A. Functions

The lifeblood of precalculus is functions. A **function** is a set of points (x, y) such that for every x, there is one and only one y. In short, in a function, the x-values cannot repeat while the y-values can. In AB Calculus, all of your graphs will come from functions.

The notation for functions is either "$y =$" or "$f(x) =$". In the $f(x)$ notation, we are stating a rule to find y given a value of x.

1. If $f(x) = x^2 - 5x + 8$, find a) $f(-6)$ b) $f\left(\frac{3}{2}\right)$ c) $\frac{f(x+h) - f(x)}{h}$

a) $f(-6) = (-6)^2 - 5(-6) + 8$
$36 + 30 + 8$
74

b) $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 8$
$\frac{9}{4} - \frac{15}{2} + 8$
$\frac{11}{4}$

c) $\frac{f(x+h) - f(x)}{h}$
$\frac{(x+h)^2 - 5(x+h) + 8 - (x^2 - 5x + 8)}{h}$
$\frac{x^2 + 2xh + h^2 - 5x - 5h + 8 - x^2 + 5x - 8}{5}$
$\frac{h^2 + 2xh - 5h}{h} = \frac{h(h + 2x - 5)}{h} = h + 2x - 5$

Functions do not always use the variable x. In calculus, other variables are used liberally.

2. If $A(r) = \pi r^2$, find a) $A(3)$ b) $A(2s)$ c) $A(r+1) - A(r)$

$A(3) = 9\pi$

$A(2s) = \pi(2s)^2 = 4\pi s^2$

$A(r+1) - A(r) = \pi(r+1)^2 - \pi r^2$
$\pi(2r+1)$

One concept that comes up in AP calculus is **composition of functions**. The format of a composition of functions is: plug a value into one function, determine an answer, and plug that answer into a second function.

3. If $f(x) = x^2 - x + 1$ and $g(x) = 2x - 1$, a) find $f(g(-1))$ b) find $g(f(-1))$ c) show that $f(g(x)) \neq g(f(x))$

$g(-1) = 2(-1) - 1 = -3$
$f(-3) = 9 + 3 + 1 = 13$

$f(-1) = 1 + 1 + 1 = 3$
$g(3) = 2(3) - 1 = 5$

$f(g(x)) = f(2x-1) = (2x-1)^2 - (2x-1) + 1$
$= 4x^2 - 4x + 1 - 2x + 1 + 1 = 4x^2 - 6x + 3$
$g(f(x)) = g(x^2 - x + 1) = 2(x^2 - x + 1) - 1$
$= 2x^2 - 2x + 1$

Finally, expect to use **piecewise functions**. A piecewise function gives different rules, based on the value of x.

4. If $f(x) = \begin{cases} x^2 - 3, & x \geq 0 \\ 2x + 1, & x < 0 \end{cases}$, find a) $f(5)$ b) $f(2) - f(-1)$ c) $f(f(1))$

$f(5) = 25 - 3 = 22$

$f(2) - f(-1) = 1 - (-1) = 2$

$f(1) = -2,\ f(-2) = -3$

A. Function Assignment

• If $f(x) = 4x - x^2$, find

1. $f(4) - f(-4)$

2. $\sqrt{f\left(\frac{3}{2}\right)}$

3. $\dfrac{f(x+h) - f(x-h)}{2h}$

• If $V(r) = \dfrac{4}{3}\pi r^3$, find

4. $V\left(\frac{3}{4}\right)$

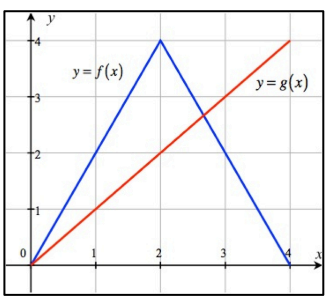
5. $V(r+1) - V(r-1)$

6. $\dfrac{V(2r)}{V(r)}$

• If $f(x)$ and $g(x)$ are given in the graph below, find

7. $(f-g)(3)$

8. $f(g(3))$



• If $f(x) = x^2 - 5x + 3$ and $g(x) = 1 - 2x$, find

9. $f(g(x))$

• If $f(x) = \begin{cases} \sqrt{x+2} - 2, & x \geq 2 \\ x^2 - 1, & 0 \leq x < 2 \\ -x, & x < 0 \end{cases}$, find

10. $f(0) - f(2)$

11. $\sqrt{5 - f(-4)}$

12. $f(f(3))$

B. Domain and Range

First, since questions in calculus usually ask about behavior of functions in intervals, understand that intervals can be written with a description in terms of $<, \leq, >, \geq$ or by using **interval notation**.

Description	Interval notation	Description	Interval notation	Description	Interval notation
$x > a$	(a,∞)	$x \leq a$	$(-\infty,a]$	$a \leq x < b$	$[a,b)$
$x \geq a$	$[a,\infty)$	$a < x < b$	(a,b) - open interval	$a < x \leq b$	$(a,b]$
$x < a$	$(-\infty,a)$	$a \leq x \leq b$	$[a,b]$ - closed interval	All real numbers	$(-\infty,\infty)$

If a solution is in one interval or the other, interval notation will use the connector $\cup$. So $x \leq 2$ or $x > 6$ would be written $(-\infty,2]\cup(6,\infty)$ in interval notation. Solutions in intervals are usually written in the easiest way to define it. For instance, saying that $x < 0$ or $x > 0$ or $(-\infty,0)\cup(0,\infty)$ is best expressed as $x \neq 0$.

The **domain of a function** is the set of allowable x-values. The domain of a function f is $(-\infty,\infty)$ except for values of x which create a zero in the denominator, an even root of a negative number or a logarithm of a non-positive number. The domain of $a^{p(x)}$ where a is a positive constant and $p(x)$ is a polynomial is $(-\infty,\infty)$.

• Find the domain of the following functions using interval notation:

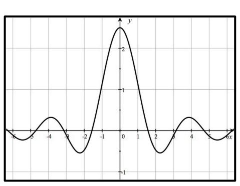
1. $f(x) = x^2 - 4x + 4$ $\boxed{(-\infty,\infty)}$
2. $y = \dfrac{6}{x-6}$ $\boxed{x \neq 6}$
3. $y = \dfrac{2x}{x^2 - 2x - 3}$ $\boxed{x \neq -1, x \neq 3}$
4. $y = \sqrt{x+5}$ $\boxed{[-5,\infty)}$
5. $y = \sqrt[3]{x+5}$ $\boxed{(-\infty,\infty)}$
6. $y = \dfrac{x^2 + 4x + 6}{\sqrt{2x+4}}$ $\boxed{(-2,\infty)}$

The **range of a function** is the set of allowable y-values. Finding the range of functions algebraically isn't as easy (it really is a calculus problem), but visually, it is the [lowest possible y-value, highest possible y-value]. Finding the range of some functions are fairly simple to find if you realize that the range of $y = x^2$ is $[0,\infty)$ as any positive number squared is positive. Also the range of $y = \sqrt{x}$ is also positive as the domain is $[0,\infty)$ and the square root of any positive number is positive. The range of $y = a^x$ where a is a positive constant is $(0,\infty)$ as constants to powers must be positive.

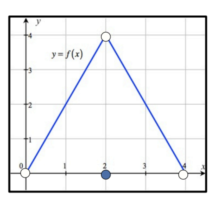
• Find the range of the following functions using interval notation:

7. $y = 1 - x^2$ $\boxed{(-\infty,1]}$
8. $y = \dfrac{1}{x^2}$ $\boxed{(0,\infty)}$
9. $y = \sqrt{x-8} + 2$ $\boxed{[2,\infty)}$

• Find the domain and range of the following functions using interval notation.

10. 

Domain: $(-\infty,\infty)$
Range: $[-0.5,2.5]$

11. 

Domain: $(0,4)$
Range: $[0,4)$

B. Domain and Range Assignment

• Find the domain of the following functions using interval notation:

1. $f(x) = 3$

2. $y = x^3 - x^2 + x$

3. $y = \dfrac{x^3 - x^2 + x}{x}$

4. $y = \dfrac{x-4}{x^2 - 16}$

5. $f(x) = \dfrac{1}{4x^2 - 4x - 3}$

6. $y = \sqrt{2x-9}$

7. $f(t) = \sqrt{t^3 + 1}$

8. $f(x) = \sqrt[5]{x^2 - x - 2}$

9. $y = 5^{x^2 - 4x - 2}$

10. $y = \log(x - 10)$

11. $y = \dfrac{\sqrt{2x+14}}{x^2 - 49}$

12. $y = \dfrac{\sqrt{5-x}}{\log x}$

Find the range of the following functions.

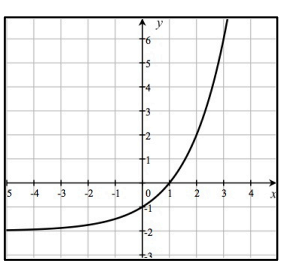
13. $y = x^4 + x^2 - 1$

14. $y = 100^x$

15. $y = \sqrt{x^2 + 1} + 1$

Find the domain and range of the following functions using interval notation.

16.



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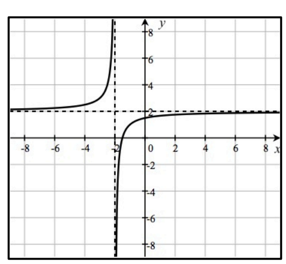

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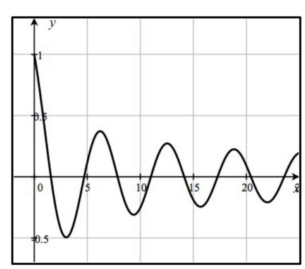
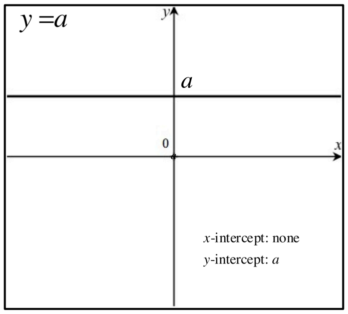


C. Graphs of Common Functions

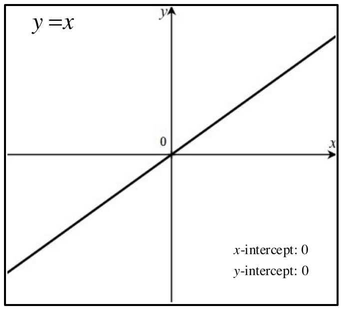
There are certain graphs that occur all the time in calculus and students should know the general shape of them, where they hit the x-axis (zeros) and y-axis (y-intercept), as well as the domain and range. There are no assignment problems for this section other than students memorizing the shape of all of these functions. In section 5, we will talk about transforming these graphs.



Function: $y = a$

Domain: $(-\infty, \infty)$

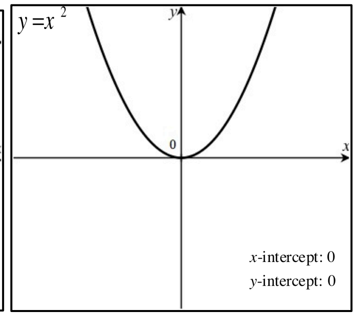
Range: $[a, a]$



Function: $y = x$

Domain: $(-\infty, \infty)$

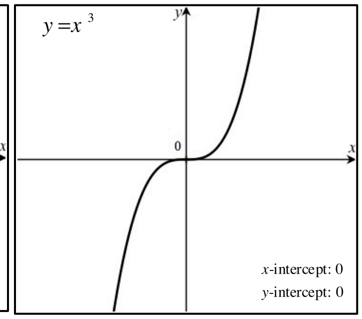
Range: $(-\infty, \infty)$



Function: $y = x^2$

Domain: $(-\infty, \infty)$

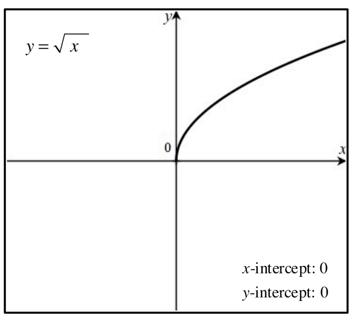
Range: $[0, \infty)$



Function: $y = x^3$

Domain: $(-\infty, \infty)$

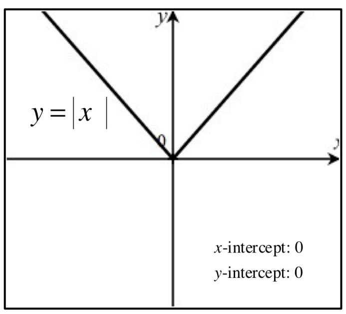
Range: $(-\infty, \infty)$



Function: $y = \sqrt{x}$

Domain: $[0, \infty)$

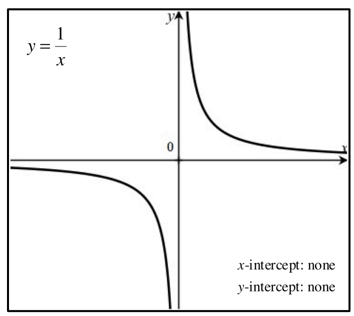
Range: $[0, \infty)$



Function: $y = |x|$

Domain: $(-\infty, \infty)$

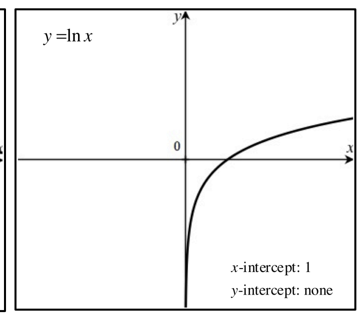
Range: $[0, \infty)$



Function: $y = \dfrac{1}{x}$

Domain: $x \neq 0$

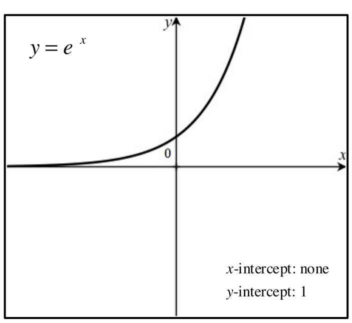
Range: $y \neq 0$



Function: $y = \ln x$

Domain: $(0, \infty)$

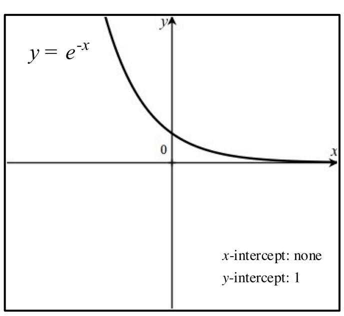
Range: $(-\infty, \infty)$



Function: $y = e^x$

Domain: $(-\infty, \infty)$

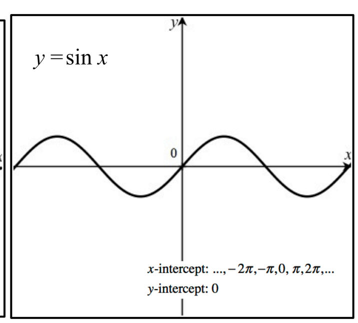
Range: $(0, \infty)$



Function: $y = e^{-x}$

Domain: $(-\infty, \infty)$

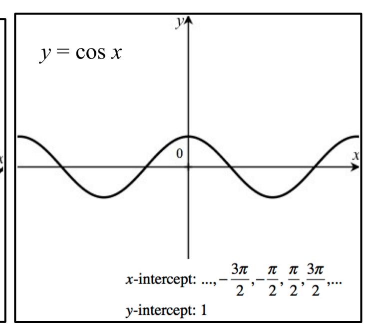
Range: $(0, \infty)$



Function: $y = \sin x$

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$



Function: $y = \cos x$

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$