calculating a Sample Variance.

WATCH THIS!
Sampling Video:
http://tiny.cc/StandardDevCalc
You can use the QR code at right, as well→

VIDEO

Example! **For the problem listed below, calculate the mean, the variance, and the standard deviation.**

You have found the following ages (in years) of all 6 lizards at your local zoo:
1, 2, 2, 1, 3, 3
What is the average age of the lizards at your zoo?

a. Calculate the Mean: $\bar{x} = \frac{1+2+2+1+3+3}{6} = \frac{12}{6} = 2$

b. Calculate the average squared distance from the mean, called the VARIANCE:

$$\sigma^2 = \frac{(2-1)^2+(2-2)^2+(2-2)^2+(2-1)^2+(2-3)^2+(2-3)^2}{6} = \frac{4}{6} = \frac{2}{3}$$

c. Square root the variance to find the STANDARD DEVIATION:

$$\sigma = \sqrt{\frac{2}{3}} \approx 0.816$$

So the lizards at the local zoo are all about 0.816 years $(0.816 * 12\ months\ =\ 9.792\ months)$ apart in age.

SELF CHECK **For the data set below:**
a. What is the average age of the lions at your zoo? b. What is the variance? c. What is the standard deviation? Utilize proper notation and round your answers to the nearest tenth.

You have found the following ages (in years) of all 6 lions at your local zoo: 13, 2, 1, 5, 2, 7

a

b.

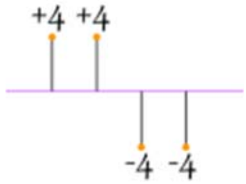
c

a. $\bar{x} = 5$ years. b. $\sigma^2 = 17$ c. $\sigma = 4.1$

Questions To Ponder

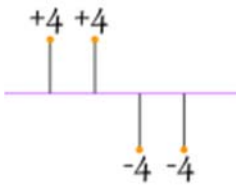
Why *square* the differences for Standard Deviation? Also, why do we need Standard Deviation when we have MAD?

If we just add up the differences from the mean ... the negatives cancel the positives:



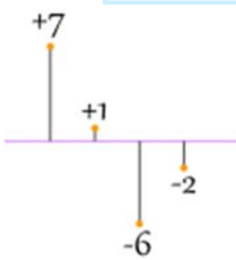
$$\frac{4+4-4-4}{4} = 0$$

So that won't work. How about we use absolute values?



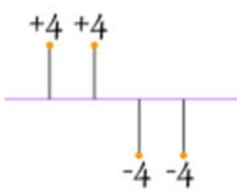
$$\frac{|4|+|4|+|-4|+|-4|}{4} = \frac{4+4+4+4}{4} = 4$$

That looks good (and is the Mean Absolute Deviation), but what about this case:

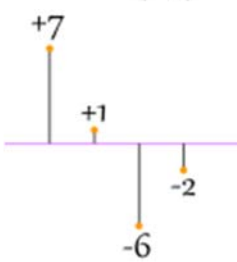


$$\frac{|7|+|1|+|-6|+|-2|}{4} = \frac{7+1+6+2}{4} = 4$$

Oh No! It also gives a value of 4, Even though the differences are more spread out.
So let us try squaring each difference (and taking the square root at the end):



$$\sqrt{\frac{4^2+4^2+4^2+4^2}{4}} = \sqrt{\frac{64}{4}} = 4$$



$$\sqrt{\frac{7^2+1^2+6^2+2^2}{4}} = \sqrt{\frac{90}{4}} = 4.74\ldots$$

That is nice! The Standard Deviation is bigger when the differences are more spread out ... just what we want. It is also easier to use algebra on squares and square roots than absolute values, which makes the standard deviation easy to use in other areas of mathematics.

Standard Deviation and Comparison of Data Sets

Name: ____________________________ Date: _________________ Period: _________ A

Match each of the following to the correct formula. (4 pts each)

1) ________ Mean	A) $\sigma^2 = \dfrac{\sum_{i=1}^{n}(x_i - \overline{X})^2}{n}$
2) ________ Population Variance	B) $\overline{X} = \dfrac{\sum_{i=1}^{n} x_i}{n}$
3) ________ Sample Variance	C) $\sigma = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{X})^2}{n}}$
4) ________Population Standard Deviation	D) $s = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{X})^2}{n-1}}$
5) ________ Sample Standard Deviation	E) $s^2 = \dfrac{\sum_{i=1}^{n}(x_i - \overline{X})^2}{n-1}$

6) Give an example of a scenario when the sample standard deviation should be used and an example where the population standard deviation should be used and justify your reasoning.

7) Coach Price is comparing the scoring ability of two of his players. The team has played 10 games this year and the number of points scored by each player is shown below.

Player 1: 14, 3, 8, 12, 6, 10, 9, 18, 12, 4

Player 2: 8, 7, 9, 10, 6, 12, 9, 6, 7, 8

a) Find $\overline{X}$, σ^2, and σ for each player.

b) Which player would you consider the better shooter? Explain your answer.

Mean Absolute Deviation, Variance, Standard Deviation: Math Award Learning Task

Name ____ Date ____________________

Your teacher has a problem and needs your input. She has to give one math award this year to a deserving student, but she can't make a decision. Here are the test grades for her two best students:

Bryce: 90, 90, 80, 100, 99, 81, 98, 82 **Brianna: 90, 90, 91, 89, 91, 89, 90, 90**

1) Create a boxplot for each student's grade distribution and record the five-number summary for each student.

Bryce:	Brianna:
Min: ___________ Max: __________	Min: ___________ Max: __________
Q1: __________ Median: __________ Q3: _______	Q1: __________ Median: __________ Q3: _______

2) Based on your display, determine which student should get the math award and discuss why they should be the one to receive it.

3) Calculate the mean ($\bar{x}$) of Bryce's grade distribution.

Calculate the mean deviation, variance, and standard deviation of Bryce's distribution. The formulas for the mean absolute deviation, variance, and standard deviation are below.

Mean Absolute Deviation:	**variance:**	**standard deviation:**
$MAD = \frac{\sum_{i=1}^{n} \lvert x_i - \bar{X} \rvert}{n}$	$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{X})^2}{n}$	$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{X})^2}{n}}$ **Note: This is the square root of the variance**

4) Fill out the table to help you calculate them by hand.

Scores for Bryce (x_i)	**Mean Deviation $x_i - \bar{x}$**	**Mean Absolute Deviation $\lvert x_i - \bar{x} \rvert$**	**Variance $(x_i - \bar{x})^2$**
90			
90			
80			
100			
99			
81			
98			
82			

MAD for Bryce: _______

Variance for Bryce: ______

Standard deviation for Bryce: _______

5) What do these measures of spread tell you about Bryce's grades?

6) Calculate the mean of Brianna 's distribution.

7) Calculate the mean deviation, variance, and standard deviation of Brianna's distribution.

Scores for Brianna (x_i)	Mean Deviation $x_i - \overline{x}$	Mean Absolute Deviation $\lvert x_i - \overline{x} \rvert$	Variance $(x_i - \overline{x})^2$
90			
90			
91			
89			
91			
89			
90			
90			

MAD for Brianna: _______

Variance for Brianna: ______

Standard deviation for Brianna: _______

8) What do these measures of spread tell you about Brianna's grades?

9) Based on this information, write down which of the two students should get the math award and discuss why they should be the one to receive it.

Pull it All Together – Restoring Lost Data

The statistician for a football team recorded data for the 5 preseason games. While he was working on the file, there was a power surge and his computer lost all of the data for the first game. Luckily, he had already calculated some statistics for the entire preseason.

The number of points the team scored for each game is shown in the table.

Game	1	2	3	4	5
Points	?	17	24	13	21

The statistician could not find the mean number of points scored per game. However, he was able to find the standard deviation for the entire preseason. Its value is $\sqrt{14}$. The statistician will use this fact to figure out the number of points scored in Game 1.

Use the formula for standard deviation, $\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$, to determine the number of points scored in game 1.

Who is the Better Athlete? Using Statistics to Compare Data

Data Set 1:

The data below shows the number of hits made by 4 different major league baseball players for each of the last ten years. Find the mean, variance & standard deviation of the data shown for each player. Then compare the results. Which player is the better hitter? Explain your reasoning.

Player 1: 216, 164, 179, 212, 179, 206, 214, 202, 188, 164

Player 2: 112, 126, 84, 129, 160, 173, 134, 109, 117, 169

Player 3: 173, 173, 183, 186, 187, 185, 177, 195, 196, 212

Player 4: 205, 197, 180, 198, 180, 188, 195, 198, 177, 181

Data Set 2:

The data below shows the pass completion rate for 4 different professional football players over a ten year time span. Find the mean, variance & standard deviation of the data shown for each player. Then compare the results. Which quarterback is the better passer? Explain your reasoning.

Player 1: 60.6, 68.4, 65.7, 66.5, 56.0, 61.3, 64.1, 65.4, 61.9, 61.6

Player 2: 65.8, 65.6, 65.9, 65.7, 63.6, 68.9, 61.8, 63.0, 60.8, 60.2

Player 3: 37.8, 66.3, 68.8, 66.8, 65.4, 65.0, 67.3, 67.6, 67.0, 66.3

Player 4: 58.9, 71.2, 68.1, 70.6, 65.0, 67.5, 64.3, 64.6, 65.5, 57.6

Data Set 3:

The data below shows the average number of points scored per game by 4 different professional basketball players over a ten year time span. Find the mean, variance & standard deviation of the data shown for each player. Then compare the results. Which player is the better shooter? Explain your reasoning.

Player 1: 27.9, 25.3, 27.0, 26.8, 28.3, 31.6, 35.4, 27.6, 24.0, 30.0

Player 2: 21.6, 23.0, 25.0, 25.9, 23.6, 24.6, 26.6, 26.1, 21.8, 25.1

Player 3: 11.6, 20.1, 20.8, 20.8, 20.9, 20.5, 17.6, 23.6, 18.0, 18.9

Player 4: 9.2, 12.0, 17.8, 13.6, 12.9, 14.2, 17.3, 20.0, 22.9, 21.5

A2.U7.C2.**D.01.vocab.EmpiricalRule**

Term	Definition	Notation	Diagram/Visual
Empirical Rule			
68-95-99.7 Rule			
Bell Curve			
Normally Distributed			

The Empirical Rule

If you knew that the prices of t-shirts sold in an online shopping site were ***normally distributed***, and had a mean cost of $10, with a **standard deviation** of $1.50, how could that information benefit you as you are looking at various t-shirt prices on the site? How could you use what you know if you were looking to make a profit by purchasing unusually inexpensive shirts to resell at prices that are more common?

A **distribution** is an evaluation of the way that points in a data set are clustered or spread across their **range** of values. A **normal distribution** is a very specific symmetrical distribution that indicates, among other things, that exactly ½ of the data is below the mean, and ½ is above, that approximately 68% of the data is within 1 standard deviation, approximately 96% of the data is within 2 standard deviations, and approximately 99.7% is within 3 standard deviations of the mean.

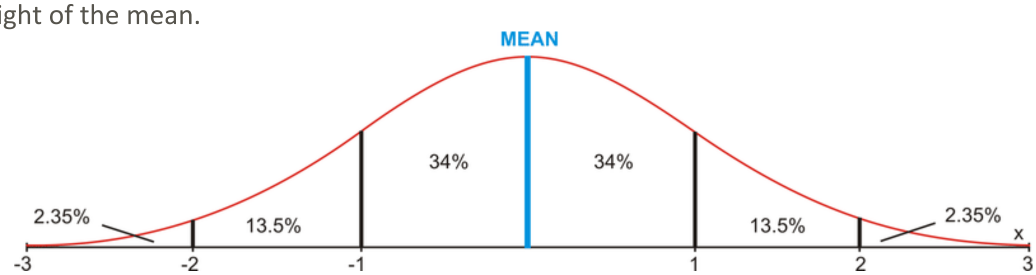
There are a number of reasons that it is important to become familiar with the normal distribution, as you will discover throughout this chapter. Examples of values associated with normal distribution:

- Physical characteristics such as height, weight, arm or leg length, etc.
- The percentile rankings of standardized testing such as the ACT and SAT
- The volume of water produced by a river on a monthly or yearly basis
- The velocity of molecules in an ideal gas

Knowing that the values in a set are exactly or approximately normally distributed allows you to get a feel for how common a particular value might be in that set. Because the values of a normal distribution are predictably clustered around the mean, you can estimate in short order the rarity of a given value in the set. In our upcoming lesson on the Empirical Rule, you will see that it is worth memorizing that normally distributed data has the characteristics mentioned above:

- 50% of all data points are above the mean and 50% are below
- ≈68% of all data points are within 1 standard deviation of the mean
- ≈95% of all data points are within 2 standard deviations of the mean
- ≈99.7% of all data points are within 3 standard deviations of the mean

As listed above, we have already learned that 68% of the data in a normal distribution lies within 1 standard deviation of the mean, 95% of the data lies within 2 standard deviations of the mean, and 99.7% of the data lies within 3 standard deviations of the mean. This is referred to as the **Empirical Rule**, which is also known as the **68-95-99.7 Rule**. To accommodate the percentages given by the Empirical Rule, there are defined values in each of the regions to the left and to the right of the mean.



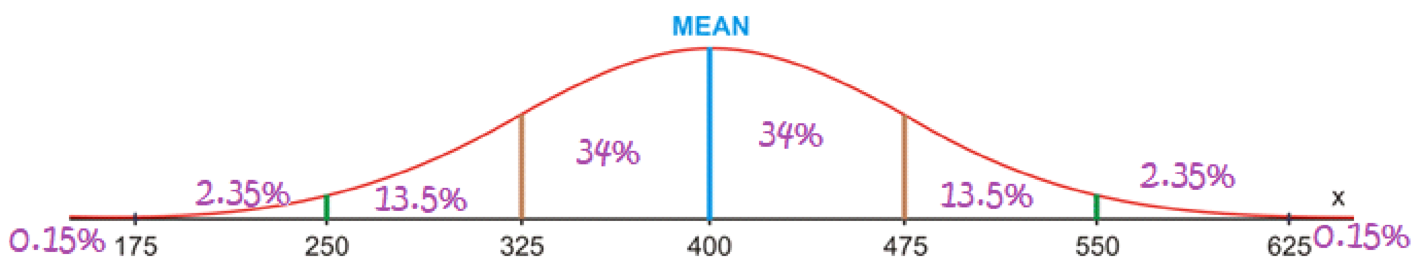
These percentages are used to answer real-world problems when both the mean and the standard deviation of a data set are known. Also keep in mind that since 99.7% of the data in a normal distribution is within 3 standard deviations of the mean, 1−99.7%=0.3% of the data does not fall within 3 standard deviations of the mean. This means that 0.3%2=0.15% of the data is beyond 3 standard deviations on either end of the normal distribution curve. This is not shown in the figure above.

Important to note: a condition necessary to calculate probabilities of distributions (using Empirical Rule or other methods) is that the distribution must be normal or **approximately normal**!

WATCH THIS!
Empirical Rule Video:
http://tiny.cc/EmpiricalRule
You can use the QR code at right, as well→

Example! **Real-World Application: Lifetime of Light Bulbs.** The lifetimes of a certain type of light bulb are normally distributed. The **mean life is 400 hours**, and the **standard deviation is 75 hours**. For a group of 5,000 light bulbs, how many are expected to last each of the following times?

Because the mean is 400 hours, 400 goes at the middle of the "bell curve." The standard deviation is 75, so you add/subtract in increments of 75 to and from the mean to find the boundaries of the bell curve.



a) between 325 hours and 475 hours

68% of the light bulbs are expected to last between 325 hours and 475 hours.
This means that 5,000×0.68=3,400 light bulbs are expected to last between 325 and 475 hours.

b) more than 250 hours

95%+2.35%+0.15%=97.5% of the light bulbs are expected to last more than 250 hours.
This means that 5,000×0.975=4,875 of the light bulbs are expected to last more than 250 hours.

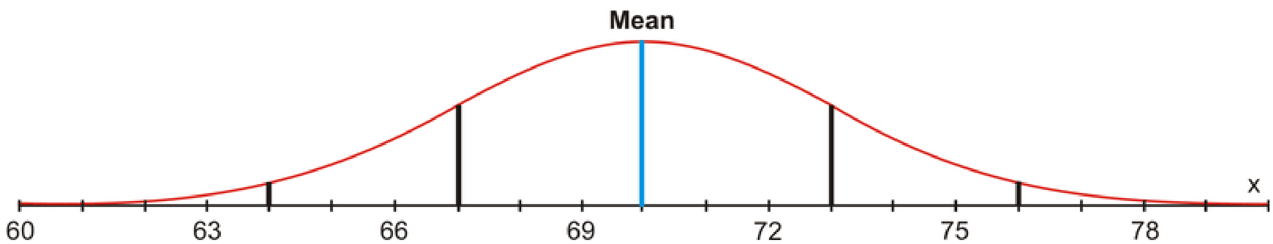
c) less than 250 hours

Only 2.35%+0.15%=2.5% of the light bulbs are expected to last less than 250 hours.
This means that 5,000×0.025=125 of the light bulbs are expected to last less than 250 hours.

Find the mean absolute deviation (MAD) of the data set

Real-World Application: Bags of Chips

A bag of chips has a mean mass of 70 g, with a standard deviation of 3 g. Assuming a normal distribution, create a normal curve, including all necessary values.



a) If 1,250 bags of chips are processed each day, how many bags will have a mass between 67 g and 73 g?

Between 67 g and 73 g lies 68% of the data. If 1,250 bags of chips are processed, 1,250×0.68=850 bags will have a mass between 67 g and 73 g.

b) What percentage of the bags of chips will have a mass greater than 64 g?

95%+2.35%+0.15%=97.5% of the bags of chips will have a mass greater than 64 grams.

WATCH THIS!

Empirical Rule Video #2 (Applications):

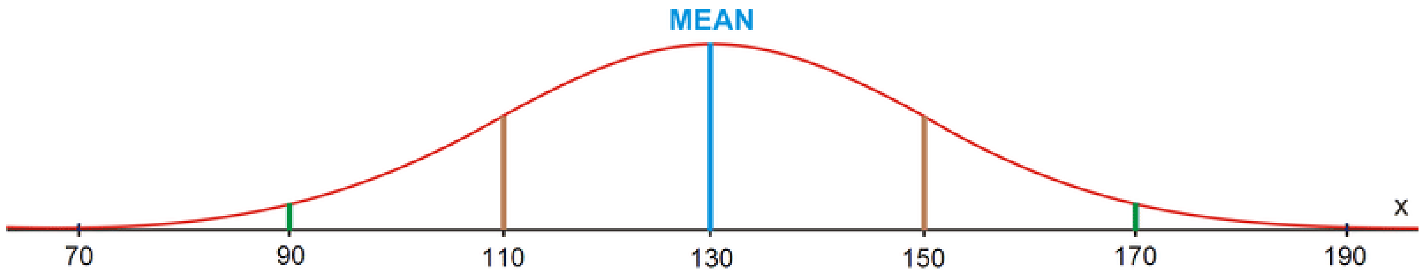
http://tiny.cc/EmpiricalRule2

You can use the QR code at right, as well→

SELF CHECK

Real-World Application: Half Marathon

The finishing times for people completing a half marathon in 2010 were normally distributed, with a mean of 130 minutes and a standard deviation of 20 minutes. If 1,400,000 people completed a half marathon in 2010, how many people had finishing times between each of the following pairs of times?



a) 90 minutes and 150 minutes

b) 110 minutes and 130 minutes

c) 130 minutes and 190 minutes

a. Between 90 minutes and 150 minutes lies 13.5%+68%=81.5% of the data. This means that 1,400,000×0.815=1,141,000 of the people who completed a half marathon in 2010 had a time between 90 minutes and 150 minutes.

b. Between 110 minutes and 130 minutes lies 34% of the data. This means that 1,400,000×0.34=476,000 of the people who completed a half marathon in 2010 had a time between 110 minutes and 130 minutes.

c. Between 130 minutes and 190 minutes lies 34%+13.5%+2.35%=49.85% of the data. This means that 1,400,000×0.4985=697,900 of the people who completed a half marathon in 2010 had a time between 130 minutes and 190 minutes.

Questions To Ponder

What conditions must apply for the Empirical Rule to be able to be used?

1. What percentage of the data in a normal distribution is between 1 standard deviation below the mean and 2 standard deviations above the mean?

2. What percentage of the data in a normal distribution is between 3 standard deviations below the mean and 1 standard deviation above the mean?

3. What percentage of the data in a normal distribution is more than 2 standard deviations above the mean?

4. What percentage of the data in a normal distribution is between 2 standard deviations below the mean and 3 standard deviations above the mean?

5. What percentage of the data in a normal distribution is between 3 standard deviations below the mean and the mean?

6. What percentage of the data in a normal distribution is more than 1 standard deviation above the mean?

7. What percentage of the data in a normal distribution is between the mean and 2 standard deviations above the mean?

8. 200 senior high students were asked how long they had to wait in the cafeteria line for lunch. Their responses were found to be normally distributed, with a mean of 15 minutes and a standard deviation of 3.5 minutes. (a) How many students would you expect to wait more than 11.5 minutes? (b) How many students would you expect to wait more than 18.5 minutes? (c) How many students would you expect to wait between 11.5 and 18.5 minutes?

9. 350 babies were born at Neo Hospital in the past 6 months. The average weight for the babies was found to be 6.8 lbs, with a standard deviation of 0.5 lbs. (a) How many babies would you expect to weigh more than 7.3 lbs? (b) How many babies would you expect to weigh more than 7.8 lbs? (c) How many babies would you expect to weigh between 6.3 and 7.8 lbs?

10. Sheldon has planted seedlings in his garden in the back yard. After 30 days, he measures the heights of the seedlings to determine how much they have grown. The differences in heights can be seen in the table below. The heights are measured in inches. Draw a normal distribution curve to represent the data. Determine what the range of the differences in heights of the seedlings is for the middle 68% of the data.

10	3	8	4	7	12	8	5	4	9	3	8
6	10	7	10	11	8	12	9	10	7	8	11

Using the Empirical Rule

Name ____ Date________________________

Under certain conditions (those you will discover during this activity) the Empirical Rule can be used to help you make a good guess of the standard deviation of a distribution.

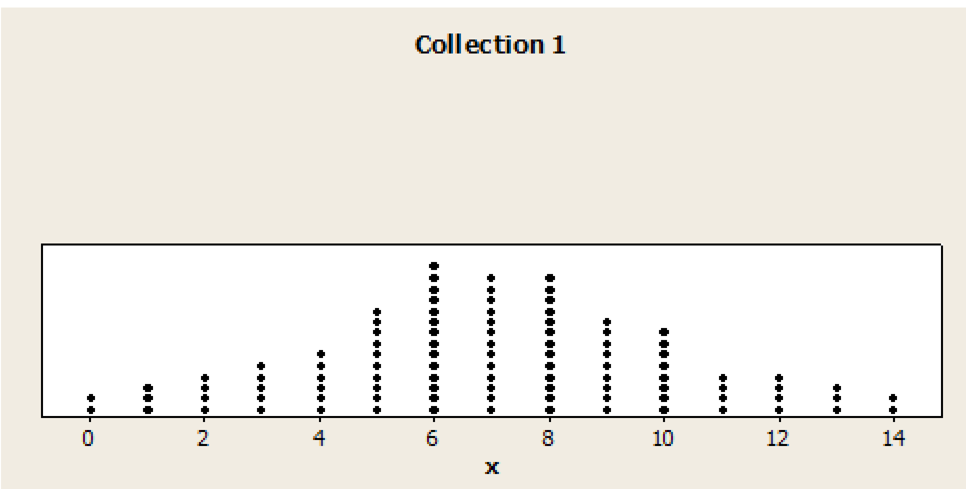
The Empirical Rule is as follows:

For certain conditions (which you will discover in this activity),

- 68% of the data will be located within one standard deviation symmetric to the mean
- 95% of the data will be located within two standard deviations symmetric to the mean
- 99.7% of the data will be located within three standard deviations symmetric to the mean

For example, suppose the data meets the conditions for which the empirical rule applies. If the mean of the distribution is 10, and the standard deviation of the distribution is 2, then about 68% of the data will be between the numbers 8 and 12 since 10-2 =8 and 10+2 = 12. We would expect approximately 95% of the data to be located between the numbers 6 and 14 since 10-2(2) = 6 and 10 + 2(2) = 14. Finally, almost all of the data will be between the numbers 4 and 16 since 10 – 3(2) = 4 and 10 + 3(2) = 16.

For each of the dotplots below, estimate the mean and the standard deviation of each of the following distributions. Then, use your calculator to determine the mean and standard deviation of each of the distributions. For your convenience, there are 100 data points for each dotplot.



Estimated mean:___________

Estimated standard deviation:_______________

Actual mean:____________

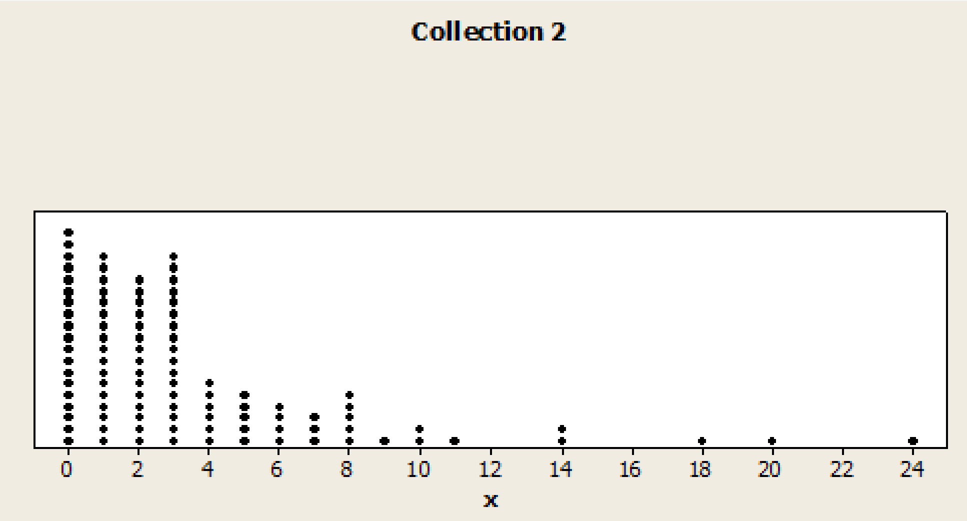
Actual standard deviation:______________

Now that you know what the actual mean and standard deviation, calculate one standard deviation below the mean and one standard deviation above the mean.

$\mu - \sigma =$ ___ $\mu + \sigma =$ ___

Locate these numbers on the dotplot above. How many dots are between these numbers?_________

Is this close to 68%?_________ Do you think that the empirical rule should apply to this distribution?________ Why or why not?



Estimated mean:______________

Estimated standard deviation:___________

Actual mean:_____________

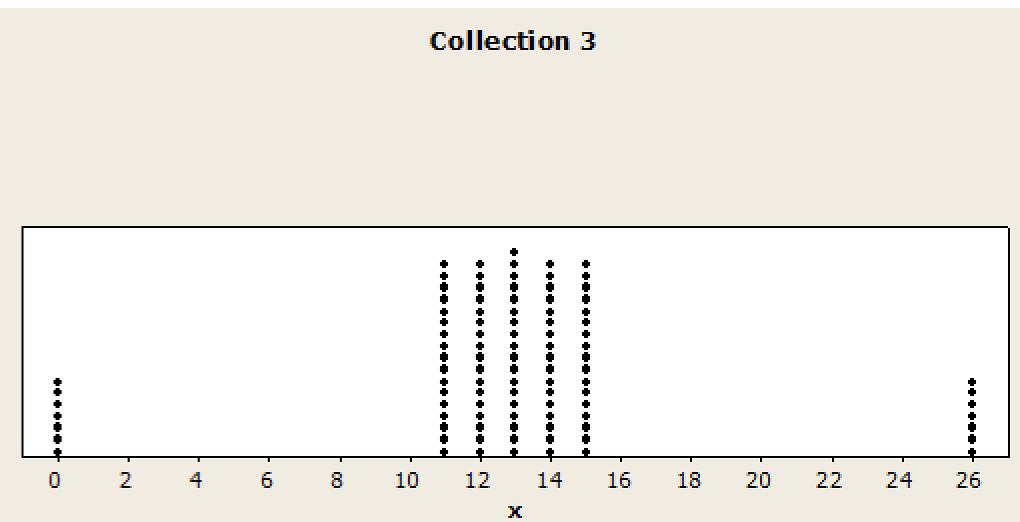
Actual standard deviation:_________

Now calculate one standard deviation below the mean and one standard deviation above the mean.

$\mu - \sigma =$ ___ $\mu + \sigma =$ ___

Locate these numbers on the dotplot above. How many dots are between these numbers?_________

Is this close to 68%?______ Do you think that the empirical rule should apply to this distribution?_________ Why or Why not?



Estimated mean:______________

Estimated standard deviation:___________

Actual mean:_____________

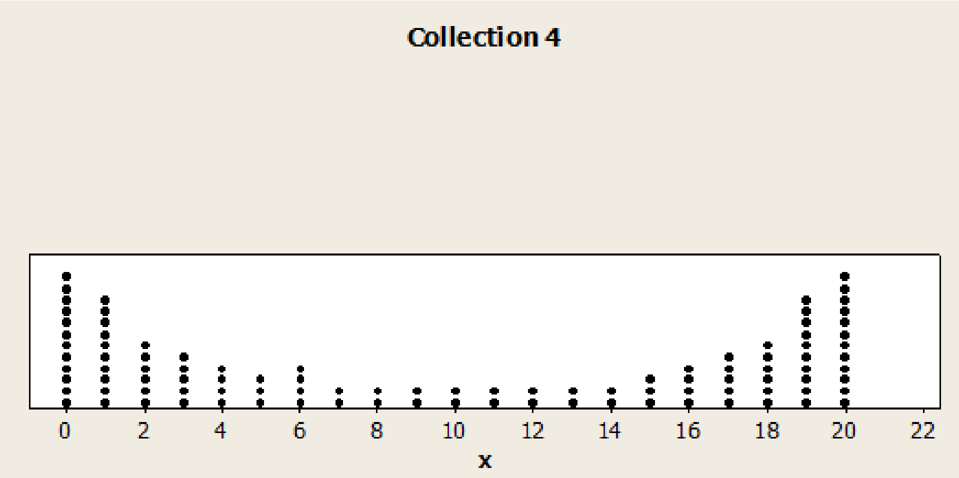
Actual standard deviation:_________

Now calculate one standard deviation below the mean and one standard deviation above the mean.

$\mu - \sigma =$ ___ $\mu + \sigma =$ ___

Locate these numbers on the dotplot above. How many dots are between these numbers?______

Is this close to 68%?______ Do you think that the empirical rule should apply to this distribution?______ Why or why not?



Estimated mean:_____________

Estimated standard deviation:__________

Actual mean:____________

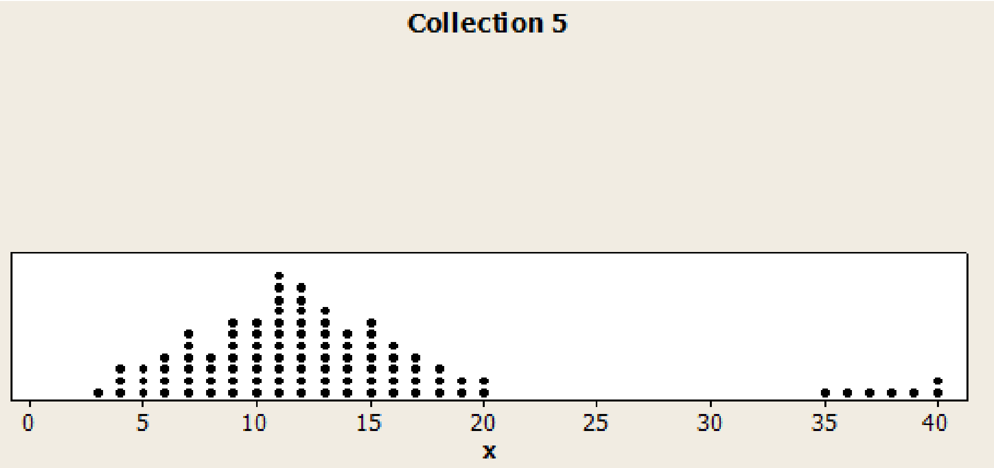
Actual standard deviation:_________

Now calculate one standard deviation below the mean and one standard deviation above the mean.

$\mu - \sigma =$ ___ $\mu + \sigma =$ ___

Locate these numbers on the dotplot above. How many dots are between these numbers?______

Is this close to 68%?______ Do you think that the empirical rule should apply to this distribution?______ Why or why not?



Estimated mean:_____________

Estimated standard deviation:__________

Actual mean:____________

Actual standard deviation:_________

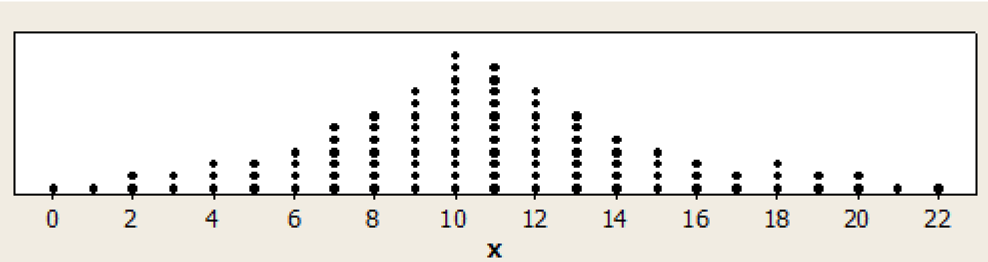
Now calculate one standard deviation below the mean and one standard deviation above the mean.

$\mu - \sigma =$ ___ $\mu + \sigma =$ ___

Locate these numbers on the dotplot above. How many dots are between these numbers?______

Is this close to 68%?______ Do you think that the empirical rule should apply to this distribution?______ Why or why not?

Collection 6



Estimated mean:_____________

Estimated std deviation:________

Actual mean:____________

Actual standard deviation:_________

Now calculate one standard deviation below the mean and one standard deviation above the mean.

$\mu - \sigma =$ ___ $\mu + \sigma =$ ___

Locate these numbers on the dotplot above. How many dots are between these numbers?______

Is this close to 68%?______ Do you think that the empirical rule should apply to this distribution?______ Why or why not?

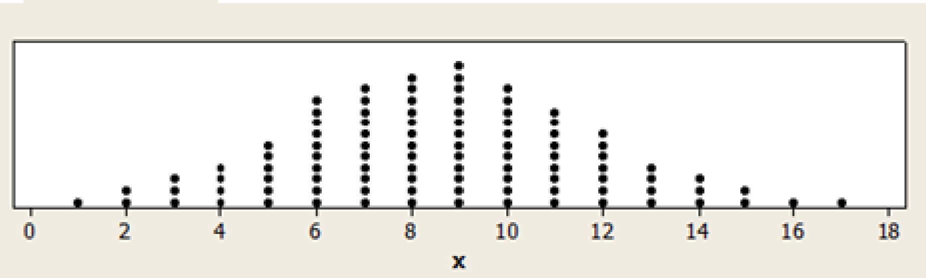
Summary Questions:

I. Which distributions had close to 68% of the data within one standard deviation of the mean? What was the general shape of these distributions?

II. For which type of distributions do you think the Empirical rule applies?

As you discovered, the empirical rule does not work unless your data is symmetrical. However, not all symmetrical graphs are normal. Apply the empirical rule to determine if the next two graphs are normal or not.

Collection 7



Mark the mean on the dotplot.

Calculate one standard deviation above and below the mean.

$\mu_X - \sigma_X =$ ______ and

$\mu_X + \sigma_X =$ ________.

Mark these points on the x-axis of the dotplot. How many data points are between these values? __________

Calculate two standard deviations below and above the mean. $\mu_X + 2\sigma_X =$_____ and $\mu_X - 2\sigma_X =$ _____. Mark these points on the x-axis of the dotplot. How many data points are between these values? ________

Calculate three standard deviations below and above the mean.

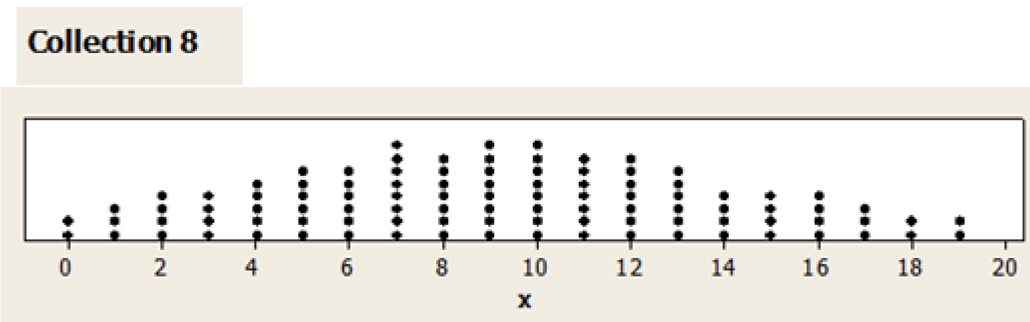
$\mu_X - 3\sigma_X$ = _____ and $\mu_X + 3\sigma_X$ = _____. Mark these points on the x-axis of the dotplot. How many data points are between these values? _______

Is it likely that this sample is from a normal population? Explain your reasoning.

Outliers are values that are beyond two standard deviations from the mean in either direction.

Which values from the data would be considered to be outliers? _______________

Using the dotplot below, calculate the mean and the standard deviation of the distribution.



Mark the mean on your dotplot.

Calculate one standard deviation above and below the mean.

$\mu_X - \sigma_X$ = ______ and $\mu_X + \sigma_X$ = _______. Mark these points on the x-axis of the dotplot. How many data points are between these values? ________

Calculate two standard deviations below and above the mean.

$\mu_X + 2\sigma_X$ = _______ and $\mu_X - 2\sigma_X$ = _______. Mark these points on the x-axis of the dotplot. How many data points are between these values? __________

Calculate three standard deviations below and above the mean.

$\mu_X - 3\sigma_X$ = _______ and $\mu_X + 3\sigma_X$ = _______. Mark these points on the x-axis of the dotplot. How many data points are between these values? _________

Is it likely that this sample is from a normal population? Explain your thinking.

Which values from the data would be considered to be outliers? _______________

Based on your observations, when is it appropriate to conclude that a data set is approximately normal?

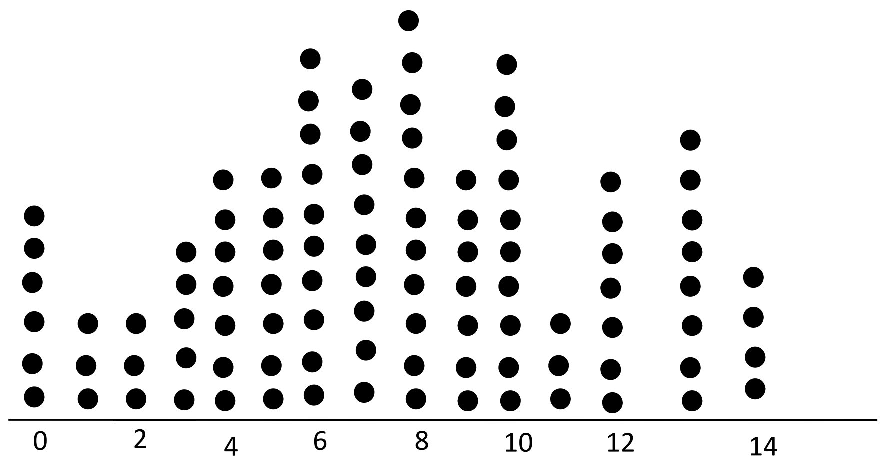
NORMAL DISTRIBUTIONS AND THE EMPIRICAL RULE

Name: ____ Date: ________ Pd: ___________

1) What is the Empirical Rule and what type of distribution does it apply to?

2) Given a set of data points where X = 80 and σ = 2.5, if the Empirical Rule applies, 99.7% of the data should fall between what two numbers? (Show your work)

3) Use the dotplot and frequency table below to answer parts a – d



x_i	F_i
0	6
1	3
2	3
3	5
4	7
5	7
6	10
7	9
8	11
9	7
10	10
11	3
12	7
13	8
14	4

a. Find $\overline{X}$, σ, Q1 and Q3.

b. What percent of the data lies within one, two and three standard deviations about the mean?

% within $\overline{X} \pm \sigma$: ____________________

% within $X \pm 2\sigma$:____________________

% within $X \pm 3\sigma$:____________________

c. Would you describe this distribution as normal? Explain your answer.

d. Find any outliers.

A2.U7.C2.**E.01.vocab.Z-Scores**

Term	Definition	Notation	Diagram/Visual
z-scores			
Standardized values			
Standard normal probability table/chart			

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

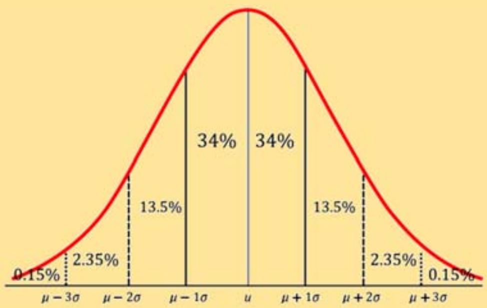
P(z < -2) = .0228

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776

Probability proportions		
Relative Frequency		
Left-tail probability		

Questions To Ponder

The mean weight of ostrich eggs is 3lbs, with a standard deviation of 0.5lbs. Label the normal curve, below, to show the weights of eggs that are exactly one standard deviation above and below the mean, two standard deviations above and below, and three standard deviations above and below the mean.



Randy, the ostrich keeper at the San Diego Zoo, has collected and weighed two different ostrich eggs.

One is exactly 3.75 pounds, and the other is 2.75 pounds. Looking at the distribution above, how many standard deviations from the mean are these weights?

Normal Distributions and Z-Scores

Using the Empirical Rule can give you a good idea of the probability of occurrence of a value that happens to be exactly one, two or three to either side of the mean, but how do you compare the probabilities of values that are in between standard deviations? In the example above, the weights fell at the exact halfway between two deviations, so we just divided a standard deviation. Partial standard deviations are called **z-scores**.

Z-scores (also called **standardized values**) are related to the Empirical Rule from the standpoint of being a method of evaluating how extreme a particular value is in a given set. **You can think of a z-score as the number of standard deviations there are between a given value and the mean of the set.** While the Empirical Rule allows you to associate the first three standard deviations with the percentage of data that each SD includes, the z-score allows you to state (as accurately as you like), just how many SD's a given value is above or below the mean. Like the empirical rule, a condition of "normal" or "approximately normal" is required.

Conceptually, the z-score calculation is just what you might expect, given that you are calculating the number of SD's between a value and the mean. You calculate the z-score by first calculating the difference between your value and the mean, and then dividing that amount by the **standard deviation** of the set.

$$z - score = \frac{(value - mean)}{standard\ deviation} = \frac{(x-\mu)}{\sigma}$$

The formula looks like this, applied to the ostrich egg question above:

The 3.75-lb ostrich egg's $z - score = \frac{(value - mean)}{standard\ deviation} = \frac{(x-\mu)}{\sigma} = \frac{(3.75-3)}{0.5} = \frac{(0.75)}{0.5} = 1.5$. Z=1.5 means the 3.75-lb ostrich egg is 1.5 standard deviations above the mean.

The 2.75-lb ostrich egg's $z - score = \frac{(value - mean)}{standard\ deviation} = \frac{(x-\mu)}{\sigma} = \frac{(2.75-3)}{0.5} = \frac{(-0.25)}{0.5} = -0.5$. $Z = -0.5$ means the 2.75-lb ostrich egg is half of a standard deviation below the mean (negative z scores means it is to the left of, or below, the mean).

In this lesson, we will practice calculating the z-score for various values, and then we will learn how to associate the z-score of a value with the probability that the value will occur using a **Standard Normal Probability Table**.

WATCH THIS!

Z-Scores Video:

http://tiny.cc/z-scores

You can use the QR code at right, as well→

Example! Finding Z-Scores

Finding the Z-Score

1. What is the *z*-score of a value of 27, given a set mean of 24, and a standard deviation of 2?

To find the z-score we need to divide the difference between the value, 27, and the mean, 24, by the standard deviation of the set, 2.

$$\begin{aligned} z\ score &= \frac{27-\mu}{\sigma} \\ &\frac{27-24}{2} \\ &\frac{3}{2} \\ z\ score\ of\ 27 &= +1.5 \end{aligned}$$

This indicates that 27 is **1.5** standard deviations above the mean.

2. What is the *z*-score of a value of 104.5, in a set with μ=125 and σ=6.2?

Find the difference between the given value and the mean, then divide it by the standard deviation.

$$\begin{aligned} z\ score &= \frac{104.5-\mu}{\sigma} \\ &\frac{104.5-125}{6.2} \\ &\frac{-20.5}{6.2} \\ z\ score\ of\ 104.5 &= -3.306 \end{aligned}$$

Note that the *z*-score is negative, since the measured value, 104.5, is less than (below) the mean, 125.

Example! Finding the Value Represented by a Z-Score (working backwards).

Find the value represented by a *z*-score of 2.403, given μ=63 and σ=4.25.

This one requires that we solve for a missing value rather than for a missing *z*-score, so we just need to fill in our formula with what we know and solve for the missing value:

$$\begin{aligned} z\ score &= \frac{x-\mu}{\sigma} \\ 2.403 &= \frac{x-63}{4.25} \\ 10.213 &= x-63 \\ 73.213 &= x \end{aligned}$$

73.213 has a ***z***-score of 2.403

SELF CHECK **Find the z-score:**

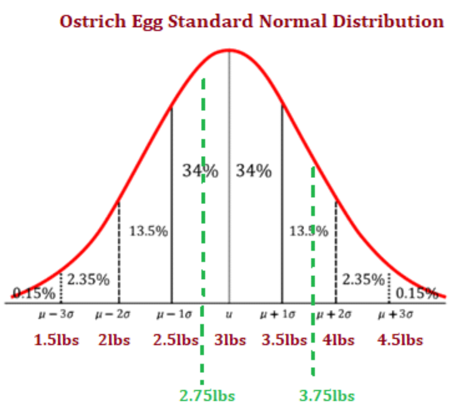
What is the *z*-score of the price of a pair of skis that cost \$247, if the mean ski price is \$279, with a standard deviation of \$16?

$\frac{247-279}{16}$
$\frac{-32}{16}$
z-score $= -2$

So what do you do with Z-Scores?

Recall the ostrich egg question at the beginning of this lesson. The 2.75-lb egg was 0.5 standard deviations below the mean ($Z =— 0.5$), and the 3.75-lb egg was 1.5 standard deviations above the mean (($Z = 1.5$). The Empirical Rule can't be used here, because their standard deviations don't fall on the 68-95-99.7 lines. This is where z-scores come in.

You can clearly see where the two odd eggs fall on the standard normal distribution for ostrich eggs…



You will use the z-scores we calculated earlier, and the **Standard Normal Probability Chart.** This chart shows us the **probability proportions (relative frequency)**, and they allow us to find the probability percentages for values other than those that fall on integer standard deviations. This chart is on the next two pages. The proportions in this chart are always the **left-tail probabilities.**

How to use the standard normal probability charts:

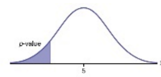
a) Calculate the z-scores, out to the nearest hundredth.
b) Separate the z-score between the tenth and the hundredths.
   For instance, $Z = -0.25$ *would be* $Z = -(0.2 + 0.05)$.
c) Use the left-most column to find the value of the truncated z-score to the tenth, and then move right until you find the appropriate hundredths value.
   For instance, $Z = -0.25$*, go down to -0.2, and then right to the 0.05 column).*
d) The value found in that box is $.4013$, which means that 2.75lbs is .4013. This number is called the **proportion**. The proportion always includes four significant figures. The proportion represents the 40.13 percentile. (This 2.75-lb ostrich egg is a little smaller than average. It is larger than only 40.13% of the population of ostrich eggs.

SELF CHECK **What about the other, larger ostrich egg?**

Calculate the 3.75-lb ostrich egg yourself using the z-score we calculated before. What percentile would that egg be? Check below.

Answer: at Z=1.5, we go down to 1.5 and over to 0.00 (the first column). That value is 0.9332, which means a 3.75-lb ostrich egg is at the 93.32 percentile, or larger than 93.32% of the population.

Table A: Standard Normal Probabilities



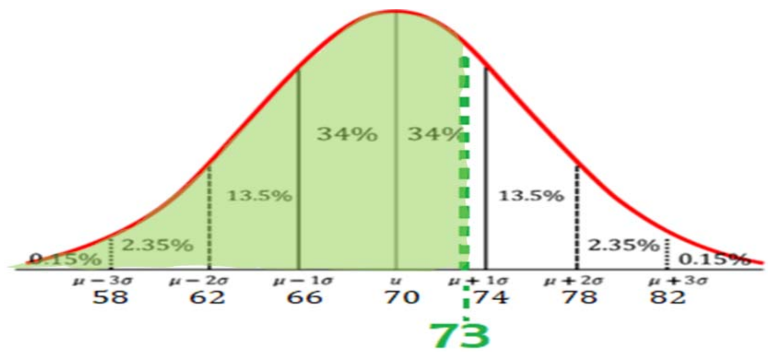
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table A: Standard Normal Probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Jonah is looking over the final exam scores of the previous year's graduates in the Engineering program from which he is about the graduate. The final exam scores of students were normally distributed with a mean of 70 and a standard deviation of 4. What percentile would Jonah be in if he scores a 73 on the final exam?

For any of the math problems involving normal distributions, it is helpful to draw out the standard normal distribution with the mean in the center of the curve, and with standard deviations steps marked, three standard deviations above the mean and three standard deviations below the mean.



$$z - score = \frac{(value - mean)}{standard\ deviation} = \frac{(x-\mu)}{\sigma} = \frac{(73-70)}{4} = \frac{(3)}{4} = 0.75$$

With a z-score of 0.75, you go to the charts, next page, and track down 0.7 first, then across to .05. This tells you that Jonah's proportion is .7734, and his percentile is 77.34. If you were asked to explain what this meant, you would state that "Jonah scored better on his final exam than 77.34% of the students who took the exam."

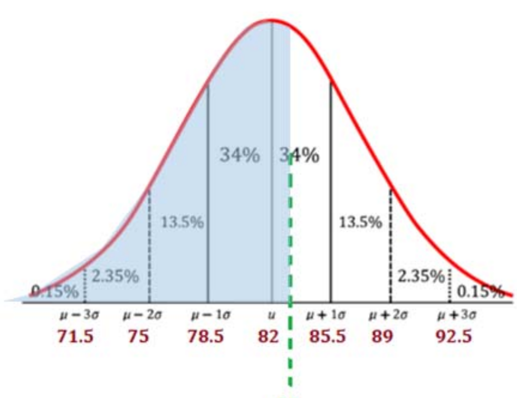
Example! You can find probabilities that scores fall within ranges, as well as the percentile.

Four recent competitors in "Battle of the Bands" received competition scores that were normally distributed with a mean of 82 and a standard deviation of 3.5. "Heavy Metal Trash Cans" will be competing this weekend. What is the probability of the band scoring between 83 and 90.5 in the competition?

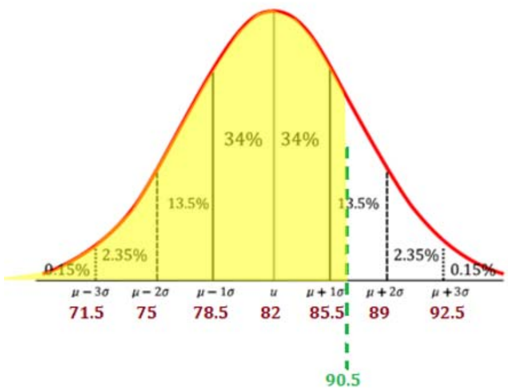
Some notation: you have used probability notation before. For this situation, you are looking for $P(83 < x < 90.5)$.

Solution:

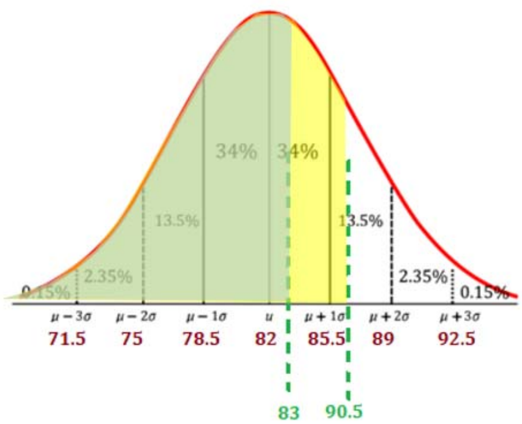
1. Calculate z-score for 83, then use the chart to find the percentile for that z-score. $Z_{83} = \frac{(83-82)}{3.5} \approx 0.29$, so $P(x < 83) = P(Z < 0.29) = .6141$



2. Calculate z-score for 90.5, then use the chart to find the percentile for that z-score. $Z_{90.5} = \frac{(90.5-82)}{3.5} \approx 2.43$, so $P(x < 90.5) = P(Z < 2.43) = 0.9925$



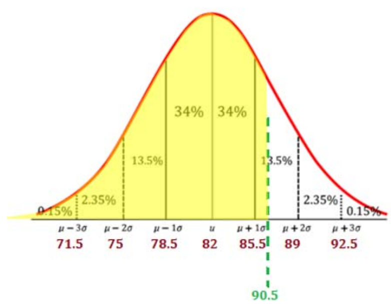
3. Lastly, The probability of the values falling between 90.5 and 83, $P(83 < x < 90.5)$, can be found by subtracting the two values. , $P(83 < x < 90.5) = 0.9925 - 0.6141 = 0.3784$.



If you were asked to explain this situation in context, you would say "the probability that any given band will score between 83 and 90.5 in the Battle of the Bands competition is about 0.3784, or 37.84%"

Example! Note that the chart is ALWAYS telling you the LEFT-hand shading (from zero to the number in question…the probability of scoring below the z-score). But you can figure out the probability that a value is GREATER by finding the complement of the left-hand side.

Revisit the previous problem: Four recent competitors in "Battle of the Bands" received competition scores that were normally distributed with a mean of 82 and a standard deviation of 3.5. "Heavy Metal Trash Cans" will be competing this weekend. What is the probability of the band scoring <u>greater than 90.5</u> in the competition?



Solution: We calculated $P(x < 90.5) = P(Z < 2.43) = 0.9925$ earlier, right? We found there is a probability proportion of 0.9925 (percentile 99.25) of scoring BELOW 90.5.

To find the probability proportion of scoring above 90.5, simply subtract the value from the chart from 1.

$P(x > 90.5) = 1 - P(x < 90.5) = 1 - 0.9925 = 0.0075.$

In terms of percentiles, if 99.25% of the population scored below 90.5, then 100% - 99.25% = 0.75% that score above 90.5. That is less than one percent!

1. Given a distribution with a mean of 70 and standard deviation of 62, find a value with a z-score of -1.82.

2. What does a z-score of 3.4 mean?

3. Given a distribution with a mean of 60 and standard deviation of 98, find the z-score of 120.76.

4. Given a distribution with a mean of 60 and standard deviation of 21, find a value with a z-score of 2.19.

5. Find the z-score of 187.37, given a distribution with a mean of 185 and standard deviation of 1.

6. What does a z-score of -3.8 mean?

7. Find the z-score of 125.18, given a distribution with a mean of 101 and standard deviation of 62.

8. Given a distribution with a mean of 117 and standard deviation of 42, find a value with a z-score of -0.94.

9. Given a distribution with a mean of 126 and standard deviation of 100, find a value with a z-score of -0.75.

10. Find the z-score of 264.16, given $\mu=188$ and $\sigma=64$.

11. Find a value with a z-score of -0.2, given $\mu=145$ and $\sigma=56$.

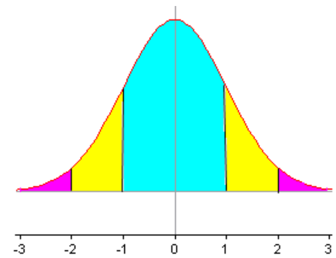
12. Find the z-score of 89.79 given $\mu=10$ and $\sigma=79$.

A2.U7.C2.**E.04.task.Z-Scores**

Let's Be Normal

Until this task, we have focused on distributions of discrete data. We will now direct our attention to continuous data. Where a **discrete variable** has a **finite number of possible values**, a continuous variable can assume all values in a given interval of values. Therefore, a **continuous random variab**le can assume an **infinite number of values**.
We will focus our attention specifically on continuous random variables with distributions that are approximately normal. Remember that normal distributions are symmetric, bell-shaped curves that follow the Empirical Rule.

The Empirical Rule for Normal Distributions states that



- 68% of the data values will fall within one standard deviation of the mean,
- 95% of the data values will fall within two standard deviations of the mean, and
- 99.7% of the data values will fall within three standard deviations of the mean.

In the last task, dot plots were used to explore this type of distribution and you spent time determining whether or not a given distribution was approximately normal. In this task, we will use probability histograms and approximate these histograms to a smooth curve that displays the shape of the distribution without the boxiness of the histogram. *We will also assume that all of the data we use is approximately normally distributed*.

Review:

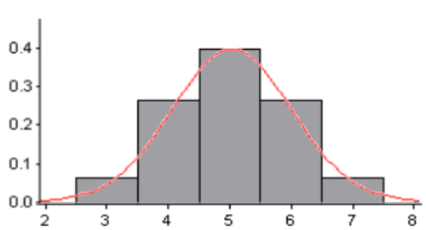
1**)** The distribution of heights of adult American women is approximately normal with a mean of 65.5 inches ($\mu = 65.5$) and a standard deviation of 2.5 inches ($\sigma = 2.5$).

a) Draw a normal curve and label the mean and points one, two, and three standard deviations above and below the mean.

b) What percent of women are taller than 70.5 inches?

c) Between what heights do the middle 95% of women fall?

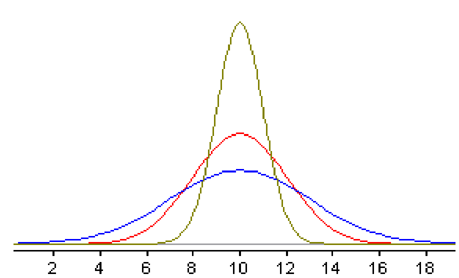
Below is an example of a probability histogram of a continuous random variable with an approximate mean of five ($\mu = 5$) and standard deviation of one ($\sigma = 1$). A normal curve with the same mean and standard deviation has been overlaid on the histogram.



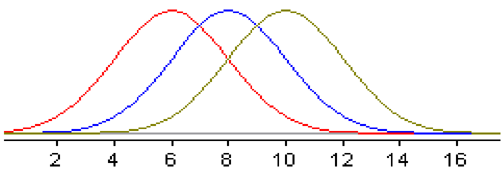
Keep in mind, normally distributed data may have any value for its mean and standard deviation.

2) Below are two graphs with three sets of normal curves.

a) Compare the means and standard deviations of the normal curves shown below. How do the difference(s) in these things affect the shape of the curve?



b) Compare the means and standard deviations of the normal curves shown below. How do the difference(s) in these things affect the shape of the curve?



3) The SAT Verbal Test in recent years follow approximately a normal distribution with a mean of 505 and a standard deviation of 110. Joey took this test made a score of 600.

The scores on the ACT English Test are approximately a normal distribution with a mean of 17 and a standard deviation of 2.5. Sarah took this test and made a score of 18.

Which student made the better score? How do you know?

The ***Standard Normal Distribution*** is a normal distribution with a mean of 0 and a standard deviation of 1. To more easily compute the probability of a particular observation given a normally distributed variable, we can transform any normal distribution to this standard normal distribution using the following formula:

$$z = \frac{X - \mu}{\sigma}$$

This is referred to as the z-score. The **z-score** is a standard score for a data value that indicates **the number of standard deviations** that a particular data value is away from its mean. We can use the z-score to find the probability of many other events.

4) a) Suppose that the mean time a typical American teenager spends doing homework each week is 4.2 hours. Assume the standard deviation is 0.9 hour. Assuming the variable is normally distributed, find the percentage of American teenagers who spend less than 3.5 hours doing homework each week.

- First, sketch a normal curve for this situation and shade the probability in which you are interested.
- Next, find the z-score for X = 3.5.
- Now, use the table of standard normal probabilities to determine the probability of P(X < 3.5).

b) The height of adult American males is normally distributed with a mean of 69.2 inches. If the standard deviation is 3.1 inches, determine the probability that a randomly selected adult American male will be at most 71 inches tall.

We can also use z-scores to find the percentage or probability of events above a given observation.

5) a) The average on the most recent test Ms. Cox gave her French students was 73 with a standard deviation of 8.2. Assuming the test scores were normally distributed, determine the probability that a student in Ms. Cox's class scored a 90 or more on the test.

b) Women's heights are approximately normally distributed with $\mu = 65.5$ inches and $\sigma = 2.5$ inches. Determine the probability of a randomly selected woman having a height of at least 64 inches.

We can also determine the probability between two values of a random variable.

6) a) According to the College Board, **Georgia** seniors graduating in 2008 had a mean Math SAT score of 493 with a standard deviation of 108. Assuming the distribution of these scores is normal, find the probability of a member of the 2008 graduating class in Georgia scoring between 500 and 800 on the Math portion of the SAT Reasoning Test.

b) According the same College Board report, the population of **American** 2008 high school graduates had a mean Math SAT score of 515 with $\sigma = 116$. What is the probability that a randomly selected senior from this population scoring between 500 and 800 on the Math portion of the SAT Reasoning Test?

Table A: Standard Normal Probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table A: Standard Normal Probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Z-Scores Practice Part 1 – Use z-scores and your Standard Normal Probabilities charts to answer the following questions.

The scores of a reference population on the Wechsler Intelligence Scale for Children (WISC) are normally distributed with μ = 100 and σ = 15.

1. What score would represent the 75th percentile? Explain.

2. Approximately what percent of the scores fall in the range from 75 to 135?

3. A score in what range would represent the top 10% of the scores?

Runner's World **reports that the times of the finishers in the New York City 10-km run are normally distributed with a mean of 61 minutes and a standard deviation of 9 minutes.**

4. Find the proportion of runners who take more than 2 hours to finish. Draw a sketch to show this proportion.

5. Find the proportion of runners who finish in less than an hour minutes. Draw a sketch to show this proportion.

Z-SCORES PRACTICE PART 2 – USE Z-SCORES AND YOUR STANDARD NORMAL PROBABILITIES CHARTS TO ANSWER THE FOLLOWING QUESTIONS.

The Chapin Social Insight Test evaluates how accurately the subject appraises other people. In the reference population used to develop the test, scores are approximately normally distributed with mean 25 and standard deviation 5. The range of possible scores is 0 to 41.

1. If a randomly selected student has a score of 39, then how many standard deviations away from the mean is that student's score? What is another name for this (what does this mean)?

2. Determine the standardized value for the score of 22.

3. What is the relative frequency corresponding to a score of 30 or more?

The Graduate Record Examinations are widely used to help predict the performance of applicants to graduate schools. The range of possible scores on a GRE is 200 to 900. The psychology department at a university finds that the scores of its applicants on the quantitative GRE are approximately normal with mean = 544 and standard deviation = 103. Use the table to find the relative frequency of applicants whose score X satisfies the following conditions: (As part of your answer, draw a standard normal curve for each, and shade the area under the curve that represented the answer to the question.

4. Determine the relative frequency of $X < 500$, and interpret this in context.

5. Determine the relative frequency of $500 < X < 700$, and interpret this in context.

6. What minimum score would a student need in order to score better than 77% of those taking the test? Explain.

Z-SCORES PRACTICE PART 3 – USE Z-SCORES AND YOUR STANDARD NORMAL PROBABILITIES CHARTS TO ANSWER THE FOLLOWING QUESTIONS.

Ziggy owns a lunch stand in the business district, and it has a mean daily gross income of \$420 with a standard deviation of \$50. Assume that his daily gross income is normally distributed.

1. If a randomly selected day has a gross income of \$495, then how many standard deviations away from the mean is that day's gross income?

2. Ziggy let his sister Xena work the lunch stand one day so he could go to a doctor's appointment. On that day, her sales were \$315. Determine the standardized value for the daily income of \$315, and describe what this means, about the day that Xena worked.

3. What is the relative frequency corresponding to a daily gross income of \$550 or more?

Using the Table (table of standard normal probabilities) or your calculator, find the proportion of observations from a standard normal distribution that satisfies each of the following statements. In each case, shade the area under the standard normal curve that is the answer to the question, and describe what this value means in context of Ziggy's lunch stand and his sales.

4. $Z < -2.25$

5. $-2.25 < Z < 1.77$